Extra Credit II - Addendum Math 116

Due Dec 4, 2012

11) As it happens, we cannot take the limit of $\|\vec{x}\|$ as $p \searrow 0$; the limit is usually infinite. However, after slightly redefining the l^p norms, we can take this limit, and it gives us something interesting. Let

$$\|\vec{x}\|_{\tilde{p}} = \left(\frac{1}{n} \sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \tag{1}$$

be the $\tilde{l}^p$-functional. Notice that

$$\|\vec{x}\|_{\tilde{p}} = n^{-\frac{1}{p}} \|\vec{x}\|_p \tag{2}$$

so this is just a scaled version of the usual l^p functional. In the case $1 \leq p < \infty$, the $\tilde{l}^p$-functionals still obey the triangle inequality, so they are still norms. Prove

$$\lim_{p \searrow 0} \|\vec{x}\|_{\tilde{p}} = \exp\left(\frac{1}{n} \sum_{\substack{i=1 \\ x_i \neq 0}}^{n} \log |x_i| \right) = \left(\prod_{i=1}^{n} |x_i| \right)^{\frac{1}{n}} \tag{3}$$

12) Define the l^0-functional to be the geometric mean functional:

$$\|\vec{x}\|_0 = \left(\prod_{i=1}^{n} |x_i| \right)^{\frac{1}{n}}. \tag{4}$$

Compute $\|\vec{x}\|_0$ for $\vec{x} = (1, 2, 3) \in \mathbb{R}^3$, $\vec{x} = (1 - i, 2i) \in \mathbb{C}^2$, and $\vec{x} = (0, 2, 1, 0) \in \mathbb{R}^4$.

Thanks to the student who pointed out an error.