Extra Credit II Math 116

Due Dec 4, 2012

Remember: No credit will be given without mathematical or logical justification.
This extra credit is worth one homework assignment.

Part 1: Young, Hölder, Minkowski

In the previous extra credit, we used Cauchy's inequality to prove Cauchy-Schwartz, which is used to prove the triangle inequality (which we did in class). Schematically,

$$Cauchy \quad \Longrightarrow \quad Cauchy-Schwarz \quad \Longrightarrow \quad Triangle$$

Finally we proved Young's inequality, and showed how Cauchy's was just a special case of Young's. Here we shall ask and answer two questions: is Cauchy-Schwartz a special case of something more general? (Yes: Hölder's inequality.) Is the triangle inequality a special case of something more general? (Yes: Minkowski's inequality.) Schematically,

$$Young's \quad \Longrightarrow \quad H\ddot{o}lder \quad \Longrightarrow \quad Minkowski$$

Throughout, we may let our vector space, denoted $\mathbb{V}^n$, be either $\mathbb{R}^n$ or $\mathbb{C}^n$; the theory is identical in either case. Given a vector $\vec{x} \in \mathbb{V}^n$, we have

$$\vec{x} \;=\; (x_1,\, \ldots,\, x_n) \tag{1}$$

where the components x_i may be either real or complex numbers. For any $p \geq 1$, we have the following norm:

$$\|\vec{x}\|_p \;\triangleq\; \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \tag{2}$$

which is called the l^p-norm. Also define the l^∞ norm by

$$\|\vec{x}\|_\infty \;\triangleq\; \max_{i\in\{1,\ldots,n\}} |x_i|. \tag{3}$$

1) Let $\vec{x} = (1,\, 2,\, 3,\, 4)$ and compute the following:

   a) $\|\vec{x}\|_1$
   b) $\|\vec{x}\|_2$
   c) $\|\vec{x}\|_3$
   d) $\|\vec{x}\|_{10}$
   e) $\|\vec{x}\|_\infty$

2) Prove that the l^2-norm on $\mathbb{V}^n$ is the same as the usual dot product norm.

3) In the previous homework, you proved *Young's inequality*: given $a, b, p, q > 0$ so that $\frac{1}{p} + \frac{1}{q} = 1$ then

$$ab \;\leq\; \frac{1}{p}a^p \;+\; \frac{1}{q}b^q. \tag{4}$$

Using this, prove the following:

$$|ab| \;\leq\; \frac{1}{p}|a|^p \;+\; \frac{1}{q}|b|^q. \tag{5}$$

where a, b are any complex (or real) numbers.

4) Using Problem (3), prove *Young's inequality for sums*:

$$\sum_{i=1}^{n} |a_i b_i| \;\leq\; \frac{1}{p}\sum_{i=1}^{p} |a_i|^p \;+\; \frac{1}{q}\sum_{i=1}^{n} |b_i|^p \tag{6}$$

where the a_i, b_i are either real or complex numbers. From this, prove *Young's inequality for vectors*:

$$\left|\vec{A}\cdot\vec{B}\right| \;\leq\; \frac{1}{p}\|\vec{A}\|_p^p \;+\; \frac{1}{q}\|\vec{B}\|_q^q \tag{7}$$

Whenever $\vec{A}, \vec{B} \in \mathbb{V}^n$ (the "$\cdot$" is either the bilinear or sesquilinear dot product, as appropriate).

5) Using Young's inequality for sums, prove *Hölder's inequality for sums*:

$$\sum_{i=1}^{n} |a_i b_i| \;\leq\; \left(\sum_{i=1}^{n} |a_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |b_i|^q\right)^{\frac{1}{q}} \tag{8}$$

(where the a_i, b_i are complex or real) and then *Hölder's inequality for vectors*:

$$\left|\vec{A}\cdot\vec{B}\right| \;\leq\; \|\vec{A}\|_p\|\vec{B}\|_q \tag{9}$$

where, as always, we take $\frac{1}{p} + \frac{1}{q} = 1$. (Hint: Use the method you used to prove Cauchy-Schwarz from Cauchy.)

6) Show that the Cauchy-Schwarz inequality is a special case of Hölder's inequality.

7) Prove *Minkowski's Inequality*: If $p \geq 1$ then

$$\|\vec{A} + \vec{B}\|_p \;\leq\; \|\vec{A}\|_p + \|\vec{B}\|_p. \tag{10}$$

("Hint": First assume $p > 1$, so setting $q = \frac{p}{p-1}$ you have $\frac{1}{p} + \frac{1}{q} = 1$. Then justify each of the following steps:

$$\begin{aligned}
\|\vec{A} + \vec{B}\|_p^p &= \sum_{i=1}^{n} |a_i + b_i|^p \\
&\leq \sum_{i=1}^{n} |a_i||a_i + b_i|^{p-1} + \sum_{i=1}^{n} |b_i||a_i + b_i|^{p-1} \\
&\leq \|\vec{A}\|_p\|\vec{A}+\vec{B}\|_p^{p-1} + \|\vec{B}\|_p\|\vec{A}+\vec{B}\|_p^{p-1} \\
\|\vec{A} + \vec{B}\|_p &\leq \|\vec{A}\|_p + \|\vec{B}\|_p.
\end{aligned} \tag{11}$$

For the case $p = 1$, take the limit as $p \searrow 1$.)

8) Formally prove that $\|\cdot\|_p$ is a norm, when $p \geq 1$.

Part 2: More on the l^p norms

9) Why is the sup-norm, $\|\cdot\|_\infty$, written as though it is an l^p norm with $p = \infty$? To answer this, prove that for any $\vec{x} \in \mathbb{V}^n$ we have

$$\|\vec{x}\|_\infty \;=\; \lim_{p\to\infty} \|\vec{x}\|_p. \tag{12}$$

10) If $0 < p < 1$, prove that the triangle inequality does *not* hold for $\|\cdot\|_p$.

11) SEE ADDENDUM

12) SEE ADDENDUM

13) Even if $p < 0$, we get some information, though $\|\vec{x}\|_p$ is nowhere close to being a norm. With $\vec{x} = (x_1, \ldots, x_n)$, assume that each x_i is non-zero, and prove that

$$\lim_{p\to-\infty} \|\vec{x}\|_p \;=\; \min_{i\in\{1,\ldots,n\}} \{|x_1|, \ldots, |x_n|\}. \tag{13}$$

and explain why it makes sense to define the $l^{-\infty}$-functional, $\|\cdot\|_{-\infty}$, by

$$\|\vec{x}\|_{-\infty} \;=\; \min_{i\in\{1,\ldots,n\}} \{|x_1|, \ldots, |x_n|\} \tag{14}$$

even if some of the x_i are zero. What is $\|(10, 12, 13, 14, 15)\|_{-\infty}$? $\|(0, 1, 2, 3, 4)\|_{-\infty}$?

Part 3: The vector spaces $\mathbb{V}^\infty$ with the l^p-norms

Let $\mathbb{V}^\infty$ be either $\mathbb{R}^\infty$ or $\mathbb{C}^\infty$. Recall that a vector $\vec{x} \in \mathbb{V}^\infty$ is an ordered list of numbers

$$\vec{x} \;=\; (x_1,\, x_2,\, \ldots,\, x_i,\, \ldots) \tag{15}$$

where the components x_i are in either $\mathbb{R}$ or $\mathbb{C}$. Recall that $\mathbb{V}^\infty$ is a vector space. On $\mathbb{V}^\infty$ we can place any of the functionals $\|\cdot\|_p$ by setting

$$\|\vec{x}\|_p \;\triangleq\; \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{\frac{1}{p}}. \tag{16}$$

However, none of the $\|\cdot\|_p$ are actually norms on $\mathbb{V}^\infty$! The reason is that the sum usually diverges.

<u>Definition</u>. The l^p-space is the subset of $\mathbb{V}^\infty$ consisting of those $\vec{x} \in \mathbb{V}^\infty$ for which the sum $\sum_{i=1}^\infty |x_i|^p$ converges.

14) Consider the following vectors in $\mathbb{V}^\infty$:

$$\begin{aligned}
\vec{x} \;&=\; (1,\, 2,\, 3,\, \ldots,\, i,\, \ldots) \\
\vec{y} \;&=\; (1,\, 1,\, 1,\, \ldots) \\
\vec{z} \;&=\; \left(1,\, \frac{1}{\sqrt{2}},\, \frac{1}{\sqrt{3}},\, \ldots,\, \frac{1}{\sqrt{i}},\, \ldots\right) \\
\vec{w} \;&=\; \left(1,\, \frac{1}{2},\, \frac{1}{3},\, \ldots,\, \frac{1}{i},\, \ldots\right) \\
\vec{v} \;&=\; \left(1,\, \frac{1}{4},\, \frac{1}{9},\, \ldots,\, \frac{1}{i^2},\right)
\end{aligned} \tag{17}$$

For each vector above, determine if it is an element of l^1, l^2, l^4, and/or l^∞.

15) We have not yet proven that l^p is a vector space: in particular, if $\vec{x}, \vec{y} \in l^p$, is $\vec{x} + \vec{y}$ also in l^p? If $\vec{x}, \vec{y} \in l^p$ and c_1, c_2 are constants, formally prove that $c_1\vec{x} + c_2\vec{y} \in l^p$. Using this, prove that each l^p is a vector space. Each norm determines a *different* infinite-dimensional vector space!

16) Prove formally that $\|\cdot\|_p$ is a norm on l^p.

17) If $\vec{A}, \vec{B} \in \mathbb{V}^\infty$, we define their (sesquilinear) dot product

$$\vec{A} \cdot \vec{B} \;=\; \sum_{i=1}^{\infty} a_i \overline{b_i} \tag{18}$$

provided the sum converges absolutely.

a) If $\vec{A} \in l^p$ and $\vec{B} \in l^q$ where $\frac{1}{p} + \frac{1}{q} = 1$, formally prove that

$$\sum_{i=1}^{\infty} a_i \overline{b_i} \tag{19}$$

converges absolutely.

b) If $\vec{A} \in l^p$ and $\vec{B} \in l^q$ where $\frac{1}{p} + \frac{1}{q} = 1$, formally prove that

$$\left|\vec{A} \cdot \vec{B}\right| \ \leq \ \frac{1}{p}\|\vec{A}\|_p^p \ + \ \frac{1}{q}\|\vec{B}\|_q^q \tag{20}$$

c) If $\vec{A} \in l^p$ and $\vec{B} \in l^q$ where $\frac{1}{p} + \frac{1}{q} = 1$, formally prove that

$$\left|\vec{A} \cdot \vec{B}\right| \ \leq \ \|\vec{A}\|_p \|\vec{B}\|_q \tag{21}$$

18) A path $\vec{f}(t)$, $t_0 \leq t \leq t_1$ is called l^p-rectifiable assuming $\vec{f}(t) \in l^p$, $\vec{f}'(t) \in l^p$, and $\int_{t_0}^{t_1} \|\vec{f}'(t)\|_p dt < \infty$.

a) Consider the path $\vec{f}(t) = (t, \frac{1}{4}t^2, \ldots, \frac{1}{i^2}t^i, \ldots)$ for $0 \leq t \leq 1$. Show that this path is l^2- and l^∞-rectifiable, but not l^1-rectifiable.

b) Consider the same path as above, but now for $0 \leq t \leq 3$. Show that this path is not rectifiable in the l^1, l^2, or l^∞ sense (instantaneously as t crosses $t = 1$, the path's speed zooms to $+\infty$ despite the fact that the l^2 speed at $t = 1$ is only $\frac{\pi^2}{6}$!).