Basic Knowledge Math 116

Ch I

Ch 1

Ch 3

Ch 4

Ch 5

Ch 9

Chapter 12

Section 12.2

- Definition of $\mathbb{R}^n$ and its vector space operations.

Section 12.3

- Definition of parallel.

Section 12.5

- Definition of the dot product in $\mathbb{R}^n$.
- Properties of the dot product in $\mathbb{R}^n$.
- The Cauchy-Schwartz inequality.

Section 12.6

- Norm, or length, of a vector.
- Properties of the norm.
- The Triangle inequality.

Section 12.7

- Definition of orthogonality.

Section 12.9

- Definition of the projection of $\vec{b}$ onto $\vec{a}$: $\text{Proj}_{\vec{a}}\vec{b}$.
- Definition of the angle between two vectors, via $\vec{v} \cdot \vec{w} = \|v\|\|w\| \cos\theta$.

Section 12.10

- Standard unit vectors for $\mathbb{R}^n$.

- Definition of linear combination.

Section 12.12

- Definition of linear span of a set of vectors.
- Definition of "span uniquely."

Section 12.13

- Definition of linear independence.
- Definition of orthogonal set.

Section 12.14

- Definition of the term "basis."
- The fact that every basis of $\mathbb{R}^n$ has precisely n vectors.

Section 12.16

- Definition of the vector space $\mathbb{C}^n$.
- Definition of the sesquilinear (or Hermitian) inner product on $\mathbb{C}^n$.
- Definition of parallel and orthogonal vectors in $\mathbb{C}^n$.
- Definition of the norm in $\mathbb{C}^n$.
- Definition of the angle between two vectors in $\mathbb{C}^n$.

Chapter 13

Sections 13.1–13.10 (Recall we worked a little differently from the book)

- The meaning of "k-plane through $\vec{P}$ in $\mathbb{R}^n$."
- The notation $M(P;\, \vec{v}_1,\, \ldots,\, \vec{v}_k)$.
- The terms *parallel* and *skew* in reference to k-planes.
- The conditions for two k-planes to coincide.
- The special names for 1-planes and $(n-1)$-planes in $\mathbb{R}^n$ (namely *lines* and *hyperplanes*).
- The special form for the equation of a hyperplane: $\vec{N} \cdot (\vec{x} - \vec{P}) \;=\; 0$.
- Vector-valued functions
- Definition of the cross product in $\mathbb{R}^3$.
- The algebra $\mathbb{H}$, and the commutator product on $\mathbb{H}$
- The relationships between quaternions and $\mathbb{R}^4$, and the purely imaginary quaternions and $\mathbb{R}^3$
- The relation between quaternions and the cross product
- The triple product and its geometric interpretation
- Determinants of $n \times n$ matrices, and interpretations

Chapter 14

Section 14.1 - 14.3

- Vector-valued functions
- Limits
- Differentiation and integration of vector-valued functions
- Properties of Integration and Differentiation (eg. product rules for dot and cross products)
- **The Chain Rule**. Super important!
- First and Second Fundamental Theorems of Calculus for vector-valued functions
- The Integral Triangle inequality: $\left\| \int_a^b \vec{f}(t)\, dt \right\| \leq \int_a^b \left\| \vec{f}(t) \right\| dt$

Section 14.6-14.8 (Basic notions)

- Position, Velocity and Acceleration vectors, and speed
- The Unit Tangent, $\vec{T}$
- The Principle Normal, $\vec{N}$
- Decomposition of the acceleration vector into tangential and normal components:
$$\vec{a}(t) = v'(t)T(t) + v(t)\|T'\|N(t)$$
- The osculating 2-plane

Section 14.10-14.12 (Arclength)

- The n-dimensional Pythagorean theorem: $\triangle s = \sqrt{(\triangle x_1)^2 + \cdots + (\triangle x_n)^2}$
- The infinitesimal Pythagorean theorem:
$$\begin{aligned} ds &= \sqrt{(dx_1)^2 + \cdots + (dx_n)^2} \\ &= \sqrt{\left(\frac{dx_1}{dt}\right)^2 + \cdots + \left(\frac{dx_n}{dt}\right)^2}\, dt \end{aligned}$$

- Arclength: given $\vec{f}(t) = (f_1(t), \ldots, f_n(t))$, the length of arc between $t = t_0$ and $t = t_1$ is

$$
\begin{aligned}
s(t_1) - s(t_0) &= \int_{t=t_0}^{t_1} ds \\
&\text{or} \\
s(t_1) - s(t_0) &= \int_{t=t_0}^{t_1} \sqrt{(df_1)^2 + \cdots + (df_n)^2} \\
&\text{or} \\
s(t_1) - s(t_0) &= \int_{t_0}^{t_1} \sqrt{\left(\frac{df_1}{dt}\right)^2 + \cdots + \left(\frac{df_n}{dt}\right)^2}\, dt \\
&\text{or} \\
s(t_1) - s(t_0) &= \int_{t_0}^{t_1} \left\| \frac{d\vec{f}}{dt} \right\| dt.
\end{aligned}
$$

Section 14.14 (Curvature)

- Definition of curvature: change in T with respect to arclength (not time!):

$$\kappa = \left\| \frac{d\vec{T}}{ds} \right\|$$

- Formula for κ with respect to time:

$$\kappa = \left\| \frac{d\vec{T}}{ds} \right\| = \left\| \frac{d\vec{T}}{dt} \right\| \frac{1}{v} = \frac{\|\dot{v}\vec{v} - v\vec{a}\|}{v^3}$$

- Formula for acceleration in terms of tangential and normal components:

$$\vec{a}(t) = \dot{v}(t)\,\vec{T}(t) + \kappa(t)\,(v(t))^2\,\vec{N}(t) \tag{1}$$

- Special formula for κ in dimension 3:

$$\kappa(t) = \frac{\|\vec{a}(t) \times \vec{v}(t)\|}{(v(t))^3}$$

Not in book (Curves in 3-dimensional space)

- The TNB frame:

$$\begin{aligned}\vec{T} &= \frac{d\vec{f}}{ds}\\ \vec{N} &= \frac{d\vec{T}/ds}{\|d\vec{T}/ds\|}\\ \vec{B} &= \vec{T}\times\vec{N}\end{aligned}$$

$\vec{T}$=Unit Tangent, $\vec{N}$=Principle Normal, $\vec{B}$=Binormal.

- Derivative formulas:

$$\begin{aligned}\frac{d\vec{T}}{ds} &= \kappa\,\vec{N}\\ \frac{d\vec{N}}{ds} &= -\kappa\,\vec{T} + \tau\,\vec{B}\\ \frac{d\vec{B}}{ds} &= -\tau\,\vec{N}\end{aligned}$$

- Computational formulas for dimension 3:

$$\begin{aligned}\vec{B} &= \frac{\dot{\vec{f}}\times\ddot{\vec{f}}}{\left\|\dot{\vec{f}}\times\ddot{\vec{f}}\right\|}\\ \kappa &= \frac{\left\|\dot{\vec{f}}\times\ddot{\vec{f}}\right\|}{\left\|\dot{\vec{f}}\right\|^3}\\ \tau &= \kappa^{-2}\,\vec{T}\cdot\left(\frac{d\vec{T}}{ds}\times\frac{d^2\vec{T}}{ds^2}\right) = \frac{\dot{\vec{f}}\cdot\left(\ddot{\vec{f}}\times\dddot{\vec{f}}\right)}{\left\|\dot{\vec{f}}\times\ddot{\vec{f}}\right\|^2}\end{aligned}$$