14

CALCULUS OF VECTOR-VALUED FUNCTIONS

14.1 Vector-valued functions of a real variable

This chapter combines vector algebra with the methods of calculus and describes some applications to the study of curves and to some problems in mechanics. The concept of a vector-valued function is fundamental in this study.

DEFINITION. *A function whose domain is a set of real numbers and whose range is a subset of n-space V_n is called a vector-valued function of a real variable.*

We have encountered such functions in Chapter 13. For example, the line through a point P parallel to a nonzero vector A is the range of the vector-valued function X given by

$$X(t) = P + tA$$

for all real t.

Vector-valued functions will be denoted by capital letters such as F, G, X, Y, etc., or by small bold-face italic letters $\boldsymbol{f}$, $\boldsymbol{g}$, etc. The value of a function F at t is denoted, as usual, by $F(t)$. In the examples we shall study, the domain of F will be an interval which may contain one or both endpoints or which may be infinite.

14.2 Algebraic operations. Components

The usual operations of vector algebra can be applied to combine two vector-valued functions or to combine a vector-valued function with a real-valued function. If F and G are vector-valued functions, and if u is a real-valued function, all having a common domain, we define new functions $F + G$, uF, and $F \cdot G$ by the equations

$$(F + G)(t) = F(t) + G(t), \qquad (uF)(t) = u(t)F(t), \qquad (F \cdot G)(t) = F(t) \cdot G(t).$$

The sum $F + G$ and the product uF are vector valued, whereas the dot product $F \cdot G$ is real valued. If $F(t)$ and $G(t)$ are in 3-space, we can also define the cross product $F \times G$ by the formula

$$(F \times G)(t) = F(t) \times G(t).$$

The operation of composition may be applied to combine vector-valued functions with real-valued functions. For example, if F is a vector-valued function whose domain includes the range of a real-valued function u, the composition $G = F \circ u$ is a new vector-valued function defined by the equation

$$G(t) = F[u(t)]$$

for each t in the domain of u.

If a function F has its values in V_n, then each vector $F(t)$ has n components, and we can write

$$F(t) = (f_1(t), f_2(t), \ldots, f_n(t)).$$

Thus, each vector-valued F gives rise to n real-valued functions $f_1, \ldots, f_n$ whose values at t are the components of $F(t)$. We indicate this relation by writing $F = (f_1, \ldots, f_n)$, and we call f_k the kth component of F.

14.3 Limits, derivatives, and integrals

The basic concepts of calculus, such as limit, derivative, and integral, can also be extended to vector-valued functions. We simply express the vector-valued function in terms of its components and perform the operations of calculus on the components.

DEFINITION. *If $F = (f_1, \ldots, f_n)$ is a vector-valued function, we define limit, derivative, and integral by the equations*

$$\lim_{t \to p} F(t) = \left(\lim_{t \to p} f_1(t), \ldots, \lim_{t \to p} f_n(t)\right),$$

$$F'(t) = (f_1'(t), \ldots, f_n'(t)),$$

$$\int_a^b F(t)\, dt = \left(\int_a^b f_1(t)\, dt, \ldots, \int_a^b f_n(t)\, dt\right),$$

whenever the components on the right are meaningful.

We also say that F is *continuous, differentiable*, or *integrable* on an interval if each component of F has the corresponding property on the interval.

In view of these definitions, it is not surprising to find that many of the theorems on limits, continuity, differentiation, and integration of real-valued functions are also valid for vector-valued functions. We state some of the theorems that we use in this chapter.

THEOREM 14.1. *If F, G, and u are differentiable on an interval, then so are $F + G$, uF, and $F \cdot G$, and we have*

$$(F + G)' = F' + G', \qquad (uF)' = u'F + uF', \qquad (F \cdot G)' = F' \cdot G + F \cdot G'.$$

If F and G have values in V_3, we also have

$$(F \times G)' = F' \times G + F \times G'.$$

Proof. To indicate the routine nature of the proofs we discuss the formula for $(uF)'$. The proofs of the others are similar and are left as exercises for the reader.

Writing $F = (f_1, \ldots, f_n)$, we have

$$uF = (uf_1, \ldots, uf_n), \qquad (uF)' = ((uf_1)', \ldots, (uf_n)').$$

But the derivative of the kth component of uF is $(uf_k)' = u'f_k + uf_k'$, so we have

$$(uF)' = u'(f_1, \ldots, f_n) + u(f_1', \ldots, f_n') = u'F + uF'.$$

The reader should note that the differentiation formulas in Theorem 14.1 are analogous to the usual formulas for differentiating a sum or product of real-valued functions. Since the cross product is not commutative, one must pay attention to the order of the factors in the formula for $(F \times G)'$.

The formula for differentiating $F \cdot G$ gives us the following theorem which we shall use frequently.

THEOREM 14.2. *If a vector-valued function is differentiable and has constant length on an open interval I, then $F \cdot F' = 0$ on I. In other words, $F'(t)$ is perpendicular to $F(t)$ for each t in I.*

Proof. Let $g(t) = \|F(t)\|^2 = F(t) \cdot F(t)$. By hypothesis, g is constant on I, and hence $g' = 0$ on I. But since g is a dot product, we have $g' = F' \cdot F + F \cdot F' = 2F \cdot F'$. Therefore we have $F \cdot F' = 0$.

The next theorem deals with composite functions. Its proof follows easily from Theorems 3.5 and 4.2 which contain the corresponding results for real-valued functions.

THEOREM 14.3. *Let $G = F \circ u$, where F is vector valued and u is real valued. If u is continuous at t and if F is continuous at $u(t)$, then G is continuous at t. If the derivatives $u'(t)$ and $F'[u(t)]$ exist, then $G'(t)$ also exists and is given by the chain rule,*

$$G'(t) = F'[u(t)]u'(t).$$

If a vector-valued function F is continuous on a closed interval $[a, b]$, then each component is continuous and hence integrable on $[a, b]$, so F is integrable on $[a, b]$. The next three theorems give basic properties of the integral of vector-valued functions. In each case, the proofs follow at once from the corresponding results for integrals of real-valued functions.

THEOREM 14.4. LINEARITY AND ADDITIVITY. *If the vector-valued functions F and G are integrable on $[a, b]$, so is $c_1F + c_2G$ for all c_1 and c_2, and we have*

$$\int_a^b (c_1F(t) + c_2G(t))\,dt = c_1\int_a^b F(t)\,dt + c_2\int_a^b G(t)\,dt.$$

Also, for each c in $[a, b]$, we have

$$\int_a^b F(t)\,dt = \int_a^c F(t)\,dt + \int_c^b F(t)\,dt.$$

THEOREM 14.5. FIRST FUNDAMENTAL THEOREM OF CALCULUS. *Assume F is a vector-valued function continuous on $[a, b]$. If $c \in [a, b]$, define the indefinite integral A to be the vector-valued function given by*

$$A(x) = \int_c^x F(t)\,dt \qquad \text{if} \quad a \le x \le b.$$

Then $A'(x)$ exists, and we have $A'(x) = F(x)$ for each x in (a, b).

THEOREM 14.6. SECOND FUNDAMENTAL THEOREM OF CALCULUS. *Assume that the vector-valued function F has a continuous derivative F' on an open interval I. Then, for each choice of c and x in I, we have*

$$F(x) = F(c) + \int_c^x F'(t)\,dt.$$

The next theorem is an extension of the property $c\int_a^b F(t)\,dt = \int_a^b cF(t)\,dt$, with multiplication by the scalar c replaced by dot multiplication by a vector C.

THEOREM 14.7. *If $F = (f_1, \ldots, f_n)$ is integrable on $[a, b]$, then for every vector $C = (c_1, \ldots, c_n)$ the dot product $C \cdot F$ is integrable on $[a, b]$, and we have*

$$C \cdot \int_a^b F(t)\,dt = \int_a^b C \cdot F(t)\,dt.$$

Proof. Since each component of F is integrable, we have

$$C \cdot \int_a^b F(t)\,dt = \sum_{i=1}^n c_i \int_a^b f_i(t)\,dt = \int_a^b \sum_{i=1}^n c_i f_i(t)\,dt = \int_a^b C \cdot F(t)\,dt.$$

Now we use Theorem 14.7 in conjunction with the Cauchy-Schwarz inequality to obtain the following important property of integrals of vector-valued functions.

THEOREM 14.8. *If F and $\|F\|$ are integrable on $[a, b]$ we have*

$$\left\| \int_a^b F(t)\,dt \right\| \le \int_a^b \|F(t)\|\,dt. \tag{14.1}$$

Proof. Let $C = \int_a^b F(t)\,dt$. If $C = O$, then (14.1) holds trivially. Assume, then, that $C \ne O$ and apply Theorem 14.7 to get

$$\|C\|^2 = C \cdot C = C \cdot \int_a^b F(t)\,dt = \int_a^b C \cdot F(t)\,dt. \tag{14.2}$$

Since the dot product $C \cdot F(t)$ is real valued, we have the inequality

$$\int_a^b C \cdot F(t)\, dt \le \int_a^b |C \cdot F(t)|\, dt \le \int_a^b \|C\|\, \|F(t)\|\, dt\,, \tag{14.3}$$

where in the last step we used the Cauchy-Schwarz inequality, $|C \cdot F(t)| \le \|C\|\, \|F(t)\|$. Combining (14.2) and (14.3), we get

$$\|C\|^2 \le \|C\| \int_a^b \|F(t)\|\, dt\,.$$

Since $\|C\| > 0$, we can divide by $\|C\|$ to get (14.1).

14.4 Exercises

Compute the derivatives $F'(t)$ and $F''(t)$ for each of the vector-valued functions in Exercises 1 through 6.

1. $F(t) = (t, t^2, t^3, t^4)$.
2. $F(t) = (\cos t, \sin^2 t, \sin 2t, \tan t)$.
3. $F(t) = (\arcsin t, \arccos t)$.
4. $F(t) = 2e^t i + 3e^t j$.
5. $F(t) = \cosh t\, i + \sinh 2t\, j + e^{-3t} k$.
6. $F(t) = \log(1 + t^2)\, i + \arctan t\, j + \dfrac{1}{1 + t^2}\, k$.
7. Let F be the vector-valued function given by

$$F(t) = \frac{2t}{1 + t^2}\, i + \frac{1 - t^2}{1 + t^2}\, j + k\,.$$

Prove that the angle between $F(t)$ and $F'(t)$ is constant, that is, independent of t.

Compute the vector-valued integrals in Exercises 8 through 11.

8. $\displaystyle\int_0^1 (t, \sqrt{t}, e^t)\, dt$.
9. $\displaystyle\int_0^{\pi/4} (\sin t, \cos t, \tan t)\, dt$.
10. $\displaystyle\int_0^1 \left(\frac{e^t}{1 + e^t}\, i + \frac{1}{1 + e^t}\, j\right) dt$.
11. $\displaystyle\int_0^1 (te^t i + t^2 e^t j + te^{-t} k)\, dt$.
12. Compute $A \cdot B$, where $A = 2i - 4j + k$ and $B = \int_0^1 (te^{2t} i + t \cosh 2t\, j + 2te^{-2t} k)\, dt$.
13. Given a nonzero vector B and a vector-valued function F such that $F(t) \cdot B = t$ for all t, and such that the angle between $F'(t)$ and B is constant (independent of t). Prove that $F''(t)$ is orthogonal to $F'(t)$.
14. Given fixed nonzero vectors A and B, let $F(t) = e^{2t}A + e^{-2t}B$. Prove that $F''(t)$ has the same direction as $F(t)$.
15. If $G = F \times F'$, compute G' in terms of F and derivatives of F.
16. If $G = F \cdot F' \times F''$, prove that $G' = F \cdot F' \times F'''$.
17. Prove that $\lim_{t \to p} F(t) = A$ if and only if $\lim_{t \to p} \|F(t) - A\| = 0$.
18. Prove that a vector-valued function F is differentiable on an open interval I if and only if for each t in I we have

$$F'(t) = \lim_{h \to 0} \frac{1}{h}[F(t + h) - F(t)]\,.$$

19. Prove the zero-derivative theorem for vector-valued functions. If $F'(t) = O$ for each t in an open interval I, then there is a vector C such that $F(t) = C$ for all t in I.

20. Given fixed vectors A and B and a vector-valued function F such that $F''(t) = tA + B$, determine $F(t)$ if $F(0) = D$ and $F'(0) = C$.
21. A differential equation of the form $Y'(x) + p(x)Y(x) = Q(x)$, where p is a given real-valued function, Q a given vector-valued function, and Y an unknown vector-valued function, is called a first-order linear vector differential equation. Prove that if p and Q are continuous on an interval I, then for each a in I and each vector B there is one and only one solution Y which satisfies the initial condition $Y(a) = B$, and that this solution is given by the formula

$$Y(t) = Be^{-q(t)} + e^{-q(t)} \int_a^t Q(x) e^{q(x)}\, dx\,,$$

where $q(x) = \int_a^x p(t)\, dt$.

22. A vector-valued function F satisfies the equation $tF'(t) = F(t) + tA$ for each $t \ge 0$, where A is a fixed vector. Compute $F''(1)$ and $F(3)$ in terms of A, if $F(1) = 2A$.
23. Find a vector-valued function F, continuous on the interval $(0, +\infty)$, such that

$$F(x) = xe^x A + \frac{1}{x} \int_1^x F(t)\, dt\,,$$

for all $x > 0$, where A is a fixed nonzero vector.

24. A vector-valued function F, which is never zero and has a continuous derivative $F'(t)$ for all t, is always parallel to its derivative. Prove that there is a constant vector A and a positive real-valued function u such that $F(t) = u(t)A$ for all t.

14.5 Applications to curves. Tangency

Let X be a vector-valued function whose domain is an interval I. As t runs through I, the corresponding function values $X(t)$ run through a set of points which we call the *graph* of the function X. If the function values are in 2-space or in 3-space, we can visualize the graph geometrically. For example, if $X(t) = P + tA$, where P and A are fixed vectors in V_3, with $A \ne O$, the graph of X is a straight line through P parallel to A. A more general function will trace out a more general graph, as suggested by the example in Figure 14.1. If X is continuous on I, such a graph is called a *curve*; more specifically, the curve described by X. Sometimes we say that the curve is described *parametrically* by X. The interval I is called a *parametric interval*; each t in I is called a *parameter*.

Properties of the function X can be used to investigate geometric properties of its graph. In particular, the derivative X' is related to the concept of tangency, as in the case of a real-valued function. We form the difference quotient

$$\frac{X(t + h) - X(t)}{h} \tag{14.4}$$

and investigate its behavior as $h \to 0$. This quotient is the product of the vector $X(t + h) - X(t)$ by the scalar $1/h$. The numerator, $X(t + h) - X(t)$, illustrated geometrically in Figure 14.2, is parallel to the vector in (14.4). If we express this difference quotient in terms of its components and let $h \to 0$, we find that

$$\lim_{h \to 0} \frac{X(t + h) - X(t)}{h} = X'(t)\,,$$

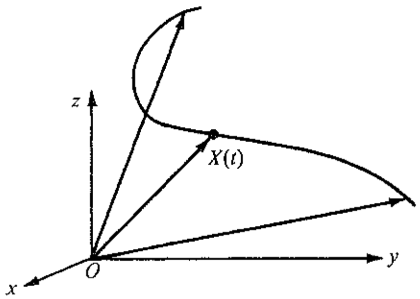


FIGURE 14.1 A curve traced out by a vector $X(t)$.

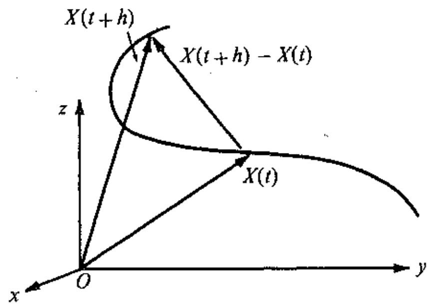


FIGURE 14.2 The vector $X(t + h) - X(t)$ is parallel to $[X(t + h) - X(t)]/h$.

assuming that the derivative $X'(t)$ exists. The geometric interpretation of this relation suggests the following definition.

DEFINITION. *Let C be a curve described by a continuous vector-valued function X. If the derivative $X'(t)$ exists and is nonzero, the straight line through $X(t)$ parallel to $X'(t)$ is called the tangent line to C at $X(t)$. The vector $X'(t)$ is called a tangent vector to C at $X(t)$.*

EXAMPLE 1. *Straight line.* For a line given by $X(t) = P + tA$, where $A \neq O$, we have $X'(t) = A$, so the tangent line at each point coincides with the graph of X, a property which we surely want.

EXAMPLE 2. *Circle.* If X describes a circle of radius a and center at a point P, then $\|X(t) - P\| = a$ for each t. The vector $X(t) - P$ is called a *radius vector;* it may be represented geometrically by an arrow from the center to the point $X(t)$. Since the radius vector has constant length, Theorem 14.2 tells us that it is perpendicular to its derivative and hence perpendicular to the tangent line. Thus, for a circle, our definition of tangency agrees with that given in elementary plane geometry.

EXAMPLE 3. *Invariance under a change of parameter.* Different functions can have the same graph. For example, suppose that X is a continuous vector-valued function defined on an interval I and suppose that u is a real-valued function that is differentiable with u' never zero on an interval J, and such that the range of u is I. Then the function Y defined on J by the equation

$$Y(t) = X[u(t)]$$

is a continuous vector-valued function having the same graph as X. Two functions X and Y so related are called *equivalent.* They are said to provide different parametric representations of the same curve. The function u is said to define a change of parameter.

The most important geometric concepts associated with a curve are those that remain invariant under a change of parameter. For example, it is easy to prove that the tangent

line is invariant. If the derivative $X'[u(t)]$ exists, the chain rule shows that $Y'(t)$ also exists and is given by the formula

$$Y'(t) = X'[u(t)]u'(t).$$

The derivative $u'(t)$ is never zero. If $X'[u(t)]$ is nonzero, then $Y'(t)$ is also nonzero, so $Y'(t)$ is parallel to $X'[u(t)]$. Therefore both representations X and Y lead to the same tangent line at each point of the curve.

EXAMPLE 4. *Reflection properties of the conic sections.* Conic sections have reflection properties often used in the design of optical and acoustical equipment. Light rays emanating from one focus of an elliptical reflector will converge at the other focus, as shown in

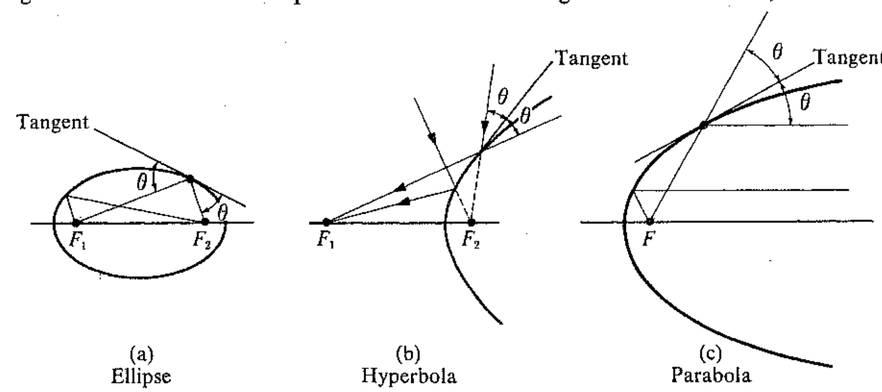


FIGURE 14.3 Reflection properties of the conic sections.

Figure 14.3(a). Light rays directed toward one focus of a hyperbolic reflector will converge at the other focus, as suggested by Figure 14.3(b). In a parabolic reflector, light rays parallel to the axis converge at the focus, as shown in Figure 14.3(c). To establish these reflection properties, we need to prove that in each figure the angles labeled θ are equal. We shall do this for the ellipse and hyperbola and ask the reader to give a proof for the parabola.

Place one focus F_1 at the origin and let $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ be unit vectors having the same direction as X and $X - F_2$, respectively, where X is an arbitrary point on the conic. (See Figure 14.4.) If $d_1 = \|X\|$ and $d_2 = \|X - F_2\|$ are the focal distances between X and the foci F_1 and F_2, respectively, we have

$$X = d_1\boldsymbol{u}_1 \quad \text{and} \quad X = d_2\boldsymbol{u}_2 + F_2 .$$

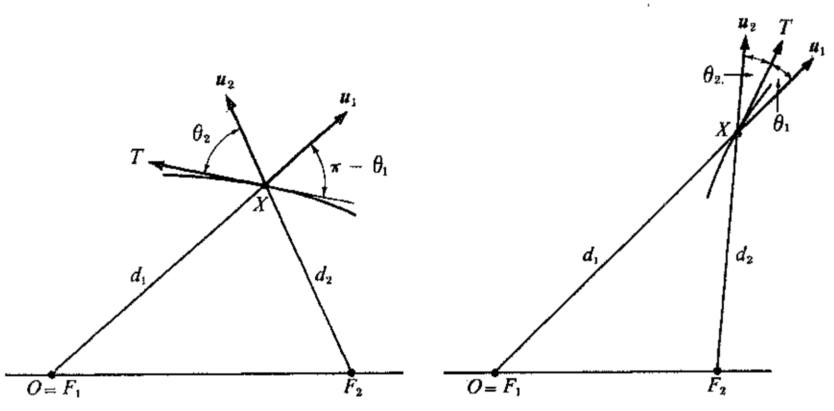
Now we think of $X, \boldsymbol{u}_1, \boldsymbol{u}_2, d_1$, and d_2 as functions defined on some interval of real numbers. Their derivatives are related by the equations

$$X' = d_1\boldsymbol{u}_1' + d_1'\boldsymbol{u}_1, \qquad X' = d_2\boldsymbol{u}_2' + d_2'\boldsymbol{u}_2 . \tag{14.5}$$

Since $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ have constant length, each is perpendicular to its derivative, so Equations (14.5) give us $X' \cdot \boldsymbol{u}_1 = d_1'$ and $X' \cdot \boldsymbol{u}_2 = d_2'$. Adding and subtracting these relations, we find that

$$X' \cdot (u_1 + u_2) = d_1' + d_2', \qquad X' \cdot (u_1 - u_2) = d_1' - d_2'. \tag{14.6}$$

On the ellipse, $d_1 + d_2$ is constant, so $d_1' + d_2' = 0$. On each branch of the hyperbola,



(a) $\theta_2 = \pi - \theta_1$ on the ellipse (b) $\theta_2 = \theta_1$ on the hyperbola

FIGURE 14.4 Proofs of the reflection properties for the ellipse and hyperbola.

$d_1 - d_2$ is constant, so $d_1' - d_2' = 0$. Therefore, Equations (14.6) give us

$$X' \cdot (u_1 + u_2) = 0 \quad \text{on the ellipse,} \qquad X' \cdot (u_1 - u_2) = 0 \quad \text{on the hyperbola.}$$

Let $T = X'/\|X'\|$ be a unit vector having the same direction as X'. Then T is tangent to the conic, and we have

$$T \cdot u_2 = -T \cdot u_1 \quad \text{on the ellipse,} \qquad T \cdot u_2 = T \cdot u_1 \quad \text{on the hyperbola.}$$

If θ_1 and θ_2 denote, respectively, the angles that T makes with u_1 and u_2, where $0 \le \theta_1 \le \pi$ and $0 \le \theta_2 \le \pi$, these last two equations show that

$$\cos\theta_2 = -\cos\theta_1 \quad \text{on the ellipse,} \qquad \cos\theta_2 = \cos\theta_1 \quad \text{on the hyperbola.}$$

Hence we have $\theta_2 = \pi - \theta_1$ on the ellipse, and $\theta_2 = \theta_1$ on the hyperbola. These relations between the angles θ_1 and θ_2 give the reflection properties of the ellipse and hyperbola.

14.6 Applications to curvilinear motion. Velocity, speed, and acceleration

Suppose a particle moves in 2-space or in 3-space in such a way that its position at time t relative to some coordinate system is given by a vector $X(t)$. As t varies through a time interval, the path traced out by the particle is simply the graph of X. Thus, the vector-valued function X serves as a natural mathematical model to describe the motion. We call

X the *position function* of the motion. Physical concepts such as velocity, speed, and acceleration can be defined in terms of derivatives of the position function.

In the following discussion we assume that the position function may be differentiated as often as is necessary without saying so each time.

DEFINITION. *Consider a motion described by a vector-valued function X. The derivative $X'(t)$ is called the velocity vector at time t. The length of the velocity vector, $\|X'(t)\|$, is called the speed. The second derivative of the position vector, $X''(t)$, is called the acceleration vector.*

Notation. Sometimes the position function X is denoted by r, the velocity vector by v, the speed by v, and the acceleration by a. Thus, $v = r'$, $v = \|v\|$, and $a = v' = r''$.

If the velocity vector $X'(t)$ is visualized as a geometric vector attached to the curve at $X(t)$, we see that it lies along the tangent line. The use of the word "speed" for the length of the velocity vector will be justified in Section 14.12 where it is shown that the speed is the rate of change of arc length along the curve. This is what the speedometer of an automobile tries to measure. Thus, the length of the velocity vector tells us how fast the particle is moving at every instant, and its direction tells us which way it is going. The velocity will change if we alter either the speed or the direction of the motion (or both). The acceleration vector is a measure of this change. Acceleration causes the effect one feels when an automobile changes its speed or its direction. Unlike the velocity vector, the acceleration vector does not necessarily lie along the tangent line.

EXAMPLE 1. *Linear motion.* Consider a motion whose position vector is given by

$$r(t) = P + f(t)A,$$

where P and A are fixed vectors, $A \ne O$. This motion takes place along a line through P parallel to A. The velocity, speed, and acceleration are given by

$$v(t) = f'(t)A, \qquad v(t) = \|v(t)\| = |f'(t)|\,\|A\|, \qquad a(t) = f''(t)A.$$

If $f'(t)$ and $f''(t)$ are nonzero, the acceleration vector is parallel to the velocity.

EXAMPLE 2. *Circular motion.* If a point (x, y) in V_2 is represented by its polar coordinates r and θ, we have

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

If r is fixed, say $r = a$, and if θ is allowed to vary over any interval of length at least 2π, the corresponding point (x, y) traces out a circle of radius a and center at the origin. If we make θ a function of time t, say $\theta = f(t)$, we have a motion given by the position function

$$r(t) = a\cos f(t)i + a\sin f(t)j.$$

The corresponding velocity vector is given by

$$v(t) = r'(t) = -af'(t)\sin f(t)i + af'(t)\cos f(t)j,$$

from which we find that the speed at time t is

$$v(t) = \|\boldsymbol{v}(t)\| = a\,|f'(t)|\,.$$

The factor $|f'(t)| = |d\theta/dt|$ is called the *angular speed* of the particle.

An important special case occurs when $\theta = \omega t$, where ω (omega) is a positive constant. In this case, the particle starts at the point $(a, 0)$ at time $t = 0$ and moves counter-clockwise around the circle with constant angular speed ω. The formulas for the position, velocity, and speed become

$$\boldsymbol{r}(t) = a\cos\omega t\,\boldsymbol{i} + a\sin\omega t\,\boldsymbol{j}\,,\qquad \boldsymbol{v}(t) = -\omega a\sin\omega t\,\boldsymbol{i} + \omega a\cos\omega t\,\boldsymbol{j}\,,\qquad v(t) = a\omega\,.$$

The acceleration vector is given by

$$\boldsymbol{a}(t) = -\omega^2 a\cos\omega t\,\boldsymbol{i} - \omega^2 a\sin\omega t\,\boldsymbol{j} = -\omega^2\boldsymbol{r}(t)\,,$$

which shows that the acceleration is always directed opposite to the position vector. When it is visualized as a geometric vector drawn at the location of the particle, the acceleration vector is directed toward the center of the circle. Because of this, the acceleration is called *centripetal* or "center-seeking," a term originally proposed by Newton.

Note: If a moving particle has mass m, Newton's second law of motion states that the force acting on it (due to its acceleration) is the vector $m\boldsymbol{a}(t)$, mass times acceleration. If the particle moves on a circle with constant angular speed, this is called a centripetal force because it is directed toward the center. This force is exerted by the mechanism that confines the particle to a circular orbit. The mechanism is a *string* in the case of a stone whirling in a slingshot, or *gravitational attraction* in the case of a satellite around the earth. The equal and opposite reaction (due to Newton's third law), that is, the force $-m\boldsymbol{a}(t)$, is said to be *centrifugal* or "center-fleeing."

EXAMPLE 3. *Motion on an ellipse.* Figure 14.5 shows an ellipse with Cartesian equation $x^2/a^2 + y^2/b^2 = 1$, and two concentric circles with radii a and b. The angle θ shown in the figure is called the *eccentric angle*. It is related to the coordinates (x, y) of a point on the ellipse by the equations

$$x = a\cos\theta\,,\qquad y = b\sin\theta\,.$$

As θ varies over an interval of length 2π, the corresponding point (x, y) traces out the ellipse. If we make θ a function of time t, say $\theta = f(t)$, we have a motion given by the position function

$$\boldsymbol{r}(t) = a\cos f(t)\boldsymbol{i} + b\sin f(t)\boldsymbol{j}\,.$$

If $\theta = \omega t$, where ω is a positive constant, the velocity, speed, and acceleration are given by

$$\boldsymbol{v}(t) = \omega(-a\sin\omega t\,\boldsymbol{i} + b\cos\omega t\,\boldsymbol{j})\,,\qquad v(t) = \omega(a^2\sin^2\omega t + b^2\cos^2\omega t)^{1/2}\,,$$

$$\boldsymbol{a}(t) = -\omega^2(a\cos\omega t\,\boldsymbol{i} + b\sin\omega t\,\boldsymbol{j}) = -\omega^2\boldsymbol{r}(t)\,.$$

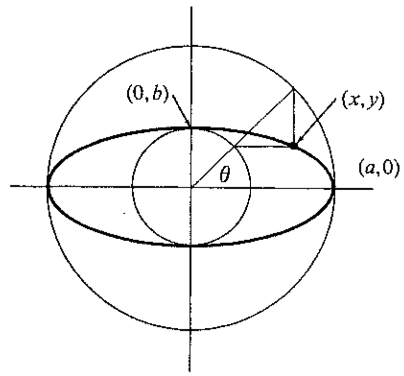


FIGURE 14.5 Motion on an ellipse.

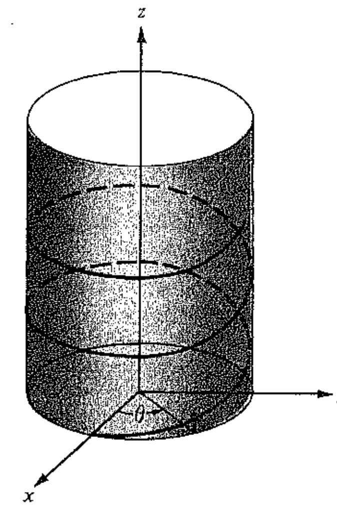


FIGURE 14.6 Motion on a helix.

Thus, when a particle moves on an ellipse in such a way that its eccentric angle changes at a constant rate, the acceleration is centripetal.

EXAMPLE 4. *Motion on a helix.* If a point (x, y, z) revolves around the z-axis at a constant distance a from it and simultaneously moves parallel to the z-axis in such a way that its z-component is proportional to the angle of revolution, the resulting path is called a *circular helix*. An example is shown in Figure 14.6. If θ denotes the angle of revolution, we have

(14.7) $$x = a\cos\theta\,,\qquad y = a\sin\theta\,,\qquad z = b\theta\,,$$

where $a > 0$, and $b \neq 0$. When θ varies from 0 to 2π, the x- and y-coordinates return to their original values while z changes from 0 to $2\pi b$. The number $2\pi b$ is often referred to as the *pitch* of the helix.

Now suppose that $\theta = \omega t$, where ω is constant. The motion on the helix is then described by the position vector

$$\boldsymbol{r}(t) = a\cos\omega t\,\boldsymbol{i} + a\sin\omega t\,\boldsymbol{j} + b\omega t\boldsymbol{k}\,.$$

The corresponding velocity and acceleration vectors are given by

$$\boldsymbol{v}(t) = -\omega a\sin\omega t\,\boldsymbol{i} + \omega a\cos\omega t\,\boldsymbol{j} + b\omega\boldsymbol{k}\,,\qquad \boldsymbol{a}(t) = -\omega^2(a\cos\omega t\,\boldsymbol{i} + a\sin\omega t\,\boldsymbol{j})\,.$$

Thus, when the acceleration vector is located on the helix, it is parallel to the xy-plane and directed toward the z-axis.

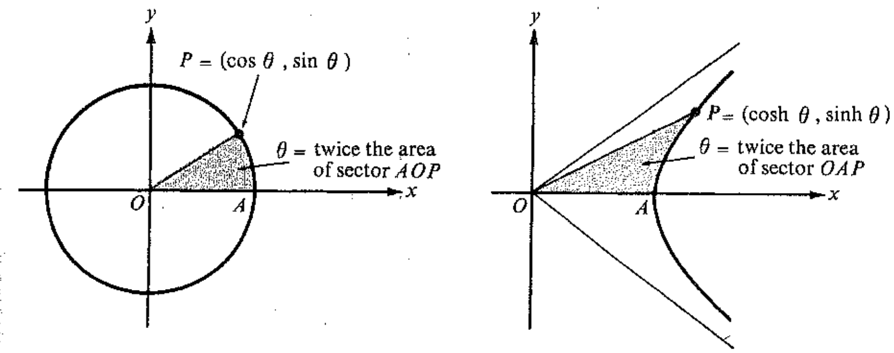
If we eliminate θ from the first two equations in (14.7), we obtain the Cartesian equation $x^2 + y^2 = a^2$ which we recognize as the equation of a circle in the xy-plane. In 3-space, however, this equation represents a surface. A point (x, y, z) satisfies the equation if and only if its distance from the z-axis is equal to a. The set of all such points is a right circular cylinder of radius a with its axis along the z-axis. The helix winds around this cylinder.

14.7 Exercises

In each of Exercises 1 through 6, $\boldsymbol{r}(t)$ denotes the position vector at time t for a particle moving on a space curve. In each case, determine the velocity $\boldsymbol{v}(t)$ and acceleration $\boldsymbol{a}(t)$ in terms of $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$; also, compute the speed $v(t)$.

1. $\boldsymbol{r}(t) = (3t - t^3)\boldsymbol{i} + 3t^2\boldsymbol{j} + (3t + t^3)\boldsymbol{k}$.
2. $\boldsymbol{r}(t) = \cos t\,\boldsymbol{i} + \sin t\,\boldsymbol{j} + e^t\boldsymbol{k}$.
3. $\boldsymbol{r}(t) = 3t\cos t\,\boldsymbol{i} + 3t\sin t\,\boldsymbol{j} + 4t\boldsymbol{k}$.
4. $\boldsymbol{r}(t) = (t - \sin t)\boldsymbol{i} + (1 - \cos t)\boldsymbol{j} + 4\sin\frac{t}{2}\boldsymbol{k}$.
5. $\boldsymbol{r}(t) = 3t^2\boldsymbol{i} + 2t^3\boldsymbol{j} + 3t\boldsymbol{k}$.
6. $\boldsymbol{r}(t) = t\,\boldsymbol{i} + \sin t\,\boldsymbol{j} + (1 - \cos t)\boldsymbol{k}$.
7. Consider the helix described by the vector equation $\boldsymbol{r}(t) = a\cos\omega t\,\boldsymbol{i} + a\sin\omega t\,\boldsymbol{j} + b\omega t\boldsymbol{k}$, where ω is a positive constant. Prove that the tangent line makes a constant angle with the z-axis and that the cosine of this angle is $b/\sqrt{a^2 + b^2}$.
8. Referring to the helix in Exercise 7, prove that the velocity $\boldsymbol{v}$ and acceleration $\boldsymbol{a}$ are vectors of constant length, and that
$$\frac{\|\boldsymbol{v} \times \boldsymbol{a}\|}{\|\boldsymbol{v}\|^3} = \frac{a}{a^2 + b^2}.$$
9. Referring to Exercise 7, let $\boldsymbol{u}(t)$ denote the unit vector $\boldsymbol{u}(t) = \sin\omega t\,\boldsymbol{i} - \cos\omega t\,\boldsymbol{j}$. Prove that there are two constants A and B such that $\boldsymbol{v} \times \boldsymbol{a} = A\boldsymbol{u}(t) + B\boldsymbol{k}$, and express A and B in terms of a, b, and ω.
10. Prove that for any motion the dot product of the velocity and acceleration is half the derivative of the square of the speed:
$$\boldsymbol{v}(t) \cdot \boldsymbol{a}(t) = \frac{1}{2}\frac{d}{dt}v^2(t).$$
11. Let $\boldsymbol{c}$ be a fixed unit vector. A particle moves in space in such a way that its position vector $\boldsymbol{r}(t)$ satisfies the equation $\boldsymbol{r}(t) \cdot \boldsymbol{c} = e^{2t}$ for all t, and its velocity vector $\boldsymbol{v}(t)$ makes a constant angle θ with $\boldsymbol{c}$, where $0 < \theta < \frac{1}{2}\pi$.
(a) Prove that the speed at time t is $2e^{2t}/\cos\theta$.
(b) Compute the dot product $\boldsymbol{a}(t) \cdot \boldsymbol{v}(t)$ in terms of t and θ.
12. The identity $\cosh^2\theta - \sinh^2\theta = 1$ for hyperbolic functions suggests that the hyperbola $x^2/a^2 - y^2/b^2 = 1$ may be represented by the parametric equations $x = a\cosh\theta$, $y = b\sinh\theta$, or what amounts to the same thing, by the vector equation $\boldsymbol{r} = a\cosh\theta\,\boldsymbol{i} + b\sinh\theta\,\boldsymbol{j}$. When $a = b = 1$, the parameter θ may be given a geometric interpretation analogous to that which holds between θ, $\sin\theta$, and $\cos\theta$ in the unit circle shown in Figure 14.7(a). Figure 14.7(b) shows one branch of the hyperbola $x^2 - y^2 = 1$. If the point P has coordinates $x = \cosh\theta$ and $y = \sinh\theta$, prove that θ equals twice the area of the sector OAP shaded in the figure.

[*Hint:* Let $A(\theta)$ denote the area of sector OAP. Show that
$$A(\theta) = \tfrac{1}{2}\cosh\theta\sinh\theta - \int_1^{\cosh\theta}\sqrt{x^2 - 1}\,dx.$$
Differentiate to get $A'(\theta) = \frac{1}{2}$.]
13. A particle moves along a hyperbola according to the equation $\boldsymbol{r}(t) = a\cosh\omega t\,\boldsymbol{i} + b\sinh\omega t\,\boldsymbol{j}$, where ω is a constant. Prove that the acceleration is centrifugal.



(a) Circle: $x^2 + y^2 = 1$ (b) Hyperbola: $x^2 - y^2 = 1$

FIGURE 14.7 Analogy between parameter for a circle and that for a hyperbola.

14. Prove that the tangent line at a point X of a parabola bisects the angle between the line joining X to the focus and the line through X parallel to the axis. This gives the reflection property of the parabola. (See Figure 14.3.)
15. A particle of mass 1 moves in a plane according to the equation $\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j}$. It is attracted toward the origin by a force whose magnitude is four times its distance from the origin. At time $t = 0$, the initial position is $\boldsymbol{r}(0) = 4\boldsymbol{i}$ and the initial velocity is $\boldsymbol{v}(0) = 6\boldsymbol{j}$.
(a) Determine the components $x(t)$ and $y(t)$ explicitly in terms of t.
(b) The path of the particle is a conic section. Find a Cartesian equation for this conic, sketch the conic, and indicate the direction of motion along the curve.
16. A particle moves along the parabola $x^2 + c(y - x) = 0$ in such a way that the horizontal and vertical components of the acceleration vector are equal. If it takes T units of time to go from the point $(c, 0)$ to the point $(0, 0)$, how much time will it require to go from $(c, 0)$ to the halfway point $(c/2, c/4)$?
17. Suppose a curve C is described by two equivalent functions X and Y, where $Y(t) = X[u(t)]$. Prove that at each point of C the velocity vectors associated with X and Y are parallel, but that the corresponding acceleration vectors need not be parallel.

14.8 The unit tangent, the principal normal, and the osculating plane of a curve

For linear motion the acceleration vector is parallel to the velocity vector. For circular motion with constant angular speed, the acceleration vector is perpendicular to the velocity. In this section we show that for a general motion the acceleration vector is a sum of two perpendicular vectors, one parallel to the velocity and one perpendicular to the velocity. If the motion is not linear, these two perpendicular vectors determine a plane through each point of the curve called the osculating plane.

To study these concepts, we introduce the *unit tangent* vector T. This is another vector-valued function associated with the curve, and it is defined by the equation
$$T(t) = \frac{X'(t)}{\|X'(t)\|}$$
whenever the speed $\|X'(t)\| \neq 0$. Note that $\|T(t)\| = 1$ for all t.

Figure 14.8 shows the position of the unit tangent geometric vector $T(t)$ for various values of t when it is attached to the curve. As the particle moves along the curve, the corresponding vector T, being of constant length, can change only in its direction. The tendency of T to change its direction is measured by its derivative T'. Since T has constant length, Theorem 14.2 tells us that T is perpendicular to its derivative T'.

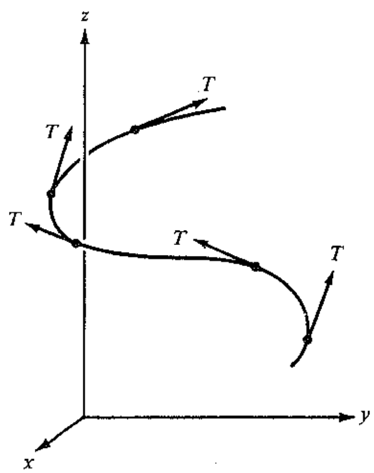


FIGURE 14.8 The unit tangent vector T.

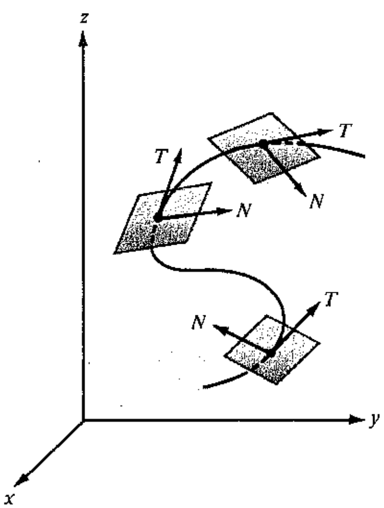


FIGURE 14.9 The osculating plane.

If the motion is linear, then $T' = O$. If $T' \neq O$, the unit vector having the same direction as T' is called the *principal normal* to the curve and it is denoted by N. Thus, N is a new vector-valued function associated with the curve and it is defined by the equation

$$N(t) = \frac{T'(t)}{\|T'(t)\|}, \qquad \text{whenever} \quad \|T'(t)\| \neq 0.$$

When the two unit geometric vectors $T(t)$ and $N(t)$ are attached to the curve at the point $X(t)$, they determine a plane known as the *osculating plane* of the curve. If we choose three values of t, say t_1, t_2, and t_3, and consider the plane determined by the three points $X(t_1)$, $X(t_2)$, $X(t_3)$, it can be shown that the position of the plane approaches the position of the osculating plane at $X(t_1)$ as t_2 and t_3 approach t_1. Because of this, the osculating plane is often called the plane that best fits the curve at each of its points. If the curve itself is a plane curve (not a straight line), the osculating plane coincides with the plane of the curve. In general, however, the osculating plane changes with t. Examples are illustrated in Figure 14.9.

The next theorem shows that the acceleration vector is a sum of two vectors, one parallel to T and one parallel to T'.

THEOREM 14.9. *For a motion described by a vector-valued function* $\boldsymbol{r}$, *let* $v(t)$ *denote the speed at time* t, $v(t) = \|\boldsymbol{r}'(t)\|$. *Then the acceleration vector* $\boldsymbol{a}$ *is a linear combination of* T *and* T' *given by the formula*

$$\boldsymbol{a}(t) = v'(t)T(t) + v(t)T'(t). \tag{14.8}$$

If $T'(t) \neq O$, *we also have*

$$\boldsymbol{a}(t) = v'(t)T(t) + v(t)\,\|T'(t)\|\,N(t). \tag{14.9}$$

Proof. The formula defining the unit tangent gives us

$$\boldsymbol{v}(t) = v(t)T(t).$$

Differentiating this product, we find that

$$\boldsymbol{a}(t) = v'(t)T(t) + v(t)T'(t),$$

which proves (14.8). To prove (14.9), we use the definition of N to write $T'(t) = \|T'(t)\|\,N(t)$.

This theorem shows that the acceleration vector always lies in the osculating plane. An example is shown in Figure 14.10. The coefficients of $T(t)$ and $N(t)$ in (14.9) are called, respectively, the *tangential* and *normal components* of the acceleration. A change in speed contributes to the tangential component, whereas a change in direction contributes to the normal component.

For a plane curve, the length of $T'(t)$ has an interesting geometric interpretation. Since T is a unit vector, we may write

$$T(t) = \cos \alpha(t)\boldsymbol{i} + \sin \alpha(t)\boldsymbol{j},$$

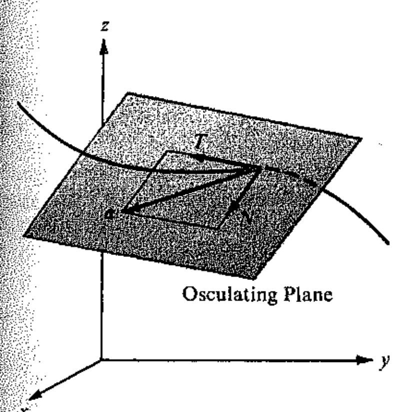


FIGURE 14.10 The acceleration vector lies in the osculating plane

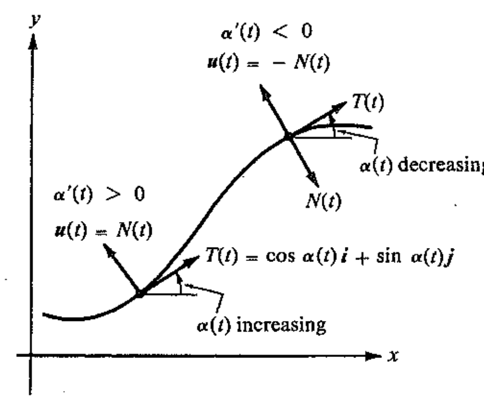


FIGURE 14.11 The angle of inclination of the tangent vector of a plane curve.

where $\alpha(t)$ denotes the angle between the tangent vector and the positive x-axis, as shown in Figure 14.11. Differentiating, we find that

$$T'(t) = -\sin \alpha(t)\, \alpha'(t) i + \cos \alpha(t)\, \alpha'(t) j = \alpha'(t) u(t),$$

where $u(t)$ is a unit vector. Therefore $\|T'(t)\| = |\alpha'(t)|$ and this shows that $\|T'(t)\|$ is a measure of the rate of change of the angle of inclination of the tangent vector. When $\alpha'(t) > 0$, the angle is increasing, and hence $u(t) = N(t)$. When $\alpha'(t) < 0$, the angle is decreasing and, in this case, $u(t) = -N(t)$. The two cases are illustrated in Figure 14.11. Note that the angle of inclination of $u(t)$ is $\alpha(t) + \frac{1}{2}\pi$ since we have

$$u(t) = -\sin \alpha(t) i + \cos \alpha(t) j = \cos\left(\alpha(t) + \frac{\pi}{2}\right) i + \sin\left(\alpha(t) + \frac{\pi}{2}\right) j .$$

14.9 Exercises

Exercises 1 through 6 below refer to the motions described in Exercises 1 through 6, respectively, of Section 14.7. For the value of t specified, (a) express the unit tangent T and the principal normal N in terms of i, j, k; (b) express the acceleration a as a linear combination of T and N.

1. $t = 2$.
2. $t = \pi$.
3. $t = 0$.
4. $t = \pi$.
5. $t = 1$.
6. $t = \frac{1}{4}\pi$.
7. Prove that if the acceleration vector is always zero, the motion is linear.
8. Prove that the normal component of the acceleration vector is $\|v \times a\|/\|v\|$.
9. For each of the following statements about a curve traced out by a particle moving in 3-space, either give a proof or exhibit a counter example.
(a) If the velocity is constant, the curve lies in a plane.
(b) If the speed is constant, the curve lies in a plane.
(c) If the acceleration is constant, the curve lies in a plane.
(d) If the velocity is perpendicular to the acceleration, the curve lies in a plane.
10. A particle of unit mass with position vector $r(t)$ at time t is moving in space under the actions of certain forces.
(a) Prove that $r \times a = O$ implies $r \times v = c$, where c is a constant vector.
(b) If $r \times v = c$, where c is a constant vector, prove that the motion takes place in a plane. Consider both $c \neq O$ and $c = O$.
(c) If the net force acting on the particle is always directed toward the origin, prove that the particle moves in a plane.
(d) Is $r \times v$ necessarily constant if a particle moves in a plane?
11. A particle moves along a curve in such a way that the velocity vector makes a constant angle with a given unit vector c.
(a) If the curve lies in a plane containing c, prove that the acceleration vector is either zero or parallel to the velocity.
(b) Give an example of such a curve (not a plane curve) for which the acceleration vector is never zero nor parallel to the velocity.
12. A particle moves along the ellipse $3x^2 + y^2 = 1$ with position vector $r(t) = f(t)i + g(t)j$. The motion is such that the horizontal component of the velocity vector at time t is $-g(t)$.
(a) Does the particle move around the ellipse in a clockwise or counterclockwise direction?
(b) Prove that the vertical component of the velocity vector at time t is proportional to $f(t)$ and find the factor of proportionality.
(c) How much time is required for the particle to go once around the ellipse?
13. A plane curve C in the first quadrant has a negative slope at each of its points and passes

through the point $(\frac{3}{2}, 1)$. The position vector r from the origin to any point (x, y) on C makes an angle θ with i, and the velocity vector makes an angle ϕ with i, where $0 < \theta < \frac{1}{2}\pi$, and $0 < \phi < \frac{1}{2}\pi$. If $3 \tan \phi = 4 \cot \theta$ at each point of C, find a Cartesian equation for C and sketch the curve.
14. A line perpendicular to the tangent line of a plane curve is called a normal line. If the normal line and a vertical line are drawn at any point of a certain plane curve C, they cut off a segment of length 2 on the x-axis. Find a Cartesian equation for this curve if it passes through the point $(1, 2)$. Two solutions are possible.
15. Given two fixed nonzero vectors A and B making an angle θ with each other, where $0 < \theta < \pi$. A motion with position vector $r(t)$ at time t satisfies the differential equation

$$r'(t) = A \times r(t)$$

and the initial condition $r(0) = B$.
(a) Prove that the acceleration $a(t)$ is orthogonal to A.
(b) Prove that the speed is constant and compute this speed in terms of A, B, and θ.
(c) Make a sketch of the curve, showing its relation to the vectors A and B.
16. This exercise describes how the unit tangent and the principal normal are affected by a change of parameter. Suppose a curve C is described by two equivalent functions X and Y, where $Y(t) = X[u(t)]$. Denote the unit tangent for X by T_X and that for Y by T_Y.
(a) Prove that at each point of C we have $T_Y(t) = T_X[u(t)]$ if u is strictly increasing, but that $T_Y(t) = -T_X[u(t)]$ if u is strictly decreasing. In the first case, u is said to *preserve orientation*; in the second case, u is said to *reverse orientation*.
(b) Prove that the corresponding principal normal vectors N_X and N_Y satisfy $N_Y(t) = N_X[u(t)]$ at each point of C. Deduce that the osculating plane is invariant under a change of parameter.

14.10 The definition of arc length

Various parts of calculus and analytic geometry refer to the arc length of a curve. Before we can study the properties of the length of a curve we must agree on a *definition* of arc length. The purpose of this section is to formulate such a definition. This will lead, in a natural way, to the construction of a function (called the arc-length function) which measures the length of the path traced out by a moving particle at every instant of its motion. Some of the basic properties of this function are discussed in Section 14.12. In particular, we shall prove that for most curves that arise in practice this function may be expressed as the integral of the speed.

To arrive at a definition of what we mean by the length of a curve, we proceed as though we had to measure this length with a straight yardstick. First, we mark off a number of points on the curve which we use as vertices of an inscribed polygon. (An example is shown in Figure 14.12.) Then, we measure the total length of this polygon with our yardstick and consider this as an approximation to the length of the curve. We soon observe that some polygons "fit" the curve better than others. In particular, if we start with a polygon P_1, and construct a new inscribed polygon P_2 by adding more vertices to those of P_1, it is clear that the length of P_2 will be larger than that of P_1, as suggested in Figure 14.13. In the same way we can form more and more polygons with successively larger and larger lengths.

On the other hand, our intuition tells us that the length of any inscribed polygon should not exceed that of the curve (since a straight line is the shortest path between two points).

In other words, when we arrive at a definition for the length of a curve, it should be a number which is an *upper bound* to the lengths of all inscribed polygons. Therefore, it certainly seems reasonable to define the length of the curve to be the *least upper bound* of the lengths of all possible inscribed polygons.

For most curves that arise in practice, this definition gives us a useful and reasonable way to assign a length to a curve. Surprisingly enough, however, there are certain pathological cases where this definition is not applicable. There are curves for which there is *no* upper bound to the lengths of the inscribed polygons. (An example is given in Exercise

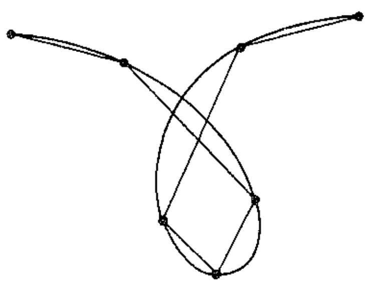


FIGURE 14.12 A curve with an inscribed polygon.

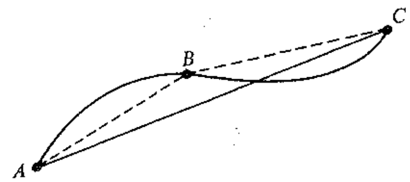


FIGURE 14.13 The polygon ABC has a length greater than the polygon AC.

22 in Section 14.13.) Therefore it becomes necessary to classify all curves into two categories: those which have a length, and those which do not. The former are called *rectifiable curves*, the latter, *nonrectifiable.*

To formulate these ideas in analytic terms, we begin with a curve in 3-space or in 2-space described by a vector-valued function $\boldsymbol{r}$, and we consider the portion of the curve traced out by $\boldsymbol{r}(t)$ as t varies over an interval $[a, b]$. At the outset, we only assume that $\boldsymbol{r}$ is continuous on the parametric interval. Later we shall add further restrictions.

Consider now any partition P of the interval $[a, b]$, say

$$P = \{t_0, t_1, \ldots, t_n\}, \qquad \text{where} \quad a = t_0 < t_1 < \cdots < t_n = b.$$

Denote by $\pi(P)$ the polygon whose vertices are the points $\boldsymbol{r}(t_0), \boldsymbol{r}(t_1), \ldots, \boldsymbol{r}(t_n)$, respectively. (An example with $n = 6$ is shown in Figure 14.14.) The sides of this polygon have lengths

$$\|\boldsymbol{r}(t_1) - \boldsymbol{r}(t_0)\|, \; \|\boldsymbol{r}(t_2) - \boldsymbol{r}(t_1)\|, \ldots, \|\boldsymbol{r}(t_n) - \boldsymbol{r}(t_{n-1})\|.$$

Therefore, the length of the polygon $\pi(P)$, which we denote by $|\pi(P)|$, is the sum

$$|\pi(P)| = \sum_{k=1}^{n} \|\boldsymbol{r}(t_k) - \boldsymbol{r}(t_{k-1})\|.$$

DEFINITION. *If there exists a positive number M such that*

$$|\pi(P)| \le M \tag{14.10}$$

for all partitions P of $[a, b]$, then the curve is said to be rectifiable and its arc length, denoted by $\Lambda(a, b)$, is defined to be the least upper bound of the set of all numbers $|\pi(P)|$. If there is no such M, the curve is called nonrectifiable.

Note that if an M exists satisfying (14.10), then, for every partition P, we have

$$|\pi(P)| \le \Lambda(a, b) \le M, \tag{14.11}$$

since the least upper bound cannot exceed any upper bound.

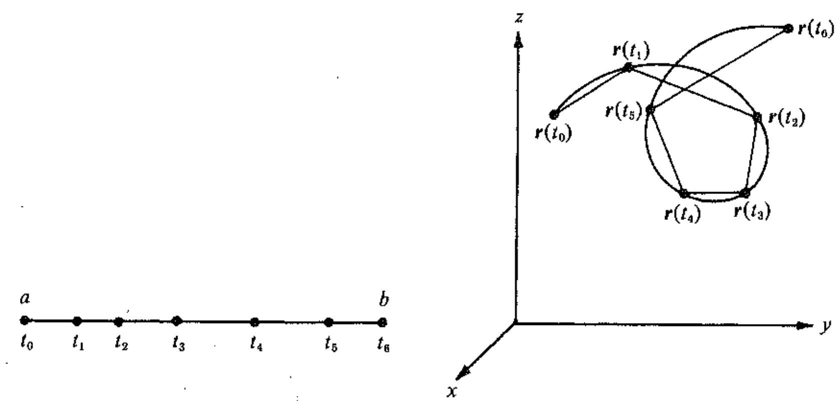


FIGURE 14.14 A partition of $[a, b]$ into six subintervals and the corresponding inscribed polygon.

It is easy to prove that a curve is rectifiable whenever its velocity vector $\boldsymbol{v}$ is continuous on the parametric interval $[a, b]$. In fact, the following theorem tells us that in this case we may use the integral of the speed as an upper bound for all numbers $|\pi(P)|$.

THEOREM 14.10. *Denote by $\boldsymbol{v}(t)$ the velocity vector of the curve with position vector $\boldsymbol{r}(t)$ and let $v(t) = \|\boldsymbol{v}(t)\|$ denote the speed. If $\boldsymbol{v}$ is continuous on $[a, b]$, the curve is rectifiable and its length $\Lambda(a, b)$ satisfies the inequality*

$$\Lambda(a, b) \le \int_a^b v(t)\, dt. \tag{14.12}$$

Proof. For each partition P of $[a, b]$, we have

$$|\pi(P)| = \sum_{k=1}^{n} \|\boldsymbol{r}(t_k) - \boldsymbol{r}(t_{k-1})\| = \sum_{k=1}^{n} \left\| \int_{t_{k-1}}^{t_k} \boldsymbol{r}'(t)\, dt \right\|$$
$$= \sum_{k=1}^{n} \left\| \int_{t_{k-1}}^{t_k} \boldsymbol{v}(t)\, dt \right\| \le \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \|\boldsymbol{v}(t)\|\, dt = \int_a^b v(t)\, dt,$$

the inequality being a consequence of Theorem 14.8. This shows that we have

$$|\pi(P)| \leq \int_a^b v(t)\, dt$$

for all partitions P, and hence the number $\int_a^b v(t)\, dt$ is an upper bound for the set of all numbers $|\pi(P)|$. This proves that the curve is rectifiable and, at the same time, it tells us that the length $\Lambda(a, b)$ cannot exceed the integral of the speed.

In a later section we shall prove that the inequality in (14.12) is, in fact, an *equality.* The proof of this fact will make use of the *additivity* of arc length, a property described in the next section.

14.11 Additivity of arc length

If a rectifiable curve is cut into two pieces, the length of the whole curve is the sum of the lengths of the two parts. This is another of those "intuitively obvious" statements whose proof is not trivial. This property is called *additivity of arc length* and it may be expressed analytically as follows.

THEOREM 14.11. *Consider a rectifiable curve of length $\Lambda(a, b)$ traced out by a vector $r(t)$ as t varies over an interval $[a, b]$. If $a < c < b$, let C_1 and C_2 be the curves traced out by $r(t)$ as t varies over the intervals $[a, c]$ and $[c, b]$, respectively. Then C_1 and C_2 are also rectifiable and, if $\Lambda(a, c)$ and $\Lambda(c, b)$ denote their respective lengths, we have*

$$\Lambda(a, b) = \Lambda(a, c) + \Lambda(c, b).$$

Proof. Let P_1 and P_2 be arbitrary partitions of $[a, c]$ and $[c, b]$, respectively. The points in P_1 taken together with those in P_2 give us a new partition P of $[a, b]$ for which we have

$$|\pi(P_1)| + |\pi(P_2)| = |\pi(P)| \leq \Lambda(a, b). \tag{14.13}$$

This shows that $|\pi(P_1)|$ and $|\pi(P_2)|$ are bounded by $\Lambda(a, b)$, and hence C_1 and C_2 are rectifiable. From (14.13), we also have

$$|\pi(P_1)| \leq \Lambda(a, b) - |\pi(P_2)|.$$

Now, keep P_2 fixed and let P_1 vary over all possible partitions of $[a, c]$. Since the number $\Lambda(a, b) - |\pi(P_2)|$ is an upper bound for all numbers $|\pi(P_1)|$, it cannot be less than their least upper bound, which is $\Lambda(a, c)$. Hence, we have $\Lambda(a, c) \leq \Lambda(a, b) - |\pi(P_2)|$ or, what is the same thing,

$$|\pi(P_2)| \leq \Lambda(a, b) - \Lambda(a, c).$$

This shows that $\Lambda(a, b) - \Lambda(a, c)$ is an upper bound for all the sums $|\pi(P_2)|$, and since it cannot be less than their least upper bound, $\Lambda(c, b)$, we have $\Lambda(c, b) \leq \Lambda(a, b) - \Lambda(a, c)$. In other words, we have

$$\Lambda(a, c) + \Lambda(c, b) \leq \Lambda(a, b). \tag{14.14}$$

Next we prove the reverse inequality. We begin with any partition P of $[a, b]$. If we adjoin the point c to P, we obtain a partition P_1 of $[a, c]$ and a partition P_2 of $[c, b]$ such that

$$|\pi(P)| \leq |\pi(P_1)| + |\pi(P_2)| \leq \Lambda(a, c) + \Lambda(c, b).$$

This shows that $\Lambda(a, c) + \Lambda(c, b)$ is an upper bound for all numbers $|\pi(P)|$. Since this cannot be less than the least upper bound, we must have

$$\Lambda(a, b) \leq \Lambda(a, c) + \Lambda(c, b).$$

This inequality, along with (14.14), implies the additive property.

14.12 The arc-length function

Suppose a curve is the path traced out by a position vector $r(t)$. A natural question to ask is this: How far has the particle moved along the curve at time t? To discuss this question, we introduce the *arc-length function s*, defined as follows:

$$s(t) = \Lambda(a, t) \quad \text{if} \quad t > a, \qquad s(a) = 0.$$

The statement $s(a) = 0$ simply means we are assuming the motion begins when $t = a$.

The theorem on additivity enables us to derive some important properties of s. For example, we have the following.

THEOREM 14.12. *For any rectifiable curve, the arc-length function s is monotonically increasing on $[a, b]$. That is, we have*

$$s(t_1) \leq s(t_2) \quad \text{if} \quad a \leq t_1 < t_2 \leq b. \tag{14.15}$$

Proof. If $a \leq t_1 < t_2 \leq b$, we have

$$s(t_2) - s(t_1) = \Lambda(a, t_2) - \Lambda(a, t_1) = \Lambda(t_1, t_2),$$

where the last equality comes from additivity. Since $\Lambda(t_1, t_2) \geq 0$, this proves (14.15).

Next we shall prove that the function s has a derivative at each interior point of the parametric interval and that this derivative is equal to the speed of the particle.

THEOREM 14.13. *Let s denote the arc-length function associated with a curve and let $v(t)$ denote the speed at time t. If v is continuous on $[a, b]$, then the derivative $s'(t)$ exists for each t in (a, b) and is given by the formula*

$$s'(t) = v(t). \tag{14.16}$$

Proof. Define $f(t) = \int_a^t v(u)\, du$. We know that $f'(t) = v(t)$ because of the first fundamental theorem of calculus. We shall prove that $s'(t) = v(t)$. For this purpose we form the difference quotient

$$\left\| \frac{r(t+h) - r(t)}{h} \right\| . \tag{14.17}$$

Suppose first that $h > 0$. The line segment joining the points $r(t)$ and $r(t+h)$ may be thought of as a polygon approximating the arc joining these two points. Therefore, because of (14.11), we have

$$\|r(t+h) - r(t)\| \le \Lambda(t, t+h) = s(t+h) - s(t) .$$

Using this in (14.17) along with the inequality (14.12) of Theorem 14.10 we have

$$\left\| \frac{r(t+h) - r(t)}{h} \right\| \le \frac{s(t+h) - s(t)}{h} \le \frac{1}{h}\int_t^{t+h} v(u)\, du = \frac{f(t+h) - f(t)}{h} .$$

A similar argument shows that these inequalities are also valid for $h < 0$. If we let $h \to 0$, the difference quotient on the extreme left approaches $\|r'(t)\| = v(t)$ and that on the extreme right approaches $f'(t) = v(t)$. It follows that the quotient $[s(t+h) - s(t)]/h$ also approaches $v(t)$. But this means that $s'(t)$ exists and equals $v(t)$, as asserted.

Theorem 14.13 conforms with our intuitive notion of speed as the distance per unit time being covered during the motion.

Using (14.16) along with the second fundamental theorem of calculus, we can compute arc length by integrating the speed. Thus, the distance traveled by a particle during a time interval $[t_1, t_2]$ is

$$s(t_2) - s(t_1) = \int_{t_1}^{t_2} s'(t)\, dt = \int_{t_1}^{t_2} v(t)\, dt .$$

In particular, when $t_1 = a$ and $t_2 = b$, we obtain the following integral for arc length:

$$\Lambda(a, b) = \int_a^b v(t)\, dt .$$

EXAMPLE 1. *Length of a circular arc.* To compute the length of an arc of a circle of radius a, we may imagine a particle moving along the circle according to the equation $r(t) = a \cos t\, i + a \sin t\, j$. The velocity vector is $v(t) = -a \sin t\, i + a \cos t\, j$ and the speed is $v(t) = a$. Integrating the speed over an interval of length θ, we find that the length of arc traced out is $a\theta$. In other words, the length of a circular arc is proportional to the angle it subtends; the constant of proportionality is the radius of the circle. For a unit circle we have $a = 1$, and the arc length is exactly equal to the angular measure.

EXAMPLE 2. *Length of the graph of a real-valued function.* The graph of a real-valued function f defined on an interval $[a, b]$ can be treated as a curve with position vector $r(t)$ given by

$$r(t) = ti + f(t)j .$$

The corresponding velocity vector is $v(t) = i + f'(t)j$, and the speed is

$$v(t) = \|v(t)\| = \sqrt{1 + [f'(t)]^2} .$$

Therefore, the arc length of the graph of f above a subinterval $[a, x]$ is given by the integral

$$s(x) = \int_a^x v(t)\, dt = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt . \tag{14.18}$$

14.13 Exercises

In Exercises 1 through 9, find the length of the path traced out by a particle moving on a curve according to the given equation during the time interval specified in each case.

1. $r(t) = a(1 - \cos t)i + a(t - \sin t)j, \quad 0 \le t \le 2\pi, \quad a > 0.$
2. $r(t) = e^t \cos t\, i + e^t \sin t\, j, \quad 0 \le t \le 2.$
3. $r(t) = a(\cos t + t \sin t)i + a(\sin t - t \cos t)j, \quad 0 \le t \le 2\pi, \quad a > 0.$
4. $r(t) = \dfrac{c^2}{a} \cos^3 t\, i + \dfrac{c^2}{b} \sin^3 t\, j, \quad 0 \le t \le 2\pi, \quad c^2 = a^2 - b^2, \quad 0 < b < a.$
5. $r(t) = a(\sinh t - t)i + a(\cosh t - 1)j, \quad 0 \le t \le T, \quad a > 0.$
6. $r(t) = \sin t\, i + t\, j + (1 - \cos t)k \quad (0 \le t \le 2\pi).$
7. $r(t) = t\, i + 3t^2 j + 6t^3 k \quad (0 \le t \le 2).$
8. $r(t) = t\, i + \log(\sec t)j + \log(\sec t + \tan t)k \quad (0 \le t \le \frac{1}{4}\pi).$
9. $r(t) = a \cos \omega t\, i + a \sin \omega t\, j + b\omega k \quad (t_0 \le t \le t_1).$
10. Find an integral similar to that in (14.18) for the length of the graph of an equation of the form $x = g(y)$, where g has a continuous derivative on an interval $[c, d]$.
11. A curve has the equation $y^2 = x^3$. Find the length of the arc joining $(1, -1)$ to $(1, 1)$.
12. Two points A and B on a unit circle with center at O determine a circular sector AOB. Prove that the arc AB has a length equal to twice the area of the sector.
13. Set up integrals for the lengths of the curves whose equations are (a) $y = e^x$, $0 \le x \le 1$; (b) $x = t + \log t$, $y = t - \log t$, $1 \le t \le e$. Show that the second length is $\sqrt{2}$ times the first one.
14. (a) Set up the integral which gives the length of the curve $y = c \cosh (x/c)$ from $x = 0$ to $x = a$ $(a > 0, c > 0)$.
    (b) Show that c times the length of this curve is equal to the area of the region bounded by $y = c \cosh (x/c)$, the x-axis, the y-axis, and the line $x = a$.
    (c) Evaluate this integral and find the length of the curve when $a = 2$.
15. Show that the length of the curve $y = \cosh x$ joining the points $(0, 1)$ and $(x, \cosh x)$ is $\sinh x$ if $x > 0$.
16. A nonnegative function f has the property that its ordinate set over an arbitrary interval has an area proportional to the arc length of the graph above the interval. Find f.
17. Use the vector equation $r(t) = a \sin t\, i + b \cos t\, j$, where $0 < b < a$, to show that the circumference L of an ellipse is given by the integral

$$L = 4a \int_0^{\pi/2} \sqrt{1 - e^2 \sin^2 t}\, dt ,$$

where $e = \sqrt{a^2 - b^2}/a$. (The number e is the eccentricity of the ellipse.) This is a special case of an integral of the form

$$E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 t}\, dt ,$$

called an *elliptic integral of the second kind*, where $0 \le k < 1$. The numbers $E(k)$ have been tabulated for various values of k.

18. If $0 < b < 4a$, let $r(t) = a(t - \sin t)i + a(1 - \cos t)j + b \sin \frac{1}{2}t\, k$. Show that the length of the path traced out from $t = 0$ to $t = 2\pi$ is $8aE(k)$, where $E(k)$ has the meaning given in Exercise 17 and $k^2 = 1 - (b/4a)^2$.
19. A particle moves with position vector

$$r(t) = tA + t^2B + 2(\tfrac{2}{3}t)^{3/2} A \times B ,$$

where A and B are two fixed unit vectors making an angle of $\pi/3$ radians with each other. Compute the speed of the particle at time t and find how long it takes for it to move a distance of 12 units of arc length from the initial position $r(0)$.
20. (a) When a circle rolls (without slipping) along a straight line, a point on the circumference traces out a curve called a *cycloid.* If the fixed line is the x-axis and if the tracing point (x, y) is originally at the origin, show that when the circle rolls through an angle θ we have

$$x = a(\theta - \sin \theta) , \qquad y = a(1 - \cos \theta) ,$$

where a is the radius of the circle. These serve as parametric equations for the cycloid.
(b) Referring to part (a), show that $dy/dx = \cot \frac{1}{2}\theta$ and deduce that the tangent line of the cycloid at (x, y) makes an angle $\frac{1}{2}(\pi - \theta)$ with the x-axis. Make a sketch and show that the tangent line passes through the highest point on the circle.
21. Let C be a curve described by two equivalent functions X and Y, where $Y(t) = X[u(t)]$ for $c \le t \le d$. If the function u which defines the change of parameter has a continuous derivative in $[c, d]$ prove that

$$\int_{u(c)}^{u(d)} \|X'(u)\|\, du = \int_c^d \|Y'(t)\|\, dt ,$$

and deduce that the arc length of C is invariant under such a change of parameter.
22. Consider the plane curve whose vector equation is $r(t) = ti + f(t)j$, where

$$f(t) = t \cos\left(\frac{\pi}{2t}\right) \quad \text{if } t \ne 0 , \qquad f(0) = 0 .$$

Consider the following partition of the interval $[0, 1]$:

$$P = \left\{0, \frac{1}{2n}, \frac{1}{2n-1}, \ldots, \frac{1}{3}, \frac{1}{2}, 1\right\}.$$

Show that the corresponding inscribed polygon $\pi(P)$ has length

$$|\pi(P)| > 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2n}$$

and deduce that this curve is nonrectifiable.

14.14 Curvature of a curve

For a straight line the unit tangent vector T does not change its direction, and hence $T' = O$. If the curve is not a straight line, the derivative T' measures the tendency of the tangent to change its direction. The rate of change of the unit tangent *with respect to arc length* is called the *curvature vector* of the curve. We denote this by dT/ds, where s represents arc length. The chain rule, used in conjunction with the relation $s'(t) = v(t)$, tells us that the curvature vector dT/ds is related to the "time" derivative T' by the equation

$$\frac{dT}{ds} = \frac{dt}{ds}\frac{dT}{dt} = \frac{1}{s'(t)}\, T'(t) = \frac{1}{v(t)}\, T'(t).$$

Since $T'(t) = \|T'(t)\|\, N(t)$, we obtain

$$\text{(14.19)} \qquad \frac{dT}{ds} = \frac{\|T'(t)\|}{v(t)}\, N(t) ,$$

which shows that the curvature vector has the same direction as the principal normal $N(t)$. The scalar factor which multiplies $N(t)$ in (14.19) is a nonnegative number called the *curvature* of the curve at t and it is denoted by $\kappa(t)$ (κ is the Greek letter kappa). Thus the curvature $\kappa(t)$, defined to be the *length of the curvature vector,* is given by the following formula:

$$\text{(14.20)} \qquad \kappa(t) = \frac{\|T'(t)\|}{v(t)} .$$

EXAMPLE 1. *Curvature of a circle.* For a circle of radius a, given by $r(t) = a \cos t\, i + a \sin t\, j$, we have $v(t) = -a \sin t\, i + a \cos t\, j$, $v(t) = a$, $T(t) = -\sin t\, i + \cos t\, j$, and $T'(t) = -\cos t\, i - \sin t\, j$. Hence we have $\|T'(t)\| = 1$ so $\kappa(t) = 1/a$. This shows that a circle has constant curvature. The reciprocal of the curvature is the radius of the circle.

When $\kappa(t) \ne 0$, its reciprocal is called the *radius of curvature* and is denoted by $\rho(t)$ (ρ is the Greek letter rho). That circle in the osculating plane with radius $\rho(t)$ and center at the tip of the curvature vector is called the *osculating circle.* It can be shown that the osculating circle is the limiting position of circles passing through three nearby points on the curve as two of the points approach the third. Because of this property, the osculating circle is often called the circle that "best fits the curve" at each of its points.

EXAMPLE 2. *Curvature of a plane curve.* For a plane curve, we have seen that $\|T'(t)\| = |\alpha'(t)|$, where $\alpha(t)$ is the angle the tangent vector makes with the positive x-axis, as shown in Figure 14.11. From the chain rule, we have $\alpha'(t) = d\alpha/dt = (d\alpha/ds)(ds/dt) = v(t)d\alpha/ds$, so Equation (14.20) implies

$$\kappa(t) = \left|\frac{d\alpha}{ds}\right|.$$

In other words, the curvature of a plane curve is the absolute value of the rate of change of α with respect to arc length. It measures the change of direction per unit distance along the curve.

EXAMPLE 3. *Plane curves of constant curvature.* If $d\alpha/ds$ is a nonzero constant, say $d\alpha/ds = a$, then $\alpha = as + b$, where b is a constant. Hence, if we use the arc length s as a parameter, we have $T = \cos(as + b)i + \sin(as + b)j$. Integrating, we find that $r = (1/a)\sin(as + b)i - (1/a)\cos(as + b)j + A$, where A is a constant vector. Therefore $\|r - A\| = 1/|a|$, so the curve is a circle (or an arc of a circle) with center at A and radius $1/|a|$. This proves that a plane curve of constant curvature $\kappa \neq 0$ is a circle (or an arc of a circle) with radius $1/\kappa$.

Now we prove a theorem which relates the curvature to the velocity and acceleration.

THEOREM 14.14. *For any motion with velocity $v(t)$, speed $v(t)$, acceleration $a(t)$, and curvature $\kappa(t)$, we have*

$$a(t) = v'(t)T(t) + \kappa(t)v^2(t)N(t). \tag{14.21}$$

This formula, in turn, implies

$$\kappa(t) = \frac{\|a(t) \times v(t)\|}{v^3(t)}. \tag{14.22}$$

Proof. To prove (14.21), we rewrite (14.20) in the form $\|T'(t)\| = \kappa(t)v(t)$, which gives us $T'(t) = \kappa(t)v(t)N(t)$. Substituting this expression for $T'(t)$ in Equation (14.8), we obtain (14.21).

To prove (14.22), we form the cross product $a(t) \times v(t)$, using (14.21) for $a(t)$ and the formula $v(t) = v(t)T(t)$ for the velocity. This gives us

$$a \times v = v'vT \times T + \kappa v^3 N \times T = \kappa v^3 N \times T \tag{14.23}$$

since $T \times T = O$. If we take the length of each member of (14.23) and note that

$$\|N \times T\| = \|N\|\,\|T\| \sin \tfrac{1}{2}\pi = 1,$$

we obtain $\|a \times v\| = \kappa v^3$, which proves (14.22).

In practice it is fairly easy to compute the vectors v and a (by differentiating the position vector r); hence Equation (14.22) provides a useful method for computing the curvature. This method is usually simpler than determining the curvature from its definition.

For a straight line we have $a \times v = O$, so the curvature is everywhere zero. A curve with a small curvature at a point has a large radius of curvature there and hence does not differ much from a straight line in the immediate vicinity of the point. Thus the curvature is a measure of the tendency of a curve to deviate from a straight line.

14.15 Exercises

1. Refer to the curves described in Exercises 1 through 6 of Section 14.9 and in each case determine the curvature $\kappa(t)$ for the value of t indicated.
2. A helix is described by the position function $r(t) = a\cos\omega t\, i + a\sin\omega t\, j + b\omega t k$. Prove that it has constant curvature $\kappa = a/(a^2 + b^2)$.

3. Two fixed unit vectors A and B make an angle θ with each other, where $0 < \theta < \pi$. A particle moves on a space curve in such a way that its position vector $r(t)$ and velocity $v(t)$ are related by the equation $v(t) = A \times r(t)$. If $r(0) = B$, prove that the curve has constant curvature and compute this curvature in terms of θ.
4. A point moves in space according to the vector equation

$$r(t) = 4\cos t\, i + 4\sin t\, j + 4\cos t\, k.$$

(a) Show that the path is an ellipse and find a Cartesian equation for the plane containing this ellipse.
(b) Show that the radius of curvature is $\rho(t) = 2\sqrt{2}\,(1 + \sin^2 t)^{3/2}$.
5. For the curve whose vector equation is $r(t) = e^t i + e^{-t} j + \sqrt{2}\, t\, k$, show that the curvature is $\kappa(t) = \sqrt{2}/(e^t + e^{-t})^2$.
6. (a) For a plane curve described by the equation $r(t) = x(t)i + y(t)j$, show that the curvature is given by the formula

$$\kappa(t) = \frac{|x'(t)y''(t) - y'(t)x''(t)|}{\{[x'(t)]^2 + [y'(t)]^2\}^{3/2}}.$$

(b) If a plane curve has the Cartesian equation $y = f(x)$, show that the curvature at the point $(x, f(x))$ is

$$\frac{|f''(x)|}{\{1 + [f'(x)]^2\}^{3/2}}.$$

7. If a point moves so that the velocity and acceleration vectors always have constant lengths, prove that the curvature is constant at all points of the path. Express this constant in terms of $\|a\|$ and $\|v\|$.
8. If two plane curves with Cartesian equations $y = f(x)$ and $y = g(x)$ have the same tangent at a point (a, b) and the same curvature at that point, prove that $|f''(a)| = |g''(a)|$.
9. For certain values of the constants a and b, the two curves with Cartesian equations $y = ax(b - x)$ and $(x + 2)y = x$ intersect at only one point P, have a common tangent line at P, and have the same curvature at P.
(a) Find all a and b which satisfy all these conditions.
(b) For each possible choice of a and b satisfying the given conditions, make a sketch of the two curves. Show how they intersect at P.
10. (a) Prove that the radius of curvature of a parabola is smallest at its vertex.
(b) Given two fixed unit vectors A and B making an angle θ with each other, where $0 < \theta < \pi$. The curve with position vector $r(t) = tA + t^2B$ is a parabola lying in the plane spanned by A and B. Determine (in terms of A, B, and θ) the position vector of the vertex of this parabola. You may use the property of the parabola stated in part (a).
11. A particle moves along a plane curve with constant speed 5. It starts at the origin at time $t = 0$ with initial velocity $5j$, and it never goes to the left of the y-axis. At every instant the curvature of the path is $\kappa(t) = 2t$. Let $\alpha(t)$ denote the angle that the velocity vector makes with the positive x-axis at time t.
(a) Determine $\alpha(t)$ explicitly as a function of t.
(b) Determine the velocity $v(t)$ in terms of i and j.
12. A particle moves along a plane curve with constant speed 2. The motion starts at the origin when $t = 0$ and the initial velocity $v(0)$ is $2i$. At every instant it is known that the curvature $\kappa(t) = 4t$. Find the velocity when $t = \frac{1}{4}\sqrt{\pi}$ if the curve never goes below the x-axis.

14.16 Velocity and acceleration in polar coordinates

Sometimes it is more natural to describe the points on a plane curve by polar coordinates rather than rectangular coordinates. Since the rectangular coordinates (x, y) are related to the polar coordinates r and θ by the equations

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

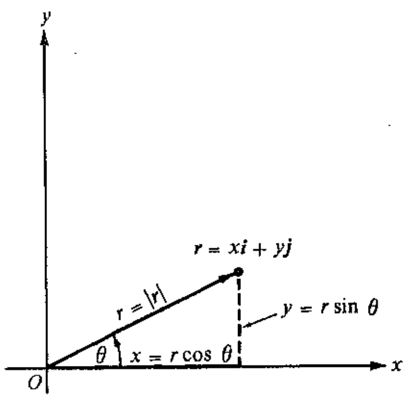


FIGURE 14.15 Polar coordinates.

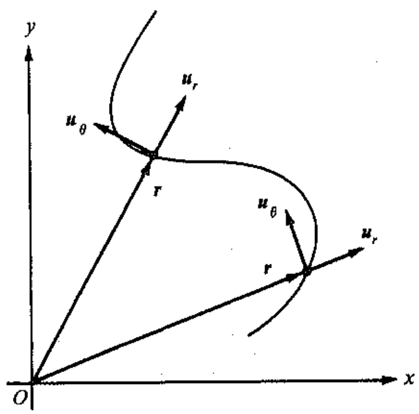


FIGURE 14.16 The unit vectors u_r and u_θ.

the position vector $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j}$ joining the origin to (x, y) is given by

$$\boldsymbol{r} = r\cos\theta\,\boldsymbol{i} + r\sin\theta\,\boldsymbol{j} = r(\cos\theta\,\boldsymbol{i} + \sin\theta\,\boldsymbol{j}),$$

where $r = \|\boldsymbol{r}\|$. This relation is illustrated in Figure 14.15.

The vector $\cos\theta\,\boldsymbol{i} + \sin\theta\,\boldsymbol{j}$ is a vector of unit length having the same direction as $\boldsymbol{r}$. This unit vector is usually denoted by $\boldsymbol{u}_r$ and the foregoing equation is written as follows:

$$\boldsymbol{r} = r\boldsymbol{u}_r, \qquad \text{where} \quad \boldsymbol{u}_r = \cos\theta\,\boldsymbol{i} + \sin\theta\,\boldsymbol{j}.$$

It is convenient to introduce also a unit vector $\boldsymbol{u}_\theta$, perpendicular to $\boldsymbol{u}_r$, which is defined as follows:

$$\boldsymbol{u}_\theta = \frac{d\boldsymbol{u}_r}{d\theta} = -\sin\theta\,\boldsymbol{i} + \cos\theta\,\boldsymbol{j}.$$

Note that we have

$$\frac{d\boldsymbol{u}_\theta}{d\theta} = -\cos\theta\,\boldsymbol{i} - \sin\theta\,\boldsymbol{j} = -\boldsymbol{u}_r.$$

In the study of plane curves, the two unit vectors $\boldsymbol{u}_r$ and $\boldsymbol{u}_\theta$ play the same roles in polar coordinates as the unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ in rectangular coordinates. Figure 14.16 shows the unit vectors $\boldsymbol{u}_r$ and $\boldsymbol{u}_\theta$ attached to a curve at some of its points.

Now suppose the polar coordinates r and θ are functions of t, say $r = f(t)$, $\theta = g(t)$. We shall derive formulas for expressing the velocity and acceleration in terms of $\boldsymbol{u}_r$ and $\boldsymbol{u}_\theta$. For the position vector, we have

$$\boldsymbol{r} = r\boldsymbol{u}_r = f(t)\boldsymbol{u}_r.$$

Since θ depends on the parameter t, the same is true of the unit vector $\boldsymbol{u}_r$ and we must take this into account when we compute the velocity vector. Thus we have

$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = \frac{d(r\boldsymbol{u}_r)}{dt} = \frac{dr}{dt}\boldsymbol{u}_r + r\frac{d\boldsymbol{u}_r}{dt}.$$

Using the chain rule, we may express $d\boldsymbol{u}_r/dt$ in terms of $\boldsymbol{u}_\theta$ by writing

$$\text{(14.24)} \qquad \frac{d\boldsymbol{u}_r}{dt} = \frac{d\theta}{dt}\frac{d\boldsymbol{u}_r}{d\theta} = \frac{d\theta}{dt}\boldsymbol{u}_\theta,$$

and the equation for the velocity vector becomes

$$\text{(14.25)} \qquad \boldsymbol{v} = \frac{dr}{dt}\boldsymbol{u}_r + r\frac{d\theta}{dt}\boldsymbol{u}_\theta.$$

The scalar factors dr/dt and $r\,d\theta/dt$ multiplying $\boldsymbol{u}_r$ and $\boldsymbol{u}_\theta$ are called, respectively, the *radial* and *transverse components* of velocity.

Since $\boldsymbol{u}_r$ and $\boldsymbol{u}_\theta$ are orthogonal unit vectors, we find that

$$\boldsymbol{v}\cdot\boldsymbol{v} = \left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2,$$

so the speed v is given by the formula

$$v = \sqrt{\left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2}.$$

Differentiating both sides of (14.25), we find that the acceleration vector is given by

$$\boldsymbol{a} = \left(\frac{d^2r}{dt^2}\boldsymbol{u}_r + \frac{dr}{dt}\frac{d\boldsymbol{u}_r}{dt}\right) + \left(r\frac{d^2\theta}{dt^2}\boldsymbol{u}_\theta + \frac{dr}{dt}\frac{d\theta}{dt}\boldsymbol{u}_\theta + r\frac{d\theta}{dt}\frac{d\boldsymbol{u}_\theta}{dt}\right).$$

The derivative $d\boldsymbol{u}_r/dt$ may be expressed in terms of $\boldsymbol{u}_\theta$ by (14.24). We may similarly express the derivative of $\boldsymbol{u}_\theta$ by the equation

$$\frac{d\boldsymbol{u}_\theta}{dt} = \frac{d\theta}{dt}\frac{d\boldsymbol{u}_\theta}{d\theta} = -\frac{d\theta}{dt}\boldsymbol{u}_r.$$

This leads to the following formula which expresses $\boldsymbol{a}$ in terms of its radial and transverse components:

$$\boldsymbol{a} = \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\boldsymbol{u}_r + \left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right)\boldsymbol{u}_\theta . \tag{14.26}$$

When $\theta = t$, the curve may be described by the polar equation $r = f(\theta)$. In this case, the formulas for velocity, speed, and acceleration simplify considerably, and we obtain

$$\boldsymbol{v} = \frac{dr}{d\theta}\boldsymbol{u}_r + r\boldsymbol{u}_\theta , \qquad v = \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}, \qquad \boldsymbol{a} = \left(\frac{d^2r}{d\theta^2} - r\right)\boldsymbol{u}_r + 2\frac{dr}{d\theta}\boldsymbol{u}_\theta .$$

14.17 Plane motion with radial acceleration

The acceleration vector is said to be *radial* if the transverse component in Equation (14.26) is always zero. This component is equal to

$$r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} = \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right).$$

Therefore, the acceleration is radial if and only if $r^2\, d\theta/dt$ is constant.

Plane motion with radial acceleration has an interesting geometric interpretation in terms of area. Denote by $A(t)$ the area of the region swept out by the position vector from a fixed time, say $t = a$, to a later time t. An example is the shaded region shown in Figure 14.17. We shall prove that the time rate of change of this area is exactly equal to $\frac{1}{2}r^2\, d\theta/dt$. That is, we have

$$A'(t) = \frac{1}{2}r^2\frac{d\theta}{dt}. \tag{14.27}$$

From this it follows that the acceleration vector is radial if and only if the position vector sweeps out area at a constant rate.

To prove (14.27), we assume that it is possible to eliminate t from the two equations $r = f(t)$, $\theta = g(t)$, and thereby express r as a function of θ, say $r = R(\theta)$. This means that there is a real-valued function R such that $R[g(t)] = f(t)$. Then the shaded region in Figure 14.17 is the radial set of R over the interval $[g(a), g(t)]$. By Theorem 2.6, the area of this region is given by the integral

$$A(t) = \tfrac{1}{2}\int_{g(a)}^{g(t)} R^2(\theta)\, d\theta .$$

Differentiating this integral by the first fundamental theorem of calculus and the chain rule, we find that

$$A'(t) = \frac{1}{2}R^2[g(t)]g'(t) = \frac{1}{2}f^2(t)g'(t) = \frac{1}{2}r^2\frac{d\theta}{dt},$$

which proves (14.27).

14.18 Cylindrical coordinates

If the x- and y-coordinates of a point $P = (x, y, z)$ in 3-space are replaced by polar coordinates r and θ, then the three numbers r, θ, z are called *cylindrical coordinates* for the point P. The nonnegative number r now represents the distance from the z-axis to the point P, as indicated in Figure 14.18. Those points in space for which r is constant are at a fixed distance from the z-axis and therefore lie on a circular cylinder (hence the name *cylindrical* coordinates).

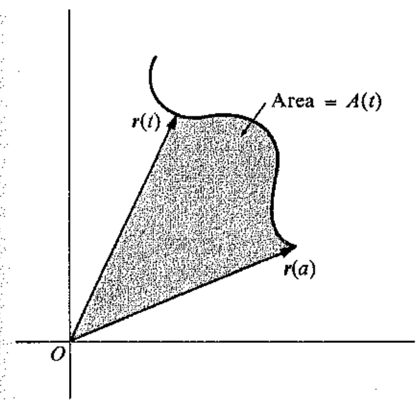


FIGURE 14.17 The position vector sweeps out area at the rate $A'(t) = \frac{1}{2}r^2\frac{d\theta}{dt}$.

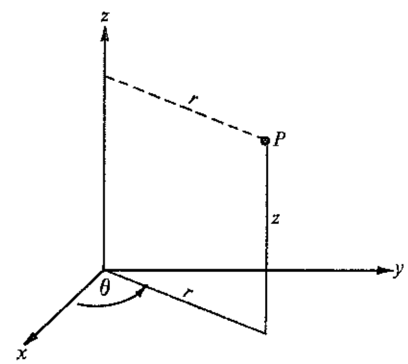


FIGURE 14.18 Cylindrical coordinates.

To discuss *space* curves in cylindrical coordinates, the equation for the position vector $\boldsymbol{r}$ must be replaced by one of the form

$$\boldsymbol{r} = r\boldsymbol{u}_r + z(t)\boldsymbol{k} .$$

Corresponding formulas for the velocity and acceleration vectors are obtained by merely adding the terms $z'(t)\boldsymbol{k}$ and $z''(t)\boldsymbol{k}$, respectively, to the right-hand members of the two-dimensional formulas in (14.25) and (14.26).

14.19 Exercises

1. A particle moves in a plane so that its position at time t has polar coordinates $r = t$, $\theta = t$. Find formulas for the velocity $\boldsymbol{v}$, the acceleration $\boldsymbol{a}$, and the curvature κ at any time t.
2. A particle moves in space so that its position at time t has cylindrical coordinates $r = t$, $\theta = t$, $z = t$. It traces out a curve called a *conical helix.*
   (a) Find formulas for the velocity $\boldsymbol{v}$, the acceleration $\boldsymbol{a}$, and the curvature κ at time t.
   (b) Find a formula for determining the angle between the velocity vector and the generator of the cone at each point of the curve.
3. A particle moves in space so that its position at time t has cylindrical coordinates $r = \sin t$, $\theta = t$, $z = \log \sec t$, where $0 \le t < \frac{1}{2}\pi$.

(a) Show that the curve lies on the cylinder with Cartesian equation $x^2 + (y - \frac{1}{2})^2 = \frac{1}{4}$.
(b) Find a formula (in terms of t) for the angle which the velocity vector makes with k.

4. If a curve is given by a polar equation $r = f(\theta)$, where $a \le \theta \le b \le a + 2\pi$, prove that its arc length is

$$\int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta .$$

5. The curve described by the polar equation $r = a(1 + \cos\theta)$, where $a > 0$ and $0 \le \theta \le 2\pi$, is called a *cardiod.* Draw a graph of the cardiod $r = 4(1 + \cos\theta)$ and compute its arc length.
6. A particle moves along a plane curve whose polar equation is $r = e^{c\theta}$, where c is a constant and θ varies from 0 to 2π.
(a) Make a sketch indicating the general shape of the curve for each of the following values of c: $c = 0, c = 1, c = -1$.
(b) Let $L(c)$ denote the arc length of the curve and let $a(c)$ denote the area of the region swept out by the position vector as θ varies from 0 to 2π. Compute $L(c)$ and $a(c)$ in terms of c.
7. Sketch the curve whose polar equation is $r = \sin^2\theta$, $0 \le \theta \le 2\pi$, and show that it consists of two loops.
(a) Find the area of region enclosed by one loop of the curve.
(b) Compute the length of one loop of the curve.

In each of Exercises 8 through 11, make a sketch of the plane curve having the given polar equation and compute its arc length.

8. $r = \theta$, $\quad 0 \le \theta \le \pi$.
9. $r = e^{\theta}$, $\quad 0 \le \theta \le \pi$.
10. $r = 1 + \cos\theta$, $\quad 0 \le \theta \le \pi$.
11. $r = 1 - \cos\theta$, $\quad 0 \le \theta \le 2\pi$.
12. If a curve has the polar equation $r = f(\theta)$, show that its radius of curvature ρ is given by the formula $\rho = (r^2 + r'^2)^{3/2}/|r^2 - rr'' + 2r'^2|$, where $r' = f'(\theta)$ and $r'' = f''(\theta)$.
13. For each of the curves in Exercises 8 through 11, compute the radius of curvature for the value of θ indicated.
(a) Arbitrary θ in Exercise 8.
(b) Arbitrary θ in Exercise 9.
(c) $\theta = \frac{1}{4}\pi$ in Exercise 10.
(d) $\theta = \frac{1}{2}\pi$ in Exercise 11.
14. Let ϕ denote the angle, $0 \le \phi \le \pi$, between the position vector and the velocity vector of a curve. If the curve is expressed in polar coordinates, prove that $v \sin\phi = r$ and $v\cos\phi = dr/d\theta$, where v is the speed.
15. A missile is designed to move directly toward its target. Due to mechanical failure, its direction in actual flight makes a fixed angle $\alpha \ne 0$ with the line from the missile to the target. Find the path if it is fired at a fixed target. Discuss how the path varies with α. Does the missile ever reach the target? (Assume the motion takes place in a plane.)
16. Due to a mechanical failure, a ground crew has lost control of a missile recently fired. It is known that the missile will proceed at a constant speed on a straight course of unknown direction. When the missile is 4 miles away, it is sighted for an instant and lost again. Immediately an anti-missile missile is fired with a constant speed three times that of the first missile. What should be the course of the second missile in order for it to overtake the first one? (Assume both missiles move in the same plane.)
17. Prove that if a homogeneous first-order differential equation of the form $y' = f(x, y)$ is rewritten in polar coordinates, it reduces to a separable equation. Use this method to solve $y' = (y - x)/(y + x)$.
18. A particle (moving in space) has velocity vector $v = \omega k \times r$, where ω is a positive constant and r is the position vector. Prove that the particle moves along a circle with constant angular speed ω. (The angular speed is defined to be $|d\theta/dt|$, where θ is the polar angle at time t.)
19. A particle moves in a plane perpendicular to the z-axis. The motion takes place along a circle with center on this axis.

(a) Show that there is a vector $\omega(t)$ parallel to the z-axis such that

$$v(t) = \omega(t) \times r(t),$$

where $r(t)$ and $v(t)$ denote the position and velocity vectors at time t. The vector $\omega(t)$ is called the *angular velocity* vector and its magnitude $\omega(t) = \|\omega(t)\|$ is called the *angular speed.*
(b) The vector $\alpha(t) = \omega'(t)$ is called the *angular acceleration* vector. Show that the acceleration vector $a(t)$ [$= v'(t)$] is given by the formula

$$a(t) = [\omega(t)\cdot r(t)]\omega(t) - \omega^2(t)r(t) + \alpha(t) \times r(t).$$

(c) If the particle lies in the xy-plane and if the angular speed $\omega(t)$ is constant, say $\omega(t) = \omega$, prove that the acceleration vector $a(t)$ is centripetal and that, in fact, $a(t) = -\omega^2 r(t)$.

20. A body is said to undergo a *rigid motion* if, for every pair of particles p and q in the body, the distance $\|r_p(t) - r_q(t)\|$ is independent of t, where $r_p(t)$ and $r_q(t)$ denote the position vectors of p and q at time t. Prove that for a rigid motion in which each particle p rotates about the z-axis we have $v_p(t) = \omega(t) \times r_p(t)$, where $\omega(t)$ is the same for each particle, and $v_p(t)$ is the velocity of particle p.

14.20 Applications to planetary motion

By analyzing the voluminous data on planetary motion accumulated up to 1600, the German astronomer Johannes Kepler (1571–1630) tried to discover the mathematical laws governing the motions of the planets. There were six known planets at that time and, according to the Copernican theory, their orbits were thought to lie on concentric spherical shells about the sun. Kepler attempted to show that the radii of these shells were linked up with the five regular solids of geometry. He proposed an ingenious idea that the solar system was designed something like a Chinese puzzle. At the center of the system he placed the sun. Then, in succession, he arranged the six concentric spheres that can be inscribed and circumscribed around the five regular solids—the octahedron, icosahedron, dodecahedron, tetrahedron, and cube, in respective order (from inside out). The innermost sphere, inscribed in the regular octahedron, corresponded to Mercury's path. The next sphere, which circumscribed the octahedron and inscribed the icosahedron, corresponded to the orbit of Venus. Earth's orbit lay on the sphere around the icosahedron and inside the dodecahedron, and so on, the outermost sphere, containing Jupiter's orbit, being circumscribed around the cube. Although this theory seemed correct to within five percent, astronomical observations at that time were accurate to a percentage error much smaller than this, and Kepler finally realized that he had to modify this theory. After much further study it occurred to him that the observed data concerning the orbits corresponded more to *elliptical* paths than the circular paths of the Copernican system. After several more years of unceasing effort, Kepler set forth three famous laws, empirically discovered, which explained all the astronomical phenomena known at that time. They may be stated as follows:

Kepler's first law: Planets move in ellipses with the sun at one focus.

Kepler's second law: The position vector from the sun to a planet sweeps out area at a constant rate.

Kepler's third law: The square of the period of a planet is proportional to the cube of its mean distance from the sun.

Note: By the *period* of a planet is meant the time required to go once around the elliptical orbit. The *mean distance* from the sun is one half the length of the major axis of the ellipse.

The formulation of these laws from a study of astronomical tables was a remarkable achievement. Nearly 50 years later, Newton proved that all three of Kepler's laws are consequences of his own second law of motion and his celebrated universal law of gravitation. In this section we shall use vector methods to show how Kepler's laws may be deduced from Newton's.

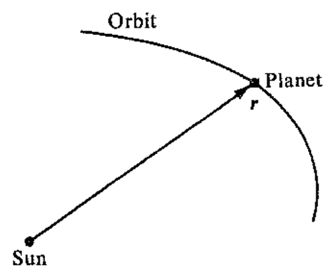


FIGURE 14.19 The position vector from the sun to a planet.

Assume we have a fixed sun of mass M and a moving planet of mass m attracted to the sun by a force F. (We neglect the influence of all other forces.) Newton's second law of motion states that

$$F = m\boldsymbol{a}, \tag{14.28}$$

where $\boldsymbol{a}$ is the acceleration vector of the moving planet. Denote by $\boldsymbol{r}$ the position vector from the sun to the planet (as in Figure 14.19), let $r = \|\boldsymbol{r}\|$, and let $\boldsymbol{u}_r$ be a unit vector with the same direction as $\boldsymbol{r}$, so that $\boldsymbol{r} = r\boldsymbol{u}_r$. The universal law of gravitation states that

$$F = -G\frac{mM}{r^2}\boldsymbol{u}_r,$$

where G is a constant. Combining this with (14.28), we obtain

$$\boldsymbol{a} = -\frac{GM}{r^2}\boldsymbol{u}_r, \tag{14.29}$$

which tells us that the acceleration is *radial.* In a moment we shall prove that the orbit lies in a plane. Once we know this, it follows at once from the results of Section 14.17 that the position vector sweeps out area at a constant rate.

To prove that the path lies in a plane we use the fact that $\boldsymbol{r}$ and $\boldsymbol{a}$ are parallel. If we introduce the velocity vector $\boldsymbol{v} = d\boldsymbol{r}/dt$, we have

$$\boldsymbol{r}\times\boldsymbol{a} = \boldsymbol{r}\times\frac{d\boldsymbol{v}}{dt} + \boldsymbol{v}\times\boldsymbol{v} = \boldsymbol{r}\times\frac{d\boldsymbol{v}}{dt} + \frac{d\boldsymbol{r}}{dt}\times\boldsymbol{v} = \frac{d}{dt}(\boldsymbol{r}\times\boldsymbol{v}).$$

Since $\boldsymbol{r}\times\boldsymbol{a} = O$, this means that $\boldsymbol{r}\times\boldsymbol{v}$ is a constant vector, say $\boldsymbol{r}\times\boldsymbol{v} = \boldsymbol{c}$.

If $\boldsymbol{c} = O$, the position vector $\boldsymbol{r}$ is parallel to $\boldsymbol{v}$ and the motion is along a straight line. Since the path of a planet is not a straight line, we must have $\boldsymbol{c} \neq O$. The relation $\boldsymbol{r}\times\boldsymbol{v} = \boldsymbol{c}$ shows that $\boldsymbol{r}\cdot\boldsymbol{c} = 0$, so the position vector lies in a plane perpendicular to $\boldsymbol{c}$. Since the acceleration is radial, $\boldsymbol{r}$ sweeps out area at a constant rate. This proves Kepler's second law.

It is easy to prove that this constant rate is exactly half the length of the vector $\boldsymbol{c}$. In fact, if we use polar coordinates and express the velocity in terms of $\boldsymbol{u}_r$ and $\boldsymbol{u}_\theta$ as in Equation (14.25), we find that

$$\boldsymbol{c} = \boldsymbol{r}\times\boldsymbol{v} = (r\boldsymbol{u}_r)\times\left(\frac{dr}{dt}\boldsymbol{u}_r + r\frac{d\theta}{dt}\boldsymbol{u}_\theta\right) = r^2\frac{d\theta}{dt}\boldsymbol{u}_r\times\boldsymbol{u}_\theta, \tag{14.30}$$

and hence $\|\boldsymbol{c}\| = |r^2\,d\theta/dt|$. By (14.27) this is equal to $2|A'(t)|$, where $A'(t)$ is the rate at which the radius vector sweeps out area.

Kepler's second law is illustrated in Figure 14.20. The two shaded regions, which are swept out by the position vector in equal time intervals, have equal areas.

We shall prove next that the path is an ellipse. First of all, we form the cross product $\boldsymbol{a}\times\boldsymbol{c}$, using (14.29) and (14.30), and we find that

$$\boldsymbol{a}\times\boldsymbol{c} = \left(-\frac{GM}{r^2}\boldsymbol{u}_r\right)\times\left(r^2\frac{d\theta}{dt}\boldsymbol{u}_r\times\boldsymbol{u}_\theta\right) = -GM\frac{d\theta}{dt}\boldsymbol{u}_r\times(\boldsymbol{u}_r\times\boldsymbol{u}_\theta) = GM\frac{d\theta}{dt}\boldsymbol{u}_\theta.$$

Since $\boldsymbol{a} = d\boldsymbol{v}/dt$ and $\boldsymbol{u}_\theta = d\boldsymbol{u}_r/d\theta$, the foregoing equation for $\boldsymbol{a}\times\boldsymbol{c}$ can also be written as follows:

$$\frac{d}{dt}(\boldsymbol{v}\times\boldsymbol{c}) = \frac{d}{dt}(GM\boldsymbol{u}_r).$$

Integration gives us

$$\boldsymbol{v}\times\boldsymbol{c} = GM\boldsymbol{u}_r + \boldsymbol{b},$$

where $\boldsymbol{b}$ is another constant vector. We can rewrite this as follows:

$$\boldsymbol{v}\times\boldsymbol{c} = GM(\boldsymbol{u}_r + \boldsymbol{e}), \tag{14.31}$$

where $GM\boldsymbol{e} = \boldsymbol{b}$. We shall combine this with (14.30) to eliminate $\boldsymbol{v}$ and obtain an equation for r. For this purpose we dot multiply both sides of (14.30) by $\boldsymbol{c}$ and both sides of (14.31) by $\boldsymbol{r}$. Equating the two expressions for the scalar triple product $\boldsymbol{r}\cdot\boldsymbol{v}\times\boldsymbol{c}$, we are led to the equation

$$GMr(1 + e\cos\phi) = c^2, \tag{14.32}$$

where $e = \|\boldsymbol{e}\|$, $c = \|\boldsymbol{c}\|$, and ϕ represents the angle between the constant vector $\boldsymbol{e}$ and the radius vector r. (See Figure 14.21.) If we let $d = c^2/(GMe)$, Equation (14.32) becomes

$$r = \frac{ed}{e\cos\phi + 1} \qquad \text{or} \qquad r = e(d - r\cos\phi)\,. \tag{14.33}$$

By Theorem 13.18, this is the polar equation of a conic section with eccentricity e and a focus at the sun. Figure 14.21 shows the directrix drawn perpendicular to e at a distance d from the sun. The distance from the planet to the directrix is $d - r\cos\phi$, and the ratio

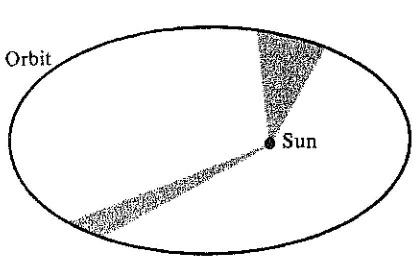


FIGURE 14.20. Kepler's second law. The two shaded regions, swept out in equal time intervals, have equal areas.

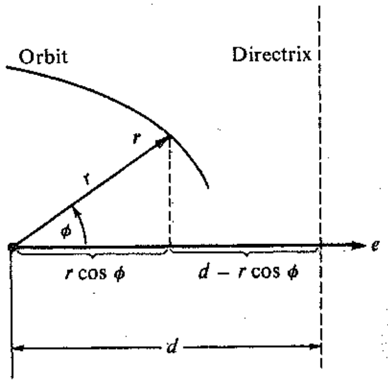


FIGURE 14.21. The ratio $r/(d - r\cos\phi)$ is the eccentricity $e = \|e\|$.

$r/(d - r\cos\phi)$ is the eccentricity e. The conic is an ellipse if $e < 1$, a parabola if $e = 1$, and a hyperbola if $e > 1$. Since planets are known to move on closed paths, the orbit under consideration must be an ellipse. This proves Kepler's first law.

Finally, we deduce Kepler's third law. Suppose the ellipse has major axis of length $2a$ and minor axis of length $2b$. Then the area of the ellipse is πab. Let T be the time it takes for the planet to go once around the ellipse. Since the position vector sweeps out area at the rate $\frac{1}{2}c$, we have $\frac{1}{2}cT = \pi ab$, or $T = 2\pi ab/c$. We wish to prove that T^2 is proportional to a^3.

From Section 13.22 we have $b^2 = a^2(1 - e^2)$, $ed = a(1 - e^2)$, so

$$c^2 = GMed = GMa(1 - e^2)\,,$$

and hence we have

$$T^2 = \frac{4\pi^2a^2b^2}{c^2} = \frac{4\pi^2a^4(1 - e^2)}{GMa(1 - e^2)} = \frac{4\pi^2}{GM}\,a^3\,.$$

Since T^2 is a constant times a^3, this proves Kepler's third law.

14.21 Miscellaneous review exercises

1. Let r denote the vector from the origin to an arbitrary point on the parabola $y^2 = x$, let α be the angle that r makes with the tangent line, $0 \le \alpha \le \pi$, and let θ be the angle that r makes with the positive x-axis, $0 \le \theta \le \pi$. Express α in terms of θ.
2. Show that the vector $T = yi + 2cj$ is tangent to the parabola $y^2 = 4cx$ at the point (x, y), and that the vector $N = 2ci - yj$ is perpendicular to T.

   [*Hint:* Write a vector equation for the parabola, using y as a parameter.]
3. Prove that an equation of the line of slope m that is tangent to the parabola $y^2 = 4cx$ can be written in the form $y = mx + c/m$. What are the coordinates of the point of contact?
4. (a) Solve Exercise 3 for the parabola $(y - y_0)^2 = 4c(x - x_0)$.
   (b) Solve Exercise 3 for the parabola $x^2 = 4cy$ and, more generally, for the parabola $(x - x_0)^2 = 4c(y - y_0)$.
5. Prove that an equation of the line that is tangent to the parabola $y^2 = 4cx$ at the point (x_1, y_1) can be written in the form $y_1y = 2c(x + x_1)$.
6. Solve Exercise 5 for each of the parabolas described in Exercise 4.
7. (a) Let P be a point on the parabola $y = x^2$. Let Q be the point of intersection of the normal line at P with the y-axis. What is the limiting position of Q as P tends to the y-axis?
   (b) Solve the same problem for the curve $y = f(x)$, where $f'(0) = 0$.
8. Given that the line $y = c$ intersects the parabola $y = x^2$ at two points. Find the radius of the circle passing through these two points and through the vertex of the parabola. The radius you determine depends on c. What happens to this radius as $c \to 0$?
9. Prove that a point (x_0, y_0) is *inside, on,* or *outside* the ellipse $x^2/a^2 + y^2/b^2 = 1$ according as $x_0^2/a^2 + y_0^2/b^2$ is *less than, equal to,* or *greater than* 1.
10. Given an ellipse $x^2/a^2 + y^2/b^2 = 1$. Show that the vectors T and N given by

$$T = -\frac{y}{b^2}\,i + \frac{x}{a^2}\,j\,, \qquad N = \frac{x}{a^2}\,i + \frac{y}{b^2}\,j$$

are, respectively, *tangent* and *normal* to the ellipse when placed at the point (x, y). If the eccentric angle of (x_0, y_0) is θ_0, show that the tangent line at (x_0, y_0) has the Cartesian equation

$$\frac{x}{a}\cos\theta_0 + \frac{y}{b}\sin\theta_0 = 1\,.$$

11. Show that the tangent line to the ellipse $x^2/a^2 + y^2/b^2 = 1$ at the point (x_0, y_0) has the equation $x_0x/a^2 + y_0y/b^2 = 1$.
12. Prove that the product of the perpendicular distances from the foci of an ellipse to any tangent line is constant, this constant being the square of the length of half the minor axis.
13. Two tangent lines are drawn to the ellipse $x^2 + 4y^2 = 8$, each parallel to the line $x + 2y = 7$. Find the points of tangency.
14. A circle passes through both foci of an ellipse and is tangent to the ellipse at two points. Find the eccentricity of the ellipse.
15. Let V be one of the two vertices of a hyperbola whose transverse axis has length $2a$ and whose eccentricity is 2. Let P be a point on the same branch as V. Denote by A the area of the region bounded by the hyperbola and the line segment VP, and let r be the length of VP.
    (a) Place the coordinate axes in a convenient position and write an equation for the hyperbola.
    (b) Express the area A as an integral and, without attempting to evaluate this integral, show that Ar^{-3} tends to a limit as the point P tends to V. Find this limit.

16. Show that the vectors $T = (y/b^2)i + (x/a^2)j$ and $N = (x/a^2)i - (y/b^2)j$ are, respectively, tangent and normal to the hyperbola $x^2/a^2 - y^2/b^2 = 1$ if placed at the point (x, y) on the curve.
17. Show that the tangent line to the hyperbola $x^2/a^2 - y^2/b^2 = 1$ at the point(x_0, y_0) is given by the equation $x_0x/a^2 - y_0y/b^2 = 1$.
18. The normal line at each point of a curve and the line from that point to the origin form an isosceles triangle whose base is on the x-axis. Show that the curve is a hyperbola.
19. The normal line at a point P of a curve intersects the x-axis at X and the y-axis at Y. Find the curve if each P is the mid-point of the corresponding line segment XY and if the point $(4, 5)$ is on the curve.
20. Prove that the product of the perpendicular distances from an arbitrary point on a hyperbola to its asymptotes is constant.
21. A curve is given by a polar equation $r = f(\theta)$. Find f if an arbitrary arc joining two distinct points of the curve has arc length proportional to (a) the angle subtended at the origin; (b) the difference of the radial distances from the origin to its endpoints; (c) the area of the sector formed by the arc and the radii to its endpoints.
22. If a curve in 3-space is described by a vector-valued function r defined on a parametric interval $[a, b]$, prove that the scalar triple product $r'(t) \cdot r(a) \times r(b)$ is zero for at least one t in (a, b). Interpret this result geometrically.

15

LINEAR SPACES

15.1 Introduction

Throughout this book we have encountered many examples of mathematical objects that can be added to each other and multiplied by real numbers. First of all, the real numbers themselves are such objects. Other examples are real-valued functions, the complex numbers, infinite series, vectors in n-space, and vector-valued functions. In this chapter we discuss a general mathematical concept, called a *linear space*, which includes all these examples and many others as special cases.

Briefly, a linear space is a set of elements of any kind on which certain operations (called *addition* and *multiplication by numbers*) can be performed. In defining a linear space, we do not specify the nature of the elements nor do we tell how the operations are to be performed on them. Instead, we require that the operations have certain properties which we take as axioms for a linear space. We turn now to a detailed description of these axioms.

15.2 The definition of a linear space

Let V denote a nonempty set of objects, called *elements*. The set V is called a linear space if it satisfies the following ten axioms which we list in three groups.

Closure axioms

AXIOM 1. CLOSURE UNDER ADDITION. *For every pair of elements x and y in V there corresponds a unique element in V called the sum of x and y, denoted by $x + y$.*

AXIOM 2. CLOSURE UNDER MULTIPLICATION BY REAL NUMBERS. *For every x in V and every real number a there corresponds an element in V called the product of a and x, denoted by ax.*

Axioms for addition

AXIOM 3. COMMUTATIVE LAW. *For all x and y in V, we have $x + y = y + x$.*

AXIOM 4. ASSOCIATIVE LAW. *For all x, y, and z in V, we have $(x + y) + z = x + (y + z)$.*