Field Axioms. A *field* is a set $\mathbb{F}$ along with two operations, “addition” and “multiplication,” that obey the following six axioms:

F-I) (Commutativity) If $x, y \in \mathbb{F}$ then $xy = yx$ and $x + y = y + x$.

F-II) (Associativity) If $x, y, z \in \mathbb{F}$ then $x(yz) = (xy)z$ and $x + (y + z) = (x + y) + z$.

F-III) (Distributivity) If $x, y, z \in \mathbb{F}$ then $x(y + z) = xy + xz$.

F-IV) (Identity) There exist two distinct elements $0, 1 \in \mathbb{F}$ (the additive identity and the multiplicative identity, respectively) so that for any $x \in \mathbb{F}$, we have $0 + x = x$ and $1x = x$.

F-V) (Additive Inverses) Given any $x \in \mathbb{F}$, there is some $y \in \mathbb{F}$ (commonly denoted $-x$) so that $x + y = 0$.

F-VI) (Multiplicative Inverses) Given any $x \in \mathbb{F}$ except $x = 0$, there is some $y \in \mathbb{F}$ (commonly denoted x^{-1}) so that $xy = 1$.

Order Axioms. A field F is called an *ordered field* if there is some subset $\mathbb{F}^+ \subset \mathbb{F}$ (called the “positive” elements of $\mathbb{F}$) so that the following three axioms hold:

O-I) If $x, y \in \mathbb{F}^+$ then $x + y \in \mathbb{F}^+$ and $xy \in \mathbb{F}^+$.

O-II) If $x \in \mathbb{F}$ and $x \neq 0$, then either $x \in \mathbb{F}^+$ or $-x \in \mathbb{F}^+$, but not both.

O-III) $0 \notin \mathbb{F}^+$.

Completeness. An ordered field $\mathbb{F}$ is called *complete* if

C-I) Every subset of $\mathbb{F}$ that has an upper bound has a least upper bound.

Theorem. There is a unique complete ordered field, which is the real numbers.

Properties of Integration

a) (Additivity of the Interval.) If $a < b < c$ and $f(x)$ is integrable on $[a, c]$ then

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx.$$

b) (Translation Invariance.) If $f(x)$ is integrable on $[a, b]$ and $c \in \mathbb{R}$ then

$$\int_a^b f(x)\,dx = \int_{a+c}^{b+c} f(x-c)\,dx.$$

c) (Scale Invariance.) If $f(x)$ is integrable on $[a, b]$ and $k \in \mathbb{R}$, $k \neq 0$, then

$$\int_a^b f(x)\,dx = \frac{1}{k}\int_{ka}^{kb} f\left(\frac{x}{k}\right)\,dx.$$

d) (Linearity.) If $f(x), g(x)$ are integrable on $[a, b]$ and c_1, c_2 are constants, then

$$\int_a^b (c_1\,f(x) + c_2\,g(x))\;dx = c_1\int_a^b f(x)\,dx + c_2\int_a^b g(x)\,dx$$

e) (Comparison.) If $g(x) \leq f(x)$ and both are integrable on $[a, b]$, then

$$\int_a^b g(x)\,dx \leq \int_a^b f(x)\,dx.$$

f) (Triangle Inequality.) If $f(x)$ is integrable on $[a, b]$, then

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\;dx$$

If $\vec{f}(t)$ is an integrable vector-valued function, then $\|\vec{f}(t)\|$ is integrable and

$$\left\|\int_a^b \vec{f}(x)\,dx\right\| \leq \int_a^b \left\|\vec{f}(x)\right\|\;dx$$

g) (Integrability of Step Functions.) If $g(x)$ is the step function subordinate to the partition $P = \{x_0, x_1, \ldots, x_N\}$ of $[a, b]$ which takes on the value g_k on the interior of the k^{th} interval, then $g(x)$ is integrable on $[a, b]$ and

$$\int_a^b g(x)\,dx = \sum_{k=1}^{N} g_k \cdot (x_k - x_{k-1}).$$

(Note that constant functions are step functions.)

Properties of the Bilinear Dot Product.

a) (Positivity) If $\vec{A} \in \mathbb{R}^n$ then $\vec{A} \cdot \vec{A} \geq 0$, with equality if an only if $\vec{A} = \mathcal{O}$

b) (Symmetry) If $\vec{A}, \vec{B} \in \mathbb{R}^n$ then $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$

c) (Bilinearity) If $\vec{A}, \vec{B}, \vec{A}_1, \vec{A}_2, \vec{B}_1, \vec{B}_2 \in \mathbb{R}^n$ and $c_1, c_2 \in \mathbb{R}$ then

$$\begin{aligned} \left(c_1\vec{A}_1 + c_2\vec{A}_2\right) \cdot \vec{B} &= c_1\left(\vec{A}_1 \cdot \vec{B}\right) + c_2\left(\vec{A}_2 \cdot \vec{B}\right) \\ \vec{A} \cdot \left(c_1\vec{B}_1 + c_2\vec{B}_2\right) &= c_1\left(\vec{A} \cdot \vec{B}_1\right) + c_2\left(\vec{A} \cdot \vec{B}_2\right) \end{aligned}$$

Properties of the Sesquilinear Dot Product.

a) (Positivity) If $\vec{A} \in \mathbb{C}^n$ then $\vec{A} \cdot \vec{A} \geq 0$, with equality if an only if $\vec{A} = \mathcal{O}$

b) (Skew Symmetry) If $\vec{A}, \vec{B} \in \mathbb{C}^n$ then $\vec{A} \cdot \vec{B} = \overline{\vec{B} \cdot \vec{A}}$

c) (Sesquilinearity) If $\vec{A}, \vec{B}, \vec{A}_1, \vec{A}_2, \vec{B}_1, \vec{B}_2 \in \mathbb{C}^n$ and $c_1, c_2 \in \mathbb{C}$ then

$$\begin{aligned} \left(c_1\vec{A}_1 + c_2\vec{A}_2\right) \cdot \vec{B} &= c_1\left(\vec{A}_1 \cdot \vec{B}\right) + c_2\left(\vec{A}_2 \cdot \vec{B}\right) \\ \vec{A} \cdot \left(c_1\vec{B}_1 + c_2\vec{B}_2\right) &= \overline{c_1}\left(\vec{A} \cdot \vec{B}_1\right) + \overline{c_2}\left(\vec{A} \cdot \vec{B}_2\right) \end{aligned}$$

Properties of the Cross Product

a) (Bilinearity) If $\vec{A}, \vec{B}, \vec{A}_1, \vec{A}_2, \vec{B}_1, \vec{B}_2 \in \mathbb{R}^3$ and $c_1, c_2 \in \mathbb{R}$ then

$$\begin{aligned} \left(c_1\vec{A}_1 + c_2\vec{A}_2\right) \times \vec{B} &= c_1\left(\vec{A}_1 \times \vec{B}\right) + c_2\left(\vec{A}_2 \times \vec{B}\right) \\ \vec{A} \times \left(c_1\vec{B}_1 + c_2\vec{B}_2\right) &= c_1\left(\vec{A} \times \vec{B}_1\right) + c_2\left(\vec{A} \times \vec{B}_2\right) \end{aligned}$$

b) (Antisymmetry) If $\vec{A}, \vec{B} \in \mathbb{R}^3$ then $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$

c) (Orthogonality) If $\vec{A}, \vec{B} \in \mathbb{R}^3$ then

$$\vec{A} \cdot \left(\vec{A} \times \vec{B}\right) = \vec{B} \cdot \left(\vec{A} \times \vec{B}\right) = 0$$

d) (Lagrange Identity) If $\vec{A}, \vec{B} \in \mathbb{R}^3$ then $\|\vec{A} \times \vec{B}\|^2 = \|\vec{A}\|^2\|\vec{B}\|^2 - (\vec{A} \cdot \vec{B})^2$

e) (Jacobi Identity) If $\vec{A}, \vec{B}, \vec{C} \in \mathbb{R}^3$ then

$$\vec{A} \times \left(\vec{B} \times \vec{C}\right) + \vec{B} \times \left(\vec{C} \times \vec{A}\right) + \vec{C} \times \left(\vec{A} \times \vec{B}\right) = 0$$

Axiomatic Definition of a Norm.
A function $\|\cdot\|$ on $\mathbb{C}^n$ (or $\mathbb{R}^n$) that takes vectors to real numbers is called a *norm* provided the following three properties hold:

N-I) (Positivity) For any $\vec{v} \in \mathbb{C}^n$ (or $\mathbb{R}^n$), we have $\|\vec{v}\| \in \mathbb{R}$, and in fact $\|\vec{v}\| \geq 0$ with equality if and only if $\vec{v} = \mathcal{O}$.

N-II) (Homogeneity) If $\vec{v}$ is a vector and c is a scalar, then $\|c\,\vec{v}\| = |c|\|\vec{v}\|$.

N-III) (The Triangle Inequality) If $\vec{v}_1, \vec{v}_2$ are vectors, then $\|\vec{v}_1 + \vec{v}_2\| \leq \|\vec{v}_1\| + \|\vec{v}_2\|$.

Curvilinear Motion. Given a path $\vec{r}(t)$ in $\mathbb{R}^n$, we have

$$\begin{array}{lll} Velocity:\ \vec{v} = \frac{d\vec{r}}{dt} & Speed:\ v = \|\vec{v}\| & Acceleration:\ \vec{a}(t) = \frac{d^2\vec{r}}{dt^2} \\ Unit\,Tangent:\ \vec{T} & Principle\,Normal:\ \vec{N} & \end{array}$$

Acceleration in terms of its normal and tangential components:

$$\vec{a}(t) \;=\; \dot{v}\,\vec{T} \;+\; \kappa\, v^2\,\vec{N}$$

Osculating plane at time t:

$$M\left(\vec{r}(t)\,;\; \vec{T}(t),\, \vec{N}(t)\right) \;=\; \left\{\vec{r}(t) \,+\, c_1\vec{T} \,+\, c_2\vec{N} \;\mid\; c_1, c_2 \in \mathbb{R}\right\}$$

Formula for curvature in an arbitrary parametrization:

$$\kappa(t) \;=\; \frac{\|v\,\vec{a} \;-\; \dot{v}\,\vec{v}\|}{v^3}$$

Special Formulas for Curvilinear Motion in $\mathbb{R}^3$

We have the *binormal*:

$$\vec{B} \;=\; \vec{T} \times \vec{N} \tag{1}$$

and the derivative equations

$$\begin{array}{rcccc} \frac{d\vec{T}}{ds} & = & & \kappa\vec{N} & \\ \frac{d\vec{N}}{ds} & = & -\kappa\vec{T} & & +\tau\vec{B} \\ \frac{d\vec{B}}{ds} & = & & -\tau\vec{N} & \end{array}$$

We have the κ, τ equations in the arclength parametrization:

$$\begin{aligned} \kappa &= \left\|\vec{T} \times \frac{d\vec{T}}{ds}\right\| \\ \tau &= \kappa^{-2}\,\vec{T} \cdot \left(\frac{d\vec{T}}{ds} \times \frac{d^2\vec{T}}{ds^2}\right) \end{aligned}$$

and the κ, τ equations in an arbitrary parametrization:

$$\begin{aligned} \kappa &= \frac{\|\vec{v} \times \vec{a}\|}{v^3} \\ \tau &= \frac{\dot{\vec{r}} \cdot \left(\ddot{\vec{r}} \times \dddot{\vec{r}}\right)}{\left\|\dot{\vec{r}} \times \ddot{\vec{r}}\right\|^2} \end{aligned}$$

Scratch

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