Extra Credit III   Math 116

Due Dec 19, 2012

Remember: No credit will be given without mathematical or logical justification.
This extra credit is worth one homework assignment.

Part 1: Relativity

Just two things are required to understand Special Relativity: vector calculus and linear algebra. In Part 1 we introduce the part of the theory that requires calculus (namely the concept of proper time), and in Part 2 we look at the part of the theory that requires linear algebra (namely the concept of reference frame).

Several formulations of Special Relativity exist; here we explore the geometric formulation. Einstein's original theory was algebraic, with various transformation laws depending on the relative velocities of various observers. Two years after Einstein's original paper, Hermann Minkowski gave a geometric formulation of the theory, which is in fact more useful and probably more intuitive than Einstein's original formulation.[1]

Space-time is $\mathbb{R}^4$, with points of the form (x_0, x_1, x_2, x_3). A particle is a path in space-time; the path itself is called the particle's *world line*.

The coordinate x^0 is called *coordinate time*. Coordinate time does not necessarily have anything to do with time that is experienced or measured. The difference between space-time and ordinary $\mathbb{R}^4$ is the way dot products are taken. Given vectors

$$\vec{X} \;=\; (x_0,\, x_1,\, x_2,\, x_3) \qquad \vec{Y} \;=\; (y_0,\, y_1,\, y_2,\, y_3) \tag{1}$$

their space-time dot product is defined to be

$$\vec{X}\cdot\vec{Y} \;=\; x_0y_0 \;-\; \frac{1}{c^2}x_1y_1 \;-\; \frac{1}{c^2}x_2y_2 \;-\; \frac{1}{c^2}x_3y_3 \tag{2}$$

where $c \approx 2.99\times 10^8 m/s$ is the speed of light. The space-time "norm-square" of a vector $\vec{X}$ is therefore

$$\|\vec{X}\|^2 \;=\; \vec{X}\cdot\vec{X} \;=\; (x_0)^2 \;-\; \frac{1}{c^2}(x_1)^2 \;-\; \frac{1}{c^2}(x_2)^2 \;-\; \frac{1}{c^2}(x_3)^2 \tag{3}$$

which can easily be zero or negative. Space-time along with this dot product is called *Minkowski space*, and is denoted $\mathbb{R}^{1,3}$.

A vector $\vec{X}\in\mathbb{R}^{1,3}$ is called

- *Time-like* if $\vec{X}\cdot\vec{X} \;>\; 0$
- *Space-like* if $\vec{X}\cdot\vec{X} \;<\; 0$
- *Light-like* if $\vec{X}\cdot\vec{X} \;=\; 0$ but $\vec{X}\neq\mathcal{O}$.

[1] Minkowski was a former math professor of Einstein's. Einstein did not do well in Minkowski's class and, according to legend, Minkowski never lost his low opinion of his former student and regarded his work on special relativity to be derivative—basically a rip-off of earlier work of Poincaré and Lorentz. It continues to be a matter of debate. Minkowski did not live to see Einstein's later work on general relativity.

A path

$$\vec{\gamma}(\tau) \;=\; (\gamma_0(\tau),\, \gamma_1(\tau),\, \gamma_2(\tau),\, \gamma_3(\tau)) \tag{4}$$

is called time-like, space-like, or light-like at $\tau = \tau_0$ if the velocity vector $\frac{d\vec{\gamma}}{d\tau}\big|_{\tau=\tau_0}$ is time-like, space-like or light-like (the parameter τ is NOT time; it is just a parameter.)

Now we come to the physical interpretation. A path $\vec{\gamma}(\tau)$ is physically admissible if $\frac{d\vec{\gamma}}{d\tau}$ is time-like or light-like at each τ (in other words, NOT space-like). In this case the quantity $\|\vec{\gamma}(\tau)\| = \sqrt{\vec{\gamma} \cdot \vec{\gamma}}$ is a real number, and the arc-length from $\tau = \tau_1$ to $\tau = \tau_2$

$$\int_{\tau_1}^{\tau_2} \left\| \frac{d\vec{\gamma}}{d\tau} \right\| \, d\tau \tag{5}$$

is interpreted to be the time the particle experiences as it travels along its path between $\tau = \tau_1$ and $\tau = \tau_2$. This is called the particle's *proper time*. It is NOT the same as coordinate time in most cases.

1) Which of the following paths are physically admissible?

   a) $\vec{\gamma}_1(\tau) \;=\; (\tau,\, 0,\, 0,\, 0)$

   b) $\vec{\gamma}_2(\tau) \;=\; (\tau,\, \tau^3,\, 0,\, 0)$

   c) $\vec{\gamma}_3(\tau) \;=\; (\tau,\, c\cos(\tau),\, c\sin(\tau),\, 0)$

2) What is the time experienced by the following two particles

$$\vec{\gamma}_4(\tau) \;=\; \left(\tau,\, \frac{c}{2}\tau,\, 0,\, 0\right) \tag{6}$$

$$\vec{\gamma}_5(\tau) \;=\; (\tau,\, c\,\tau,\, 0,\, 0) \tag{7}$$

   between $\tau = 0$ and $\tau = 1$?

3) Letting

$$\vec{\eta}(\tau) = (\eta_0(\tau),\, \eta_1(\tau),\, \eta_2(\tau),\, \eta_3(\tau)) \tag{8}$$

   be a path, we have already discussed the interpretation of its velocity 4-vector. But how would you measure the particle's speed? Relative to a given frame, the particle's *velocity 3-vector* is the vector

$$\vec{v} \;=\; \left(\frac{d\eta_1/d\tau}{d\eta_0/d\tau},\, \frac{d\eta_2/d\tau}{d\eta_0/d\tau},\, \frac{d\eta_3/d\tau}{d\eta_0/d\tau} \right) \tag{9}$$

   and the particle's speed is

$$\sqrt{\left|\frac{d\eta_1/d\tau}{d\eta_0/d\tau}\right|^2 + \left|\frac{d\eta_2/d\tau}{d\eta_0/d\tau}\right|^2 + \left|\frac{d\eta_3/d\tau}{d\eta_0/d\tau}\right|^2} \tag{10}$$

   (ie. the usual vector norm for 3-vectors). In different frames, velocity and speed will be different (see below to learn by what is meant by "frame").

   a) Compute the speeds of the paths from problems (1) and (2). Which path represents a particle that isn't moving at all? Which represents a particle that is moving at the speed of light? Do any of the particles move faster than light?

b) In terms of 3-velocity, what does it mean for a particle to have a space-like velocity vector?

8) Show that the path $\vec{\gamma}(\tau) = (\tau, 0, 0, 0)$ is a the path of a stationary particle (meaning the 3-velocity is zero). In the case of this stationary particle, show that the particle's coordinate time is the same as its proper time.

9) Suppose you are a stationary observer, so your world-line is given by $\vec{\gamma}(\tau) = (\tau, 0, 0, 0)$. Consider a particle that has world-line

$$\vec{\eta}(\tau) \;=\; (\tau,\, c\sin(\tau),\, 0,\, 0)\,. \tag{11}$$

a) According to your measurements, what are the particle's maximum and minimum speeds?

b) After you have measured exactly π seconds of time elapsed (starting from $\tau = 0$), how much time has the particle experienced?

d) Assume $\vec{\eta}(s)$ is the path taken by some kind of space ship. After you have aged by $10,000,000 \times \pi$ many seconds (about 1 year), how much have the ship's occupants aged?

e) Now assume $\vec{\eta}(\tau) = (\tau, \frac{1}{2}c\tau, 0, 0)$. Show the ship's speed is half the speed of light. After you have measured $10,000,000 \times \pi$ many seconds go by, how many seconds has the ship measured go by?

Part 2. Change of Frame

General Theory

The standard basis for $\mathbb{R}^{1,3}$ is

$$\vec{E}_0 = (1,0,0,0) \quad \vec{E}_1 = (0,1,0,0) \quad \vec{E}_2 = (0,0,1,0) \quad \vec{E}_3 = (0,0,0,1) \tag{12}$$

This is *orthonormal* in the sense that

$$\begin{aligned} \vec{E}_0 \cdot E_0 \;&=\; 1 \\ \vec{E}_1 \cdot \vec{E}_1 \;&=\; \vec{E}_2 \cdot \vec{E}_2 \;=\; \vec{E}_3 \cdot \vec{E}_3 \;=\; -c^{-2} \\ \vec{E}_i \cdot \vec{E}_j \;&=\; 0 \quad for\ i \neq j. \end{aligned} \tag{13}$$

In this frame, coordinate time and proper time are equal for stationary observers, and for stationary observers only.

However, the meaning of the word "stationary" is relative! An unaccelerated observer will also consider him or herself to be stationary, regardless of how their velocity is measured by others. Such an observer will construct space-time using different unit vectors. Mathematically, converting from one observer to another is simply a change of the orthonormal basis (aka the frame) of $\mathbb{R}^{1,3}$.

Definition. *A* **frame** *is a selection of basis vectors $\vec{F}_0$, $\vec{F}_1$, $\vec{F}_2$, $\vec{F}_3$ for $\mathbb{R}^{1,3}$ that obey the orthonormality relations:*

$$\begin{aligned} \vec{F}_0 \cdot \vec{F}_0 \;&=\; 1 \\ \vec{F}_1 \cdot \vec{F}_1 \;&=\; \vec{F}_2 \cdot \vec{F}_2 \;=\; \vec{F}_3 \cdot \vec{F}_3 \;=\; -c^{-2} \\ \vec{F}_i \cdot \vec{F}_j \;&=\; 0 \quad for\ i \neq j. \end{aligned} \tag{14}$$

Definition. *If a particle is un-accelerated (moving at constant velocity), a* **reference frame** *is any frame in which that particle's 3-velocity is 0.*

Given two choices of frame

$$\begin{aligned} \mathbf{E} &= \left\{ \vec{E}_0,\, \vec{E}_1,\, \vec{E}_2,\, \vec{E}_3 \right\} \\ \mathbf{F} &= \left\{ \vec{F}_0,\, \vec{F}_1,\, \vec{F}_2,\, \vec{F}_3 \right\} \end{aligned} \tag{15}$$

we can express points in space-time with respect to either of the frames. Given a point $\vec{P}$, we write

$$\vec{P} \;=\; (a_0,\, a_1,\, a_2,\, a_3)_E \;=\; a_0\vec{E}_0 + a_1\vec{E}_1 + a_2\vec{E}_2 + a_3\vec{E}_3 \tag{16}$$

to indicate the coordinates of $\vec{P}$ in the **E**-frame and

$$\vec{P} \;=\; (a_0,\, a_1,\, a_2,\, a_3)_F \;=\; a_0\vec{F}_0 + a_1\vec{F}_1 + a_2\vec{F}_2 + a_3\vec{F}_3 \tag{17}$$

to indicate the coordinates of $\vec{P}$ in the **F**-frame.

10) Let **E** be a frame for $\mathbb{R}^{1,3}$, and set

$$\begin{aligned} \vec{F}_0 &= \sqrt{2}\vec{E}_0 + c\vec{E}_1 \\ \vec{F}_1 &= \frac{1}{c}\vec{E}_0 + \sqrt{2}\vec{E}_1 \\ \vec{F}_2 &= \vec{E}_2 \\ \vec{F}_3 &= \vec{E}_3 \end{aligned} \tag{18}$$

Prove that $\mathbf{F} = \{\vec{F}_0, \vec{F}_1, \vec{F}_2, \vec{F}_3\}$ is a frame. Show that $\vec{E}_0 = \sqrt{2}\vec{F}_0 - c\vec{F}_1$ and $\vec{E}_1 = -\frac{1}{c}\vec{F}_0 + \sqrt{2}\vec{F}_1$. These are the *change of frame equations*.

11) Let **E**, **F** be the frames from Problem 10. Given the path

$$\vec{\gamma}(\tau) \;=\; (\tau,\, 0, 0, 0)_F \tag{19}$$

expressed in the **F**-basis, express $\vec{\gamma}(\tau)$ in the **E**-basis. In the **F**-frame, how fast is $\vec{\gamma}$ moving? (That is, compute the 3-velocity and corresponding speed.) How fast is $\vec{\gamma}$ moving in the **E**-frame? (Again, compute the 3-velocity.)

Reduction to $\mathbb{R}^{1,1}$

If we pretend space is 1-dimensional so that space-time, $\mathbb{R}^{1,1}$, is 2-dimensional, then we can actually graph things. Let $\mathbf{E} = \{\vec{E}_0,\, \vec{E}_1\}$ and $\mathbf{F} = \{\vec{F}_0, \vec{F}_1\}$ be two frames, with change-of-frame equations

$$\begin{aligned} \vec{E}_0 &= \sqrt{2}\vec{F}_0 - c\vec{F}_1 \\ \vec{E}_1 &= -\frac{1}{c}\vec{F}_0 + \sqrt{2}\vec{F}_1 \end{aligned} \tag{20}$$

$$\begin{aligned} \vec{F}_0 &= \sqrt{2}\vec{E}_0 + c\vec{E}_1 \\ \vec{F}_1 &= \frac{1}{c}\vec{E}_0 + \sqrt{2}\vec{E}_1. \end{aligned} \tag{21}$$

12) Set

$$\begin{aligned} \vec{\gamma}_1(\tau) &= (\tau, 0)_E \\ \vec{\gamma}_2(\tau) &= (\sqrt{2}\tau,\, c\tau)_E \end{aligned} \tag{22}$$

and

$$\begin{aligned} \vec{\eta}_1(\tau) &= (\tau, c\tau)_E \\ \vec{\eta}_2(\tau) &= (\tau, -c\tau)_E. \end{aligned} \tag{23}$$

a) Prove $\vec{\gamma}_1$, $\vec{\gamma}_2$ are time-like, and prove $\vec{\eta}_1$, $\vec{\eta}_2$ are light-like.

b) Plot all four paths in the $\vec{E}_0$-$\vec{E}_1$ plane, and label each path appropriately.

c) At coordinate-time 1, how much time has $\vec{\gamma}_1$ measured? How much time has γ_2 measured?

13) Now we make the change to the **F**-frame.

a) Express the four paths $\vec{\gamma}_1$, $\vec{\gamma}_2$, $\vec{\eta}_1$, $\vec{\eta}_2$ in the **F**-frame.

b) Graph and label all four paths on the $\vec{F}_0$-$\vec{F}_1$ plane. Notice that the light-like paths are the only paths that have not changed their locations.

c) At coordinate-time 1, how much time have $\vec{\gamma}_1$ and $\vec{\gamma}_2$ measured?

d) Note the "paradox" here: When either $\vec{\gamma}_1$, $\vec{\gamma}_2$ have measured 1 second going by, *each* sees the other as having had *less* than 1 second go by. There is in fact no paradox here at all. Explain why.

14) What is the reference frame for the particle whose path is $\vec{\gamma}_1$? whose path is $\vec{\gamma}_2$?