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Operator algebras and conformal field theory

III. Fusion of positive energy representations of $LSU(N)$ using bounded operators

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Contents

1. Introduction

This is one of a series of papers devoted to the study of conformal field theory from the point of view of operator algebras (see [41] and [42] for an overview of the whole series). In order to make the paper accessible and self-contained, we have not assumed a detailed knowledge of either operator algebras or conformal field theory, including short-cuts and direct proofs wherever possible. This research programme was originally motivated by V. Jones' suggestion that there might be a deeper 'operator algebraic' explanation of the coincidence between certain unitary representations of the infinite braid group that had turned up independently in the theory of subfactors, exactly solvable models in statistical mechanics and conformal field theory (CFT). To understand why there should be any link between these subjects, recall that, amongst other things, the classical 'additive' theory of von Neumann algebras [26] was developed to provide a framework for studying unitary representations of Lie groups. In concrete examples, for example the Plancherel theorem for semisimple groups, this abstract framework had to be complemented by a considerably harder analysis of intertwining operators and associated differential equations. The link between CFT and operator algebras comes from the recently developed 'multiplicative' (quantum?) theory of von Neumann algebras. This theory has three basic sources: firstly the algebraic approach to quantum field theory (QFT) of Doplicher, Haag and Roberts [10]; then in Connes' theory of bimodules and their tensor products of fusion [9]; and lastly in Jones' theory of subfactors [18]. Our work reconciles these ideas with the theory of primary fields, one of the fundamental concepts in CFT. Our work has the following consequences, some of which will be taken up in subsequent papers:

(1) Several new constructions of subfactors.

(2) Non-trival algebraic QFT's in $1+1$ dimensions with finitely many sectors and non-integer statistical (or quantum) dimension ("algebraic CFT").

(3) A definition of quantum invariant theory without using quantum groups at roots of unity.

(4) A computable and manifestly unitary definition of fusion for positive energy representations ("Connes fusion") making them into a tensor category.

(5) Analytic properties of primary fields ("constructive CFT").

To our knowledge, no previous work has suceeded in integrating the theory of primary fields with the ideas of algebraic QFT nor in revealing the very simple analytic structure of primary fields. As we explain below, the main thrust of our work is the explicit computation of Connes fusion of positive energy representations. Finiteness of statistical dimension (or Jones index) is a natural consequence, not a technical mathematical inconvenience. It is perhaps worth emphasising that the theory of operator algebras only provides a framework for studying CFT. As in the case of group representations, it must be complemented by a detailed analysis of certain interwining operators, the primary fields, and their associated differential equations. As we discuss later, however, the operator algebraic point of view can be used to reveal basic positivity and unitarity properties in CFT that have previously seem to have been overlooked.

Novel features of our treatment are the construction of representations and primary fields from fermions. This makes unitarity of the representations and boundedness properties of smeared vector primary fields obvious. The only formal "vertex algebra" aspects of the theory of primary fields borrowed from [39] are the trivial proof of uniqueness and the statement of the Knizhnik-Zamolodchikov equation; our short derivation of the KZ equation circumvents the well-known contour integral proof implicit but not given in [39]. The proof that the axioms of algebraic QFT are satisfied in the non-vacuum sectors is new and relies heavily on our fermionic construction; the easier properties in the vacuum sector have been known for some time [7, 15]. The treatment of braiding relations for smeared primary fields is new but inspired by the Bargmann-Hall-Wightman theorem [20, 36]. To our knowledge, the application of Connes fusion to a non-trivial model in QFT is quite new. Our definition is a slightly simplified version of Connes' original definition, tailor-made for CFT because of the "four-point function formula"; no general theory is required.

The finite-dimensional irreducible unitary representations of $SU(N)$ and their tensor product rules are well known to mathematicians and physicists. The representations V_f are classified by signatures or Young diagrams $f_1 \geq f_2 \geq \cdots \geq f_N$ and, if $V_{[k]} = \lambda^k \mathbf{C}^N$, we have the tensor product rule $V_f \otimes V_{[k]} = \bigoplus_{g >_k f} V_g$, where g ranges over all diagrams that can be obtained by adding k boxes to f with no two in the same row. For the infinite-dimensional loop group $LSU(N) = C^\infty(S^1, SU(N))$, the appropriate unitary representations to consider in place of finite-dimensional representations are the projective unitary representations of positive energy. Positive energy representations form one of the most important foundation stones of conformal field theory [5, 12, 23]. The classification of positive energy representations is straightforward and has been known for some time now. A positive energy representation H_f is classified by its level ℓ, a positive integer, and its signature f, which must satisfy the permissibility condition $f_1 - f_N \leq \ell$. Extending the tensor product rules to representations of a fixed level, however, presents a problem. It is already extremely difficult just giving a coherent definition of the tensor product, since the naive one fails hopelessly because it does not preserve the level. On the other hand physicists have known for years how to 'fuse' representations in terms of short range expansions of products of associated quantum fields (primary fields). We provide one solution to this 'problem of fusion' in conformal field theory by giving a mathematically sound definition of the tensor product that ties up with the intuitive picture of physicists. Our solution relates positive energy representations of loop groups to bimodules over von Neumann algebras. Connes defined a tensor product operation on such bimodules – "Connes fusion" – which translates directly into a definition of fusion for positive energy representations. The general fusion rules follow from the particular rules $H_f \boxtimes H_{[k]} = \bigoplus_{g >_k f} H_g$, where g must now also be permissible. In this way the level ℓ representations of $LSU(N)$ exhibit a structure similar to that of the irreducible representations of a finite group. There are several other approaches to fusion of positive energy representations, notably those of Segal [35] and Kazhdan & Lusztig [22]. Our picture seems to be a unitary boundary value of Segal's holomorphic proposal for fusion, based on a disc with two smaller discs removed. When the discs shrink to points on the Riemann sphere, Segal's definition should degenerate to the algebraic geometric fusion of Kazhdan & Lusztig. We now give an informal summary of the paper.

Fermions. Let Cliff(H) be the Clifford algebra of a complex Hilbert space H, generated by a linear map $f \mapsto a(f)$ $(f \in H)$ satisfying $a(f)a(g) + a(g)a(f) = 0$ and $a(f)a(g)^* + a(g)^*a(f) = (f, g)$. It acts irreducibly on Fock space ΛH via $a(f)\omega = f \wedge \omega$. Other representations of Cliff(H) arise by considering the real linear map $c(f) = a(f) + a(f)^*$ which satisfies

$c(f)c(g)+c(g)c(f)=2\mathrm{Re}(f,g)$; note that $a(f)=\frac{1}{2}(c(f)-ic(if))$. Since c relies only on the underlying real Hilbert space $H_{\mathbb{R}}$, complex structures on $H_{\mathbb{R}}$ commuting with i give new irreducible representations of $\mathrm{Cliff}(H)$. The structures correspond to projections P with multiplication by i given by i on PH and $-i$ on $(PH)^{\perp}$. The corresponding representation π_P is given by $\pi_P(a(f))=\frac{1}{2}(c(f)-ic(i(2P-I)f))$. Using ideas that go back to Dirac and von Neumann, we give our own short proof of I. Segal's equivalence criterion: if $P-Q$ is a Hilbert-Schmidt operator, then π_P and π_Q are unitarily equivalent. On the other hand if $u\in U(H)$, then $a(uf)$ and $a(ug)$ also satisfy the complex Clifford algebra relations. Thus $a(f)\to a(uf)$ gives an automorphism of $\mathrm{Cliff}(H)$. We say that this "Bogoliubov" automorphism is implemented in π_P iff $\pi_P(a(uf))=U\pi_P(a(f))U^*$ for some unitary U. This gives a projective representation of the subgroup of implementable unitaries $U_P(H)$. Segal's equivalence criterion leads immediately to a quantisation criterion: if $[u,P]$ is a Hilbert-Schmidt operator, then $u\in U_P(H)$.

Positive energy representations. Let $G=SU(N)$ and let $LG=C^\infty(S^1,G)$ be the loop group, with the rotation group $\mathrm{Rot}\ S^1$ acting as automorphisms. If $H=L^2(S^1,\mathbb{C}^N)$ and P is the projection onto Hardy space $H^2(S^1,\mathbb{C}^N)$, $LSU(N)\rtimes\mathrm{Rot}\ S^1\subset U_P(H)$ so we get a projective representation $\pi_P^{\otimes\ell}: LU(N)\rtimes\mathrm{Rot}\ S^1\to PU(\mathscr{F}^{\otimes\ell})$ where $\mathscr{F}$ denotes Fock space ΛH_P. Now $\mathrm{Rot}\ S^1$ acts with positive energy, where an action U_θ on H is said to have positive energy if $H=\bigoplus_{n\geq 0}H(n)$ with $U_\theta\xi=e^{in\theta}\xi$ for $\xi\in H(n)$, $H(n)$ is finite-dimensional and $H(0)\neq(0)$. This implies that $\mathscr{F}^{\otimes\ell}$ splits as a direct sum of irreducibles H_i, called the level ℓ positive energy representations. The H_i's are classified by their lowest energy subspaces $H_i(0)$, which are irreducible modules for the constant loops $SU(N)$. Their signatures $f_1\geq f_2\geq\cdots\geq f_N$ must satisfy $f_1-f_N\leq\ell$, so $\mathscr{F}_V^{\otimes\ell}$ has only finitely many inequivalent irreducible summands. This classification is achieved by defining an infinitesimal action of the algebraic Lie algebra $L^0\mathfrak{g}$ on the finite energy subspace $H^0=\sum H(n)$ using bilinear terms $a(f)a(g)^*$. Our main contribution here is to match up these operators with the skew-adjoint operators predicted by analysis. The quantisation criterion also implies that the Möbius transformations of determinant 1 act projectively on each positive energy representation compatibly with LG. The vacuum representation H_0 corresponds to the trivial representation of G; the Möbius transformations of determinant -1 also act on H_0, but this time by conjugate-linear isometries. This presentation of the theory of positive energy representations is adequate for the needs of this paper; in [42] we show from scratch that any irreducible positive energy representation of $LSU(N)\rtimes\mathrm{Rot}\ S^1$ arises as a subrepresentation of some $\mathscr{F}_V^{\otimes\ell}$.

von Neumann algebras. We briefly summarise those parts of the general theory of operator algebras that are background for this paper. (They will serve only as motivation, since all the advanced results we need will be proved directly for local fermion or loop group algebras.) A *von Neumann* *algebra* is simply the commutant $\mathscr{S}' = \{T \in B(H) : Tx = xT \text{ for all } x \in \mathscr{S}\}$ of a subset $\mathscr{S}$ of $B(H)$ with $\mathscr{S}^* = \mathscr{S}$. Typically $\mathscr{S}$ will be a *-subalgebra of $B(H)$ or a subgroup of $U(H)$; the von Neumann algebra generated by $\mathscr{S}$ is then just $\mathscr{S}''$. A von Neumann algebra M is called a *factor* if its centre contains only scalar operators. *Modules* over a factor were classified by Murray and von Neumann [26] using a *dimension function*, the range of values giving an invariant of the factor: the non-negative integers (type I), the non-negative reals (type II) and $\{0, \infty\}$ (type III). Further structure comes from the modular operators Δ^{it} and J of Tomita-Takesaki [8]: if Ω is a cyclic vector for M and M' and $S = J\Delta^{1/2}$ is the polar decomposition of the map $S : M\Omega \to M\Omega, a\Omega \mapsto a^*\Omega$, then $JMJ = M'$ and $\Delta^{it}M\Delta^{-it} = M$. On the one hand the operators Δ^{it} provide a further invariant of type III factors, the Connes spectrum $\bigcap_\Omega \operatorname{Spec} \Delta_\Omega^{it}$, a closed subgroup of $\mathbb{R}$ [9]; see also [42]; while on the other hand J makes the underlying Hilbert space H_0 into a bimodule over M, the *vacuum bimodule*, with the action of the opposite algebra M^{op} given by $a \mapsto Ja^*J$. Bimodules are closely related to subfactors and endomorphisms: a bimodule defines a subfactor by the inclusion $M^{\mathrm{op}} \subset M'$; and an endomorphism $\rho : M \to M$ can be used to define a new bimodule structure on H_0. *Connes fusion* [9] gives an associative tensor product operation on bimodules that generalises composition of endomorphisms: given bimodules X and Y, their fusion $X \boxtimes Y$ is the completion of $\mathrm{Hom}_{M^{\mathrm{op}}}(H_0, X) \otimes \mathrm{Hom}_M(H_0, Y)$ with respect to the pre-inner product $\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle = (x_2^* x_1 y_2^* y_1 \Omega, \Omega)$. Roughly speaking Jones, Ocneanu and Popa [18, 19, 29, 42] proved that an *irreducible bimodule* is classified by the *tensor category* it generates under fusion, provided the category contains only finitely many isomorphism classes of irreducible bimodules.

Modular theory for fermions. For fermions and bosons, modular theory provides the most convenient framework for proving the much older result in algebraic quantum field theory known as "Haag-Araki duality". This deals with the symmetry between observables in a region and its (space-like) complement. As in [24], we consider more generally a *modular subspace* K of a complex Hilbert space H, i.e. a closed real subspace such that $K \cap iK = (0)$ and $K + iK$ is dense in H. (Thus $K = \overline{M_{\mathrm{sa}}\Omega}$ in Tomita-Takesaki theory.) If $S = J\Delta^{1/2}$ is the polar decomposition of the map $S : K + iK \to K + iK, \xi + i\eta \mapsto \xi - i\eta$, then $JK = iK^\perp$ and $\Delta^{it}K = K$; in the text following [33] we avoid unbounded operators by taking the equivalent definitions $J = \mathrm{phase}(E - F)$ and $\Delta^{it} = (2I - E - F)^{it}(E + F)^{it}$, where E and F are the projections onto K and iK. The modular operators J and Δ^{it} are uniquely characterised by the Kubo-Martin-Schwinger (KMS) condition: commuting operators J and Δ^{it} give the modular operators if $\Delta^{it}K = K$ and, for each $\xi \in K$, $f(t) = \Delta^{it}\xi$ extends to a bounded continuous function on the strip $-\frac{1}{2} \leq \operatorname{Im} z \leq 0$, holomorphic in the interior, with $f(t - i/2) = Jf(t)$.

This theory can be used to prove an abstract result, implicit in the work of Araki [1, 2]. Let K be a modular subspace of H and let $M(K)$ be the von

Neumann algebra on ΛH generated by the operators $c(\xi)$ for $\xi \in H$. Then $M(K^\perp)$ is the graded commutant of $M(K)$ ("Araki duality") and the modular operators for $M(K)$ on ΛH come from the quantisations of the corresponding operators for K. This reduces computations to "one-particle states", i.e. the prequantised Hilbert space. We then perform the prequantised computation explicitly when $H = L^2(S^1, V)$ and $K = L^2(I, V)$, with I a proper subinterval of S^1 with complement I^c. We deduce that if $M(I)$ is the von Neumann algebra on ΛH_P generated by $a(f)$'s with $f \in L^2(I, V)$, then $M(I^c)$ is the graded commutant of $M(I)$ (Haag-Araki duality) Δ^{it} and J come from the Möbius flow and flip fixing the end points of I.

Local loop groups. Let $L_I SU(N)$ be the subgroup of $LSU(N)$ consisting of loops equal to 1 off I. The von Neumann algebra $N(I)$ generated by $L_I G$ is a subalgebra of the local fermion algebra $M(I)$ invariant under conjugation by the modular group Δ^{it}, since it is geometric. The modular operators of $N(I)$ can therefore be read off from those of $M(I)$ by a result in [37] ("Takesaki devissage"); we give our own short proof of a slightly modified version of Takesaki's result. We deduce the following properties of the local subgroups, predicted by the Doplicher-Haag-Roberts axioms [10]. The use of devissage, relating different models, is new and seems unavoidable in proving factoriality and local equivalence.

1. *Locality* In any positive energy representation $L_I SU(N)$ and $L_{I^c} SU(N)$ commute.
2. *Factoriality*. $\pi_i(L_I SU(N))''$ is a factor if (π_i, H_i) is an irreducible positive energy representation.
3. *Local equivalence*. There is a unique *-isomorphism $\pi_i : \pi_0(L_I G)'' \to \pi_i(L_I G)''$ sending $\pi_0(g)$ to $\pi_i(g)$ for $g \in L_I G$ such that $Ta = \pi_i(a)T$ for all $T \in \mathrm{Hom}_{L_I G}(H_0, H_i)$.
4. *Haag duality*. If π_0 is the vacuum representation at level ℓ, then $\pi_0(L_I SU(N))'' = \pi_0(L_{I^c} SU(N))'$.
5. *Irreducibility*. Let A be a finite subset of S^1 and let $L^A SU(N)$ be the subgroup of $LSU(N)$ of loops trivial to all orders at points of A. If π is positive energy, then $\pi(L^A SU(N))' = \pi(LSU(N))'$, so the irreducible positive energy representations of $LSU(N)$ stay irreducible and inequivalent when restricted to $L^A SU(N)$.

Vector primary fields. Let P_i and P_j be projections onto the irreducible summands H_i and H_j of $\pi_P^{\otimes \ell}$ and fix an $SU(N)$-equivariant embedding of $\mathbb{C}^N$ in $\mathbb{C}^N \otimes \mathbb{C}^\ell$. If $f \in L^2(S^1, \mathbb{C}^N) \subset L^2(S^1, \mathbb{C}^N \otimes \mathbb{C}^\ell)$, we may "compress" the smeared fermion field $a(f)$ to get an operator $\phi_{ij}(f) = P_i a(f) P_j \in \mathrm{Hom}(H_j, H_i)$. By construction $\phi_{ij}(f)$ satisfies a group covariance relation $g\phi(f)g^{-1} = \phi(g \cdot f)$ for $g \in LSU(N) \rtimes \mathrm{Rot}\, S^1$ as well as the L^2 bound $\|\phi(f)\| \leq \|f\|_2$. If f is supported in I^c, then $\phi(f)$ gives a concrete element in $\mathrm{Hom}_{L_I SU(N)}(H_j, H_i)$; this space of intertwiners is known to be non-zero by local equivalence. Clearly ϕ defines a map $L^2(S^1, \mathbb{C}^N) \otimes H_j \to H_i$ which intertwines the action of $LSU(N) \rtimes \mathrm{Rot}\, S^1$. The modes $\phi(v, n) = \phi(z^{-n} v)$

satisfy Lie algebra covariance relations $[D, \phi(v,n)] = -\phi(v,n)$, $[X(m), \phi(v,n)] = \phi(Xv, n+m)$. Exactly as in [39], the field ϕ is uniquely determined by these relations and its initial term $\phi(v,0)$ in $\mathrm{Hom}_G(V_j \otimes V, V_i)$. Our main new result is that all vector primary fields arise by compressing fermions and therefore satisfy the L^2 bound above.

Braiding relations. If f and g have disjoint supports in S^1, then $a(f)a(g) = -a(g)a(f)$ and $a(f)a(g)^* = -a(g)^*a(f)$. Similar but more complex "braiding relations" hold for vector primary fields and their adjoints. These may be summarised as follows. Let $a, b \in L^2(S^1, \mathbb{C}^N)$ be supported in intervals I and J in $S^1\backslash\{1\}$ with J anticlockwise after I. Define $a_{gf} = \phi_{gf}^{\square}(e_{-\alpha}a)$ and $b_{gf} = \phi_{gf}^{\square}(e_{-\alpha}b)$, with $\alpha = (\Delta_g - \Delta_f - \Delta_{\square})/2(N+\ell)$ and $e_\alpha(e^{i\theta}) = e^{i\alpha\theta}$. Then

$$b_{gf}a_{fh} = \sum \mu_{f_1} a_{gf_1} b_{f_1 h}, \quad b_{gf}a_{g_1 f}^* = \sum \nu_h a_{hg}^* b_{hg_1},$$

with all coefficients non-zero. The proof of these relations is similar to that of the Bargmann-Hall-Wightman theorem [11, 20, 36]. To prove the first for example let $F_k(z) = \sum(\phi_{ik}(u,n)\phi_{kj}(v,-n)v_j, v_i)z^n$, a power series convergent for $|z| < 1$ with values in $W = \mathrm{Hom}_{SU(N)}(V_j \otimes U \otimes V, V_i)$. To prove the braiding relation, it suffices to show that F_k extends continuously to $S^1\backslash\{1\}$ and $F_k(e^{i\theta}) = \sum c_{kh} e^{i\mu_{kh}\theta} F_h(e^{-i\theta})$ there. Using Sugawara's formula for D, we show directly that the F_k's satisfy the Knizhnik-Zamolodchikov ODE [23]

$$\frac{dF}{dz} = \frac{PF}{z} + \frac{QF}{1-z},$$

where $P, Q \in \mathrm{End}(W)$ (the original proof in [23], referred to in [39], is different and less elementary). In all cases we need, the matrix P has distinct eigenvalues, none of which differ by positive integers, and Q is a non-zero multiple of a rank one idempotent in general position with respect to P. For two vector primary fields this ODE reduces to the classical hypergeometric ODE and the required relation on $S^1\backslash\{1\}$ follows from Gauss' formula for transporting solutions at 0 to ∞. In general the ODE can be related to the generalised hypergeometric ODE for which the corresponding transport relations were first obtained by Thomae [38] in 1867 in terms of products of gamma functions. Such a link exists because there is a basis of W for which P and $P - Q$ are both in rational canonical form. In this basis, the ODE is just the matrix form of the generalised hypergeometric ODE.

Transport formulas. The operator $a_{\square 0}^* a_{\square 0}$ on H_0 commutes with $L_{I^c}SU(N)$, so lies in $\pi_0(L_I SU(N))''$. Therefore, by local equivalence, we have the right to consider its image under π_f. We obtain the fundamental "transport formula": $\pi_f(a_{\square 0}^* a_{\square 0}) = \sum \lambda_g a_{gf}^* a_{gf}$, with $\lambda_g > 0$. Thus for $T \in \mathrm{Hom}_{L_I G}(H_0, H_f)$, we have

$$T a_{\square 0}^* a_{\square 0} = \sum \lambda_g a_{gf}^* a_{gf} T.$$

We will prove the transport formula by induction using the braiding relations; the original proof in [43] used the transport relations between 0 and 1 of the basic ODE above.

Definition of Connes fusion. We develop the ideas of fusion directly at the level of loop groups without appeal to the general theory of bimodules over von Neumann algebras [9, 34, 42, 43]. Let X, Y be positive energy representations of $LSU(N)$ at level ℓ. Let $\mathscr{X} = \mathrm{Hom}_{L_{I^c}SU(N)}(H_0, X)$ and $\mathscr{Y} = \mathrm{Hom}_{L_I SU(N)}(H_0, Y)$. These spaces of bounded intertwiners or fields replace vectors or states in X and Y. Thus $x \in \mathscr{X}$ "creates" the state $x\Omega$ from the vacuum Ω. The fusion $X \boxtimes Y$ is defined to be the completion of $\mathscr{X} \otimes \mathscr{Y}$ with respect to the pre-inner product $\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle = (x_2^* x_1 y_2^* y_1 \Omega, \Omega)$, a four-point function. $X \boxtimes Y$ admits a natural action of $L_I SU(N) \times L_{I^c} SU(N)$. The map $\mathscr{X} \otimes \mathscr{Y} \to X \boxtimes Y$ extends to continuous maps $X \otimes \mathscr{Y} \to X \boxtimes Y$ and $\mathscr{X} \otimes Y \to X \boxtimes Y$. This implies that if $\mathscr{X}_0 \subset \mathscr{X}$ and $\overline{\mathscr{X}_0 \Omega} = X$, then $\mathscr{X}_0 \otimes \mathscr{Y}$ has dense image in $X \boxtimes Y$. Fusion is associative and $X \boxtimes H_0 = X = H_0 \boxtimes X$.

Explicit computation of fusion. We use the transport formula to prove the fusion rule $H_\square \boxtimes H_f = \bigoplus H_g$ where g ranges over permissible signatures obtained by adding a box to f. The transport formula is still true if a_{gf} is replaced by linear combinations x_{gf} of intertwiners $\pi_g(h) a_{gf}$ with $h \in L_I G$. But then for $y \in \mathrm{Hom}_{L_I SU(N)}(H_0, H_f)$ we have $(x^* x y^* y \Omega, \Omega) = (y^* \pi_f(x^* x) y\Omega, \Omega) = \sum \lambda_g \|x_{gf} y \Omega\|^2$. Thus $U(x \otimes y) = \bigoplus \lambda_g^{1/2} x_{gf} y \Omega$ gives the required unitary intertwiner from $H_\square \boxtimes H_f$ onto $\bigoplus H_g$. Similar reasoning can be used to prove that $H_f \boxtimes H_{[k]} \leq \bigoplus_{g >_k f} H_g$, where g runs over all permissible signatures that can be obtained by adding k boxes to f with no two boxes in the same row. This time a transport formula must be proved with $a_{\square 0}$ replaced by a path $a_{k,k-1} a_{k-1,k-2} \cdots a_{\square 0}$ indexed by exterior powers. This device of considering products of vector primary fields means that we can avoid the use of smeared primary fields corresponding to the exterior powers $\lambda^k \mathbb{C}^N$ which need not be bounded [43].

The fusion ring. It follows immediately from the fusion rule with $H_\square$ that the H_f's are closed under fusion. Moreover, if R denotes the operator corresponding to rotation by 180°, then the formula $B(x \otimes y) = R^*[RyR^* \otimes RxR^*]$ gives a unitary intertwining $X \boxtimes Y$ and $Y \boxtimes X$; this is a less refined form of the braiding operation that makes the level ℓ representations into a braided tensor category [44]. Thus the representation ring $\mathscr{R}$ of formal sums $\sum m_f H_f$ becomes a commutative ring. For each permitted signature h, let $z_h \in SU(N)$ be the diagonal matrix with entries $\exp(2\pi i (h_k + N - k - H)/(N + \ell))$ where $H = (\sum h_k + N - k)/N$; these give a subset $\mathscr{T}$. Let $\mathscr{S} \subset \mathbb{C}^{\mathscr{T}}$ be the image of $R(SU(N))$ under the map of restriction of characters. Our main result asserts that the natural $\mathbb{Z}$-module isomorphism ch $: \mathscr{R} \to \mathscr{S}$ defined by $[H_f] \mapsto [V_f]$ is a ring isomorphism. This completely determines the fusion rules. They agree with the well-known "Verlinde formulas" [40, 21], in which the usual tensor product rules for $SU(N)$ are modified by an action of the affine Weyl group.

Discussion. Many of the early versions of the results in Chapter II were worked out in discussions with Jones in 1989–1990 (see [19] and [42]). We were mainly interested in the inclusion $\pi_i(L_I G)'' \subseteq \pi_i(L_{I^c} G)'$ defined by the "failure of Haag duality". Algebraic quantum field theory [15] provided a series of predictions about these local loop group algebras which we interpreted (in the language of [30]) and verified. In particular two of the main theorems, Haag-Araki duality and loop group irreducibility, were originally obtained with Jones. In the case of geometric modular theory for fermions on S^1, each of us came up with different proofs which appear in simplified form here (see also [42]). The original proofs of irreducibility have been superseded by the simpler and more widely applicable method described above. One of our original proofs followed from the stronger result that $L^A G$ is dense in LG in the natural topology on $U_P(H)$, so that $\pi(L^A G)$ is strong operator dense in $\pi(LG)$ for any positive energy representation; the analogous result fails for Diff S^1 and its discrete series representations. The geometric method of descent from local fermion algebras to local loop group algebras and its application to Haag duality and local equivalence were first suggested by me, but it was Jones who pointed out that this approach tacitly assumed Takesaki's result [37] ("Takesaki devissage").

The first paper of this series [42] is an expanded version of expository lectures given in the Borel seminar in Bern in 1994. Since it was intended as an introduction to the general theory, we included a complete treatment of the whole theory of fusion, braiding and subfactors for the important special case of $LSU(2)$. In the second paper of the series [43] we made a detailed study of primary fields from several points of view. (See Jones' Séminaire Bourbaki talk [48] for a detailed summary.) We constructed all primary fields as compressions of tensor products of fermionic operators, thus establishing their analytic properties. To do so, we had to complete and extend the Lie algebraic approach of Tsuchiya and Kanie [39] and in particular prove the conjectured four-point property of physicists. Fusion of positive energy representations was computed using the braiding properties of primary fields. The braiding coefficients appeared as transport coefficients between different singular points of the basic ODE studied here; these coefficients were derived using Karamata's Tauberian theorem and a unitary trick. Since the smeared primary fields could be unbounded, their action had to be controlled by Sobolev norms; and a detailed argument had to be supplied for extending the braiding relations to arbitrary bounded intertwiners.

In this paper we give a more elementary approach to fusion using only vector primary fields and their adjoints. It is not possible to overemphasise the central rôle (prophesied by Connes) played by the fermionic model in our work, nor the importance of considering the relationships between different models (stressed by P. Goddard). The boundedness of the corresponding smeared fields is very significant. Not only does it simplify the analysis, but more importantly it can be seen to lie at the heart of the crucial irreducibility result (due to the duality between smeared primary fields and loop group observables). This is an example of Goddard's philosophy that "vertex operators tell you what to do." With the important exception of the Lie algebra operators (indispensable for proving the KZ equation), we have tried to keep exclusively to bounded operators. This is in line with Rudolf Haag's philosophy that quantum field theory can and should be understood in terms of (algebras of) bounded operators [15]. Here, because of the boundedness of vector primary fields, there is no choice.

In the fourth paper of this series [44] we explain how the positive energy representations at a fixed level become a braided tensor category. We have already seen a simplified version of the braiding operation when proving that Connes fusion is commutative. The key to understanding this braiding structure lies in the "monodromy" action of the braid group on products of vector primary fields. The important feature of braiding allows us to make contact with the subfactors of the hyperfinite type II_1 factor defined by Jones and Wenzl [18, 19, 45] using special traces on the infinite braid group. In particular this explains the coincidence between the monodromy representation of the braid group in [39] and the Hecke algebra representations of Jones and Wenzl. Further developments include understanding the "modularity" of the category, the property which allows 3-manifold invariants to be defined. This involves studying the elliptic KZ equations as well as finding and versifying precise versions of the axioms for a CFT; the ideas behind our computation of fusion seem to give a general method for understanding unitarity and positivity properties of quite general CFTs. In addition the analytic properties of primary fields implied by our construction (such as the fact that $q^{L_0}\phi(z)$ is a Hilbert-Schmidt operator for $|q| < 1$) should allow primary fields to be interpreted as morphisms corresponding to 3-holed spheres or trinions in Segal's language. This should yield a precise analytic version of Segal's "modular functor", using the "operator formalism" for trinion decompositions of Riemann surfaces.

The braiding properties of vector primary fields can also be developed through a more systematic use of the *conformal inclusion* $SU(N)\times SU(\ell) \subset SU(N\ell)$. The level one representations and vector primary fields of $SU(N\ell)$, when restricted to $SU(N) \times SU(\ell)$ and decomposed into tensor products, yield all representations and vector primary fields of $SU(N)$ at level ℓ and $SU(\ell)$ at level N. The level one representations of $LSU(N\ell)$ arise by restricting the fermionic representation of $LU(N\ell)$ to $LSU(N\ell) \times LU(1)$ (here $U(1)$ is the centre of $U(N\ell)$). Our fermionic construction of primary fields for $LSU(N)$ in this and the previous paper have been a simplification of the more sophisticated picture provided by the above conformal inclusion, first considered from this point of view by Tsuchiya & Nakanishi [27]. Here we have ignored the rôle of the group $SU(\ell)$. If it is brought into play, it is possible to give a less elementary but more conceptual non-computational proof that all the braiding coefficients are non-zero, based on the Abelian braiding of fermions or vector primary fields at level one. This approach, which will be taken up in detail when we consider subfactors defined by conformal inclusions, has the advantage firstly that it makes the non-vanishing of the coefficients manifest and secondly that it does not require the explicit solutions of the KZ ODE and their monodromy properties that we have used here and in the second paper. It therefore extends to other groups where less information about the KZ ODE is available at present.

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I. Positive energy representations of LSU(N)

2. *Irreducible representations of $SU(N)$*

We give a brief account of the representation theory of $SU(N)$ from a point of view relevant to this paper. This account closely parallels our development of the classification and fusion of positive energy representations of $LSU(N)$, so provides a simple prototype. Let $V = \mathbb{C}^N$ be the vector representation. We shall consider irreducible representations of $SU(N)$ appearing in tensor powers $V^{\otimes m}$. Let $R(SU(N))$ denote the representation ring of $SU(N)$, the ring of formal integer combinations of such irreducible representations. Let $\mathfrak{g}$ be the Lie algebra of $SU(N)$, the traceless skew-adjoint matrices. Thus $\mathfrak{g}$ acts on $V^{\otimes m}$, hence each irreducible representation W, and $\mathrm{End}_G(W) = \mathrm{End}_{\mathfrak{g}}(W)$. This representation of $\mathfrak{g}$ extends linearly to a *-representation of its complexification $\mathfrak{g}_{\mathbb{C}}$, the traceless matrices. $\mathfrak{g}_{\mathbb{C}}$ is spanned by the elementary matrices E_{ij} $(i \neq j)$ and traceless diagonal matrices. Let T denote the subgroup of diagonal matrices $z = (z_1, z_2, \ldots, z_N)$ in $SU(N)$. Given an irreducible representation $SU(N) \to U(W)$, we can write $W = \bigoplus_{g \in \mathbb{Z}^N} W_g$ with $\pi(z)v = z^g v$ for $v \in W_g$, $z \in T$. We call g a *weight* and W_g a *weight space*; g is only determined up to addition of a vector $(a, a, \ldots, a)$ for $a \in \mathbb{Z}$. The monomial matrices in $SU(N)$ permute the weight spaces by permuting the entries of $g = (g_1, \ldots, g_N)$, so there is always a weight with $g_1 \geq g_2 \geq \ldots \geq g_N$. Such a weight is called a *signature*. If the weights are ordered lexicographically, the raising operators $\pi(E_{ij})$ $(i < j)$ carry weight spaces into weight spaces of higher weight; their adjoints $\pi(E_{ij})$ $(i > j)$ are called lowering operators and decrease weight.

Clearly every irreducible representation W contains a highest weight vector v. Now W is irreducible for $\mathfrak{g}_{\mathbb{C}}$ and every monomial A of operators in $\mathfrak{g}_{\mathbb{C}}$ is a sum of products LDR where L is a product of lowering operators, D is a product of diagonal operators and R is a product of raising operators.

Since $LDRv$ is proportional to v or has lower weight, v is unique up to a multiple. On the other hand (A_1v, A_2v) is uniquely determined by the weight of v and the A_i's, since $A_2^*A_1$ can be written as a sum of operators LDR and $(LDRv, v) = (DRv, L^*v)$ with L^* a raising operator. Thus if W' is another irreducible representation with the same highest weight and corresponding vector v', $Av \mapsto Av'$ is a unitary $W \to W'$ intertwining g and hence $G = \exp(\mathrm{g})$. Thus irreducible representations are classified by their signatures. Every signature occurs: if $f_1 \geq f_2 \geq \cdots \geq f_N \geq 0$, the vector $e_f = e_1^{\otimes(f_1-f_2)} \otimes (e_1 \wedge e_2)^{\otimes(f_2-f_3)} \otimes \cdots \otimes (e_1 \wedge e_2 \wedge \cdots \wedge e_N)^{\otimes f_N}$ is the unique highest weight vector in $\lambda^1 V^{\otimes(f_1-f_2)} \otimes \lambda^2 V^{\otimes(f_2-f_3)} \otimes \cdots \otimes \lambda^N V^{\otimes f_N} \subseteq V^{\otimes} \left(\sum f_i\right)$. By uniqueness, e_f generates an irreducible submodule.

A signature f with $f_N \geq 0$ is represented by a Young diagram with at most N rows and f_i boxes in the ith row. Thus V corresponds to the diagram $\square$ and $\lambda^k V$ to the diagram $[k]$ with k rows, with one box in each row. We write $g > f$ if g can be obtained by adding one box to f. More generally we write $g >_k f$ if g can be obtained by adding k boxes to f with no two in the same row.

Lemma. $\mathrm{Hom}_G(V_f \otimes V_{[k]}, V_g)$ *is at most one-dimensional and only non-zero if* $g >_k f$. *When* $k = 1$, *it is non-zero iff* $g > f$. *Hence* $V_f \otimes V_\square = \bigoplus_{g>f} V_g$ *and* $V_f \otimes \lambda^k V \leq \bigoplus_{g>_k f} V_g$.

Proof. Let v_f and v_g be highest weight vectors in V_f and V_g. If $T \in \mathrm{Hom}_G(V_f \otimes V_{[k]}, V_g)$ with $T(v_f \otimes v) = 0$ for all $v \in \lambda^k V$, then applying lowering operators we see that $T = 0$. If $T \neq 0$, we take $w = e_{i_1} \wedge \ldots \wedge e_{i_k}$ of highest weight such that $T(v_f \otimes w) \neq 0$. Applying raising operators, we see that $T(v_f \otimes w)$ is highest weight in V_g so is proportional to v_g. So the weight of $v_f \otimes w$ is a signature and $g >_k f$. If S is another non-zero intertwiner, we may choose α such that $R = S - \alpha T$ satisfies $R(v_f \otimes w) = 0$. If $R \neq 0$, we may choose w' of highest weight such that $R(v_f \otimes w) \neq 0$. But this gives a contradiction, since $R(v_f \otimes w)$ would be annihilated by all raising operators and have weight lower than v_g. So $\mathrm{Hom}_G(V_f \otimes V_{[k]}, V_g)$ is at most one-dimensional.

If g is obtained by adding a box to the ith row of f, then the map

$$\begin{aligned} T : \lambda^1 V^{\otimes(f_1-f_2)} \otimes \lambda^2 V^{\otimes(f_2-f_3)} \otimes \cdots \otimes \lambda^N V^{\otimes f_N} \bigotimes V \\ \to \lambda^1 V^{\otimes(g_1-g_2)} \otimes \lambda^2 V^{\otimes(g_2-g_3)} \otimes \cdots \otimes \lambda^N V^{\otimes g_N} \end{aligned}$$

given by exterior multiplication by V on the $(f_1 - f_i)$th copy of ΛV commutes with G and satisfies $T(e_f \otimes e_i) = e_g$. Thus if P and Q denote the projections onto the submodules generated by e_f and e_g respectively, $QT(P \otimes I)$ gives a non-zero intertwiner $V_f \otimes V \to V_g$.

For $z_i \in \mathbb{C}$ and a signature f, we define the symmetric function $X_f(z) = \det(z_j^{f_i+n-i})/\det(z_j^{n-i})$. The denominator here is a Vandermonde determinant given by $\prod_{i<j}(z_i - z_j)$. If $X_k(z) = \sum_{i_1<\cdots i_k} z_{i_1} \ldots z_{i_k}$, then it is elementary to show that $X_f X_k = \sum_{g>_k f} X_g$ for $k = 1, \ldots, N$. In particular

$X_k(z)$ coincides with $X_{[k]}(z)$; and it follows, by induction on $f_1 - f_N$ and the number of boxes in the f_1th column, that each $X_f(z)$ is an integral polynomial in the $X_k(z)$'s.

Theorem. (1) $V_f \otimes V_{[k]} = \bigoplus_{g>_h f} V_g$.

(2) $R(SU(N))$ *is generated by the exterior powers and the map* $\mathrm{ch} : [V_f] \to X_f$ *gives a ring isomorphism between* $R(SU(N))$ and $\mathscr{S}_N$, *the ring of symmetric integral polynomials in z, where* $\prod z_i = 1$.

(3) (*Weyl's character formula* [*44*]) $\chi_f(z) \equiv \mathrm{Tr}(\pi_f(z)) = X_f(z)$ for all f.

Proof. (1) We know that $V_f \otimes \lambda^k V \leq \bigoplus_{g>_k f} V_g$. We prove by induction on $|f| = \sum f_j$ that $V_f \otimes V_k = \bigoplus_{g_1>_k f} V_{g_1}$. It suffices to show that if this holds for f then it holds for all g with $g > f$. Now, comparing the coefficients of X_h in $(X_f X_k) X_\square = (X_f X_\square) X_k$, we see that $|\{g_1 : h >_k g_1 > f\}| = |\{g_2 : h > g_2 >_k f\}|$. Tensoring by $V_\square$, we deduce that $\oplus_{g>f} V_g \otimes V_{[k]} = \oplus_{g_1>_k f} \oplus_{h>g_1} V_h = \oplus_{g>f} \oplus_{h>_k g} V_h$. Since $V_g \otimes V_{[k]} \leq \bigoplus_{h>_k g} V_h$, we must have equality for all g, completing the induction.
(2) Let ch be the $\mathbb{Z}$-linear isomorphism $\mathrm{ch} : R(SU(N)) \to \mathscr{S}_N$ extending $\mathrm{ch}(V_f) = X_f$. Then by (1), $\mathrm{ch}(V_{[k]} V_f) = X_k X_f$. This implies that ch restricts to a ring homomorphism on the subring of $R(SU(N))$ generated by the exterior powers. On the other hand the X_k's generate $\mathscr{S}_N$, so the image of this subring is the whole of $\mathscr{S}_N$. Since ch is injective, the ring generated by the exterior powers must be the whole of $R(SU(N))$ and ch is thus a ring homomorphism, as required.
(3) The maps $[V_f] \to \chi_f(z)$ and $[V_f] \mapsto X_f(z)$ define ring homomorphisms $R(SU(N)) \to \mathbb{C}$. These coincide on the exterior powers and therefore everywhere.

3. *Fermions and quantisation*

Given a complex Hilbert space H, the complex Clifford algebra Cliff(H) is the unital *-algebra generated by a complex linear map $f \mapsto a(f)$ $(f \in H)$ satisfying the anticommutation relations $a(f)a(g) + a(g)a(f) = 0$ and $a(f)a(g)^* + a(g)^* a(f) = (f, g)$ (complex Clifford algebra relations). The Clifford algebra has a natural action π on ΛH (fermionic Fock space) given by $\pi(a(f))\omega = f \wedge \omega$, called the complex wave representation. The complex wave representation is irreducible. For Ω is the unique vector such that $a(f)^*\Omega = 0$ for all f (this condition is equivalent to orthogonality to $\sum_{k\geq 1} \lambda^k H$) and Ω is cyclic for the $a(f)$'s. Thus if $T \in \mathrm{End}(\Lambda H)$ commutes with all $a(f)^*$'s, $T\Omega = \lambda\Omega$ for $\lambda \in \mathbb{C}$; and if T also commutes with all $a(f)$'s, $T = \lambda I$.

To produce other irreducible representations of Cliff(H), we introduce the operators $c(f) = a(f) + a(f)^*$. Thus c satisfies $c(f) = c(f)^*$, $f \mapsto c(f)$ is real-linear and $c(f)c(g) + c(g)c(f) = 2\mathrm{Re}(f, g)$ (real Clifford algebra relations). The equations $c(f) = a(f) + a(f)^*$ and $a(f) = (c(f) - ic(if))/2$

give a correspondence between complex and real Clifford algebra relations. Since c relies only on the underlying real Hilbert space $H_{\mathbb{R}}$, complex structures on $H_{\mathbb{R}}$ commuting with i give new irreducible representations of $\mathrm{Cliff}(H)$. These complex structures correspond to projections P in H: multiplication by i is given by i on PH and $-i$ on $(PH)^{\perp}$. Unravelling this definition, we find that the projection P defines an irreducible representation π_P of $\mathrm{Cliff}(H)$ on fermionic Fock space $\mathscr{F}_P = \Lambda PH \widehat{\otimes} \Lambda(P^{\perp}H)^*$ given by $\pi_P(a(f)) = a(Pf) + a((P^{\perp}f)^*)^*$. (Equivalently $\pi_p(a(f)) = (c(f) - ic(i(2P-I)f))/2$ on ΛH.)

Theorem (Segal's equivalence criterion [3]). *Two irreducible representations π_P and π_Q are unitarily equivalent if $P - Q$ is a Hilbert-Schmidt operator.*

Remark. The converse also holds [3, 42], but will not be needed.

Proof. If PH (or $P^{\perp}H$) is finite-dimensional, then so is QH (or $Q^{\perp}H$) and the representations are easily seen to be equivalent to the irreducible representation on ΛH (or ΛH^*). So we may assume that $\dim PH = \dim P^{\perp}H = \infty$.

The operator $T = (P-Q)^2$ is compact, so by the spectral theorem $H = \bigoplus_{\lambda \geq 0} H_\lambda$ where $T\xi = \lambda\xi$ for $\xi \in H_\lambda$. Moreover $\dim H_\lambda < \infty$ for $\lambda > 0$ while $P = Q$ on H_0. Now T commutes with P and Q, so that each H_λ is invariant under P and Q. Thus H can be written as a direct sum of finite-dimensional irreducible submodules V_i for P and Q, with $(P-Q)^2$ a scalar λ on each. Since the images of P and Q (and I) should generate $\mathrm{End}(V_i)$, the identity $(P-Q)^2 = \lambda I$ forces $\dim \mathrm{End}(V_i) \leq 4$. Hence $\dim V_i = 1$ or 2.

Pick an orthonormal basis $(e_i)_{i \geq -a}$ of $P^{\perp}H$ with each e_i lying in some V_j. We may assume that $Q^{\perp}e_{-1} = Q^{\perp}e_{-2} = \cdots = Q^{\perp}e_{-a} = 0$ and that $Q^{\perp}e_i \neq 0$ for $i \geq 0$. Complete (e_i) to an orthonormal basis $(e_i)_{i \in \mathbb{Z}}$ by adding remaining vectors from the V_j's. We can also choose an orthonormal basis $(f_j)_{j \geq -b}$ of $Q^{\perp}H$ with f_i lying in the same V_j as e_i if $i \geq 0$; we shall even pick f_i so that $(e_i, f_i) > 0$ in this case. A simple computation shows that if $(P-Q)^2 = \lambda_i I$ on V_j, then $(e_i, f_i) = \sqrt{1-\lambda_i}$ (so $\lambda_i = 0$ when $\dim V_j = 1$). Note that, using these bases, we get $\|P-Q\|_2^2 = \mathrm{Tr}(P-Q)^2 = a + b + 2\sum \lambda_i$, so that $\sum \lambda_i < \infty$.

The "Dirac sea" model $\mathscr{H}$ for ΛH_P is the Hilbert space with orthonormal basis given by all symbols $e_{i_1} \wedge e_{i_2} \wedge e_{i_3} \wedge \cdots$ where $i_1 < i_2 < i_3 < \cdots$ and $i_{k+1} = i_k + 1$ for k sufficiently large. If $A(e_i)$ denotes exterior multiplication by e_i, then $A(e_i)A(e_j) + A(e_j)A(e_i) = 0$ and $A(e_i)A(e_j)^* + A(e_j)^*A(e_i) = \delta_{ij}I$. By linearity and continuity, these extend to operators $A(f)$ ($f \in H$) satisfying the complex Clifford algebra relations so give a representation π of $\mathrm{Cliff}(H)$. Let $\xi = e_{-a} \wedge e_{-a+1} \wedge \cdots$. Then the $A(f)$ and $A(f)^*$'s act cyclically on ξ and $(A(f_1)\ldots A(f_m)\xi, A(g_1)\cdots A(g_n)\xi) = \delta_{mn}\det(Pf_i, g_j)$. On the other hand $(\pi_P(a(f_1)) \ldots \pi_P(a(f_m))\Omega_P, \pi_P(a(g_1)) \ldots \pi_P(a(g_n))\,\Omega_P) = \delta_{mn}$ $\det(Pf_i, g_j)$, where Ω_P is the vacuum vector in ΛH_P. Thus $(\pi(a)\xi, \xi) = (\pi_P(a)\Omega_P, \Omega_P)$ for $a \in \mathrm{Cliff}(H)$. Replacing a by a^*a and recalling that ξ and Ω_P are cyclic, we see that $U(\pi_P(a)\Omega_P) = \pi(a)\xi$ defines a unitary from ΛH_P onto $\mathscr{H}$ such that $\pi(a) = U\pi_P(a)U^*$. The same "Gelfand-Naimark-Segal" argument shows that unitary equivalence of π_P and π_Q will follow as soon as we find $\eta \in \mathscr{H}$ such that $(\pi(a)\eta,\eta) = (\pi_Q(a)\Omega_Q,\Omega_Q)$. (Note that η is automatically cyclic, since $\mathscr{H} \cong \Lambda H_P$ is irreducible.)

Let $\eta_N = f_{-b} \wedge \cdots \wedge f_{-1} \wedge f_0 \wedge \cdots \wedge f_N \wedge e_{N+1} \wedge e_{N+2} \wedge \cdots$. Clearly if a lies in the *-algebra generated by the $a(e_i)$'s, then $(\pi(a)\eta_N,\eta_N) = (\pi_Q(a)\Omega_Q,\Omega_Q)$ for N sufficiently large. Thus it will suffice to show that η_N has a limit η, i.e. (η_N) is a Cauchy sequence. Since $\|\eta_N\| = 1$, this follows if $\mathrm{Re}(\eta_M,\eta_N) \to 1$ as $M \le N \to \infty$. But $(\eta_M,\eta_N) = \prod_{i=M+1}^{N}(e_i,f_i) = \prod_{i=M+1}^{N}\sqrt{1-\lambda_i}$ and, as $\sum \lambda_i < \infty$, this tends to 1 if $M,N \to \infty$, as required.

Corollary of proof. *If π_P and π_Q are unitarily equivalent and Ω_Q is the image of the vacuum vector in $\mathscr{F}_Q$* in *$\mathscr{F}_Q$, then $|(\Omega_P,\Omega_Q)|^2 = \prod(1-\mu_i)$ where μ_i are the eigenvalues of $(P-Q)^2$.*

Proof. We have $|(\Omega_P,\Omega_Q)| = |(\xi,\eta)| = \lim|(\xi,\eta_N)| = \prod(1-\mu_i)^{1/2}$.

Any $u \in U(H)$ gives rise to a Bogoliubov automorphism of $\mathrm{Cliff}(H)$ via $a(f) \mapsto a(uf)$. This automorphism is said to be implemented in π_P (or on $\mathscr{F}_P$) if $\pi_P(a(uf)) = U\pi_P(a(f))U^*$ for some unitary $U \in U(\mathscr{F}_P)$ unique up to a phase. Since $\pi_P(a(uf)) = \pi_Q(a(f))$ with $Q = u^*Pu$, we immediately deduce:

Corollary (Segal's quantisation criterion [3, 30,42]). *u is implemented in $\mathscr{F}_P$ if $[u,P]$ is a Hilbert-Schmidt operator.*

We define the restricted unitary group $U_P(H) = \{u \in U(H)\colon [u, P]$ $\mathit{Hilbert-Schmidt}\}$, a topological group under the strong operator topology combined with the metric $d(u,v) = \|[u-v,P]\|_2$. By the corollary, there is a homomorphism π of $U_P(H)$ into $PU(\mathscr{F}_P)$, called the *basic* projective representation.

Lemma. *The basic representation is continuous.*

Proof. It is enough to show continuity at the identity. Thus if $u_n \xrightarrow{s} I$ and $\|[u_n,P]\|_2 \to 0$, we must find a lift $U_n \in U(\mathscr{F}_P)$ of $\pi(u_n)$ such that $U_n \xrightarrow{s} I$. Now $\|[u_n,P]\|_2 = \|P-Q_n\|_2$ where $Q_n = u_n^*Pu_n$. So $\mathrm{Tr}(P-Q_n)^2 \to 0$. On the other hand $|(\Omega_P,\Omega_{Q_n})|^2 = \prod(1-\mu_i)$ where μ_i are the (non-zero) eigenvalues of $(P-Q_n)^2$. Since $\mathrm{Tr}(P-Q_n)^2 = \sum \mu_i$ and $\prod(1-\mu_i) \ge \exp(-2\sum\mu_i)$ for $\sum\mu_i$ small, it follows that $|(\Omega_P,\Omega_{Q_n})| \to 1$ as $n \to \infty$. If u_n is implemented by U_n in $\mathscr{F}_P$, then $U_n\Omega_P$ and Ω_{Q_n} are equal up to a phase. So $|(U_n\Omega_P,\Omega_P)| \to 1$. Adjusting U_n by a phase, we may assume $(U_n\Omega_P,\Omega_P) > 0$ eventually so that $U_n\Omega_P \to \Omega_P$. Now, taking operator norms, $\|U_n\pi(a(f))U_n^* - \pi(a(f))\| = \|\pi(a(u_nf-f))\| \le \|u_nf-f\|$. It follows that $\|U_naU_n^*-a\| \to 0$ for any $a \in \pi_P(\mathrm{Cliff}\,H)$. Thus $U_na\Omega_P = (U_naU_n^*)\,(U_n\Omega_P) \to a\Omega_P$ as $n \to \infty$. Since vectors $a\Omega_P$ are dense in $\mathscr{F}_P$, we get $U_n \xrightarrow{s} I$, as required.

Note that if $[u,P]=0$, so that u commutes with P, then u is canonically implemented in Fock space $\mathscr{F}_P$ and we may refer to the *canonical* *quantisation* of u. If on the contrary $uPu^* = I - P$, then u is canonically implemented by a conjugate-linear isometry in Fock space, also called the canonical quantisation of u. Thus the canonical quantisations correspond to unitaries that are complex-linear or conjugate-linear for the complex structure defined by P.

4. The fundamental representation

Let $G = SU(N)$ (or $U(N)$) and define the loop group $LG = C^\infty(S^1, G)$, the smooth maps of the circle into G. Let $H = L^2(S^1) \otimes V$ ($V = \mathbb{C}^N$) and let P be the projection onto the Hardy space $H^2(S^1) \otimes V$ of functions with vanishing negative Fourier coefficients (or equivalently boundary values of functions holomorphic in the unit disc). Now LG acts unitarily by multiplication on H. In fact if $f \in C^\infty(S^1, \mathrm{End}\ V)$ and $m(f)$ is the corresponding multiplication operator, then it is easy to check, using the Fourier coefficients of f, that $\|[P, m(f)]\|_2 \leq \|f'\|_2$. In particular LG satisfies Segal's quantisation criterion for P and we therefore get a projective representation of $LU(N)$ on $\mathscr{F}_P$ [30, 42], continuous for the C^∞ topology on $LU(N) \subset C^\infty(S^1, \mathrm{End}\ V)$. The rotation group Rot S^1 acts by automorphisms on LG by $(r_\alpha f)(\theta) = f(\theta + \alpha)$. The same formula defines a unitary action on $L^2(S^1) \otimes V$ which leaves $H^2(S^1) \otimes V$ invariant. Therefore this action of Rot S^1 is canonically quantised and we thus get a projective representation of $LG \rtimes \mathrm{Rot}\ S^1$ on $\mathscr{F}_P$ which restricts to an ordinary representation on Rot S^1.

Let

$$SU_\pm(1,1) = \left\{ \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} : |\alpha|^2 - |\beta|^2 = \pm 1 \right\}$$

and let $SU_+(1,1) = SU(1,1)$ and $SU_-(1,1)$ denote the elements with determinant $+1$ or -1. Thus $SU_-(1,1)$ is a coset of $SU_+(1,1)$ with representative $F = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$, for example. The matrices $g \in SU_\pm(1,1)$ act by fractional linear transformations on S^1, $g(z) = (\alpha z + \beta)/(\overline{\beta} z + \overline{\alpha})$. This leads to a unitary action on $L^2(S^1, V)$ via $(V_g \cdot f)(z) = (\alpha - \overline{\beta} z)^{-1} f(g^{-1}(z))$. Since $(\alpha - \overline{\beta} z)^{-1}$ is holomorphic for $|z| < 1$ and $|\alpha| > |\beta|$, it follows that V_g commutes with the Hardy space projection P for $g \in SU_+(1,1)$. The matrix F acts on $L^2(S^1, V)$ via $(F \cdot f)(z) = z^{-1} f(z^{-1})$ and clearly satisfies $FPF = I - P$. It follows that F is canonically implemented in fermionic Fock space $\mathscr{F}_V$ by a conjugate-linear isometry fixing the vacuum vector. Since $SU_-(1,1) = SU_+(1,1)F$, the same holds for each $g \in SU_-(1,1)$. Thus we get an orthogonal representation of $SU_\pm(1,1)$ for the underlying real inner product on $\mathscr{F}_V$ with $SU_+(1,1)$ preserving the complex structure and $SU_-(1,1)$ reversing it. The same is true in $\mathscr{F}_V^{\otimes \ell}$.

Let U_z denote the canonically quantised action of the gauge group $U(1)$ on $\mathscr{F}_V$, corresponding to multiplication by z on H. The $\mathbb{Z}_2$-grading on $\mathscr{F}_V$ is given by the operator $U = U_{-I}$.

Lemma. $\pi(g)U_z\pi(g)^* = U_z$ *for all* $g \in LSU(N)$ *and* $z \in U(1)$.

Proof. The group $SU(N)$ is simply connected, so the group $LSU(N)$ is connected (any path can be smoothly contracted to a constant path and $SU(N)$ is connected). The map $U(H) \times U(H) \to U(H)$, $(u, v) \mapsto uvu^*v^*$ is continuous and descends to $PU(H) \times PU(H)$. So $(u, v) \mapsto uvu^*v^*$ defines a continuous map $PU(H) \times PU(H) \to U(H)$. Since g and z commute on the prequantised space H, $\pi(g)$ and U_z commute in $PU(H)$. Hence $\pi(g)U_z\pi(g)^*U_z^* = \lambda(g, z)$ where $\lambda(g, z) \in \mathbb{T}$ depends continuous on g and z. Writing this equation as $\pi(g)U_z\pi(g)^* = \lambda(g, z)U_z$, we see that $\lambda(g, \cdot)$ defines a character λ_g of $U(1)$. Clearly $\lambda_g\lambda_h = \lambda_{gh}$, so we get a continuous homomorphism of $LSU(N)$ into $\widehat{U(1)}$, the group of characters of $U(1)$. Since $\widehat{U(1)} = \mathbb{Z}$ and $LSU(N)$ is connected, $\lambda_g = 1$ for all g. So $\lambda(g, z) = 1$ for all g, z as required.

Corollary. *Each operator* $\pi(g)$ *with* $g \in LSU(N)$ *is even* (*it commutes with* $U = U_{-1}$).

5. *The central extension* $\mathscr{L}G$

We introduce the central extension of LG

$$1 \to \mathbb{T} \to \mathscr{L}G \to LG \to 1$$

obtained by pulling back the central extension $1 \to \mathbb{T} \to U(\mathscr{F}_V) \to PU(\mathscr{F}_V) \to 1$ under the map $\pi : LG \to PU(\mathscr{F}_V)$. In other words it is the closed subgroup of $LG \times U(\mathscr{F}_V)$ given by $\{(g, u) : \pi(g) = [u]\}$: it contains $\mathbb{T} = 1 \times \mathbb{T}$ as a central subgroup and has quotient LG. By definition $\mathscr{L}G$ has a unique unitary representation π on $\mathscr{F}_V$ given by $\pi(g, u) = u$. This extension is compatible with the action of $SU_\pm(1, 1)$ and Rot S^1.

Lemma. *If* $\pi(\gamma)$ *denotes the canonical quantisation of* $\gamma \in SU_\pm(1, 1)$*on fermionic Fock space* $\mathscr{F}_V$ *and* $\mathscr{L}G = \{(g, u) : \pi(g) = [u]\}$, *then the operators* $(\gamma, \pi(\gamma))$ *normalise* $\pi(\mathscr{L}G)$ *acting on the centre* $\mathbb{T}$ *as the identity if* $\gamma \in SU_+(1, 1)$ *and as complex conjugation if* $\gamma \in SU_-(1, 1)$.

Proof. This follows because $\pi(\gamma)\pi(g)\pi(\gamma)^{-1}$ has the same image as $\pi(g \cdot \gamma^{-1})$ in $PU(\mathscr{F}_V)$.

6. *Positive energy representations*

We may consider the decomposition of $\mathscr{F}_P = \Lambda(PH) \otimes \Lambda(P^\perp H)^*$ into weight spaces of Rot $S^1 = \mathbb{T}$, writing $\mathscr{F}_P = \bigoplus_{n\geq 0} \mathscr{F}_P(n)$, where $z \in \mathbb{T}$ acts on $\mathscr{F}_P(n)$ as multiplication by z^n. Since Rot S^1 acts with finite multiplicity and only non-negative weight spaces on PH and $(P^\perp H)^*$, it is easy to see that $\mathscr{F}_P(n)$ is finite-dimensional for $n \geq 0$ and $\mathscr{F}_P(n) = (0)$ for $n < 0$. Moreover

$\mathscr{F}_P(0) = \Lambda(V)$. We define a representation of $\mathbb{T}$ on H to have *positive energy* if in the decomposition $H = \bigoplus H(n)$ we have $H(n) = 0$ for $n < 0$ and $H(n)$ finite-dimensional for $n \geq 0$. (Usually we will also insist on the normalisation $H(0) \neq (0)$, which can always be achieved through tensoring by a character of $\mathbb{T}$.) Thus Rot S^1 acts on $\mathscr{F}_V$ with positive energy.

Proposition. *Suppose that* Γ *is a subgroup of* $U(H)$ *and that* $\mathbb{T}$ *acts on* H *with positive energy normalising* Γ. *Let* U_t *be the action* (*with* $t \in [0, 2\pi]$).

(a) *If* H *is irreducible as an* $\Gamma \rtimes \mathbb{T}$*-module, then it is irreducible as a* Γ*-module.*

(b) *If* H_1 *and* H_2 *are irreducible* $\Gamma \rtimes \mathbb{T}$*-modules which are isomorphic as* Γ*-modules, then one is obtained from the other by tensoring with a character of* $\mathbb{T}$.

(c) *If* H *is the cyclic* Γ*-module generated by a lowest energy vector, it contains an irreducible* $\Gamma \rtimes \mathbb{T}$*-module generated by some lowest energy vector.*

(d) *Any positive energy representation is a direct sum of irreducibles.*

Proof. (a) Let $M = \Gamma'$, the commutant of Γ, so that $M = \{T : Tg = gT$ for all $g \in \Gamma\}$. By Schur's lemma, $M \cap \langle U_t \rangle' = \mathbb{C}I$ since Γ and $\mathbb{T}$ act irreducibly. Note that U_t normalises M, since it normalises Γ. Let v be a lowest energy vector in H. v is cyclic for Γ and $\mathbb{T}$ and hence Γ, so $av \neq 0$ for $a \neq 0$ in M. If $M \neq \mathbb{C}$, there is a non-scalar self-adjoint element $T \in M$. Define $T_n \in B(H)$ by $(T_n\xi, \eta) = (2\pi)^{-1} \int_0^{2\pi} e^{-int}(U_t T_n U_t^* \xi, \eta)dt$. Then $T_n \in M$, $U_t T U_t^* = e^{int} T_n$, $T_n^* = T_{-n}$ and $Tv = \oplus T_n v$. By assumption T_0 must be a scalar. Since $T \notin \mathbb{C}I$, Tv cannot be a multiple of v and therefore $T_n \neq 0$ for some $n \neq 0$. Since $T_n^* = T_{-n}$, we may assume $n < 0$. But then $T_n v \neq 0$ gives a vector of lower energy than v. So $M = \mathbb{C}$ and Γ acts irreducibly.
(b) Let $T : H_1 \to H_2$ be a unitary intertwiner for Γ. Then $V_t^* T U_t$ is also a unitary intertwiner, so must be of the form $\lambda(t)T$ for $\lambda(t) \in \mathbb{T}$ by Schur's lemma. Since $TU_tT^* = \lambda(t)V_t$, $\lambda(t)$ must be a character of $\mathbb{T}$.
(c) Let V be the subspace of lowest energy. Let K be any $\Gamma \rtimes \mathbb{T}$-invariant subspace of H with corresponding projection $p \in \Gamma'$. Since $H = \overline{\mathrm{lin}}(\Gamma V)$, $K = pH = \overline{\mathrm{lin}}(\Gamma p V)$. But $pV \subseteq V$, since p commutes with $\mathbb{T}$. Choosing pV in V of smallest dimension, we see that $K = \overline{\mathrm{lin}}(\Gamma p V)$ must be irreducible as a $\Gamma \rtimes \mathbb{T}$-module and hence as a Γ-module. Thus H contains an irreducible submodule K generated by any non-zero pv with $v \in V$.
(d) Take the cyclic module generated by a vector of lowest energy. This contains an irreducible submodule generated by another vector of lowest energy H_1 say. Now repeat this process for $H_1^\perp$, to get H_2, H_3, etc. The positive energy assumption shows that $H = \bigoplus H_i$.

Corollary. *If* $\pi : LG \rtimes \mathrm{Rot}\, S^1 \to PU(H)$ *is a projective representation which restricts to an ordinary positive energy representation of* Rot S^1, *then* H *decomposes as a direct sum* $\bigoplus H_i \otimes K_i$ *where the* H_i*'s are representations of* $LG \rtimes \mathrm{Rot}\, S^1$ *irreducible on* LG *with* $H_i(0) \neq (0)$ *and the multiplicity spaces are positive energy representations of* Rot S^1.

We apply this result to the positive energy representation $\mathscr{F}_P^{\otimes \ell}$ of $LG \rtimes \mathrm{Rot}\, S^1$. The irreducible summands of $\mathscr{F}_P^{\otimes \ell}$ are called the level ℓ irreducible representations of LG. By definition any positive energy representation extends to $LG \rtimes \mathrm{Rot}\, S^1$. More generally the vacuum representation at level ℓ extends (canonically) to $LG \rtimes SU_{\pm}(1,1)$. In fact, since $SU_{\pm}(1,1)$ fixes the vacuum vector and this generates the vacuum representation at level ℓ as an LG-module, it follows that the vacuum representation at level ℓ admits a compatible orthogonal representation of $SU_{\pm}(1,1)$, unitary on $SU_{+}(1,1)$ and antiunitary on $SU_{-}(1,1)$. We also need the less obvious fact that $SU(1,1)$ is implemented by a projective unitary representation in any level ℓ representation; this follows from a global form of the Goddard-Kent-Olive construction [12].

Lemma (coset construction). *Let $H = \bigoplus H_i \otimes K_i$ and let $M = \bigoplus B(H_i) \otimes I$. Let $\pi : \mathscr{G} \to PU(H)$ be a projective unitary representation of the connected topological group $\mathscr{G}$ such that $\pi(g) M \pi(g)^* = M$ for all $g \in \mathscr{G}$. Then there exist projective unitary representations π_i and σ_i of $\mathscr{G}$ on H_i and K_i such that $\pi(g) = \oplus \pi_i(g) \otimes \sigma_i(g)$.*

Proof. $\mathscr{G}$ acts by automorphisms on M through conjugation. It therefore preserves the centre and hence the minimal central projections. Since $\mathscr{G}$ is connected and the action strong operator continuous, it must fix the central projections. Thus it fixes each block $H_i \otimes K_i$. It also normalises $B(H_i)$. If W_i denotes the restriction of $\pi(g)$ to $H_i \otimes K_i$, then $\mathrm{Ad}\, W_i$ restricts an automorphism α_i of $B(H_i)$. But, if K is a Hilbert space, any automorphism α of $B(K)$ is inner: indeed if ξ is a fixed unit vector in K and P_ξ is the rank one projection onto $\mathbb{C}\xi$, then $\alpha(P_\xi) = P_\eta$ for some unit vector η and $U(T\xi) = \alpha(T)\eta$ $(T \in B(K))$ defines a unitary with $\alpha = \mathrm{Ad}\, U$. Hence $\alpha_i = \mathrm{Ad}\, U_i$ for $U_i \in U(H_i)$. But then $(U_i^* \otimes I) W_i$ commutes with $B(H_i) \otimes I$ and hence lies in $I \otimes B(K_i)$. Hence $(U_i^* \otimes I) W_i = I \otimes V_i$, so that $W_i = U_i \otimes V_i$. Thus we get the required homomorphism $\mathscr{G} \to \prod PU(H_i) \times PU(V_i)$, which is clearly continuous.

Corollary. *There is a* (*unique*) *projective representation π_i of $SU(1,1)$ on H_i satisfying $\pi_i(\gamma)\pi_i(g)\pi_i(\gamma)^* = \pi_i(g \cdot \gamma^{-1})$ for $g \in \mathscr{L}G$ and $\gamma \in SU(1,1)$.*

Proof. If $H = \mathscr{F}_V^{\otimes \ell}$, we may write $H = \bigoplus H_i \otimes K_i$ where the H_i's are the distinct level ℓ irreducible representations of $\mathscr{L}G$ and the K_i's are multiplicity spaces. Then $\pi(\mathscr{L}G)'' = \bigoplus B(H_i) \otimes I$ and the unitary representation of $SU(1,1)$ normalises this algebra. By the coset construction, each $\gamma \in SU(1,1)$ has a decomposition $\pi(\gamma) = \bigoplus \pi_i(\gamma) \otimes \sigma_i(\gamma)$, where $\tau_i(\gamma) = \pi_i(\gamma) \otimes \sigma_i(\gamma)$ is an ordinary representation of $SU(1,1)$ on $H_i \otimes K_i$. But $\pi_i(g \cdot \gamma^{-1}) \otimes I = \tau_i(\gamma)(\pi_i(g) \otimes I)\tau_i(\gamma)^* = \pi_i(\gamma)\pi_i(g)\pi_i(\gamma)^*$. Hence $\pi_i(\gamma)\pi_i(g)$ $\pi_i(\gamma)^* = \pi_i(g \cdot \gamma^{-1})$. So, as before, the representation of $SU(1,1)$, now projective, is compatible with the central extension $\mathscr{L}G$.

7. Infinitesimal action of $L^0\mathrm{g}$ on finite energy vectors

If $\mathrm{g} = \mathrm{Lie}\ (G)$, then $\mathrm{Lie}\ (LG) = L\mathrm{g} = C^\infty(S^1, \mathrm{g})$. Let $L^0\mathrm{g}$ be the algebraic Lie algebra consisting of trigonometric polynomials with values in g. Its complexification is spanned by the functions $X_n(\theta) = e^{-in\theta}X$ with $X \in \mathrm{g}$. Rot S^1 and its Lie algebra act on $L^0\mathrm{g}$. The Lie algebra of Rot S^1 is generated by id where $[d, f](\theta) = -if'(\theta)$ for $f \in L^0\mathrm{g}$. Thus d may be identified with the operator $-id/d\theta$. We obtain the Lie algebra relations $[X_n, Y_m] = [X, Y]_{n+m}$ and $[d, X_n] = -nX_n$. For $v \in V$, let $v(n) = a(v_n)$ where $v_n \in L^2(S^1, V)$ is given by $v_n(\theta) = e^{-in\theta}v$. In particular, if (e_i) is an orthonormal basis of V, then we have fermions $e_i(n)$ for all n. If Ω denotes the vacuum vector in $\mathscr{F}_V$, then it is easy to see from its description as an exterior algebra that an orthonormal basis of $\mathscr{F}_V$ is given by

$$e_{i_1}(n_1)e_{i_2}(n_2)\cdots e_{i_p}(n_p)e_{j_1}(m_1)^*e_{j_2}(m_2)^*\ldots e_{j_q}(m_q)^*\Omega$$

where $n_i \leq 0$ and $m_j > 0$. Moreover $e_i(n)\Omega = 0$ for $n \geq 0$ and $e_i(n)^*\Omega = 0$ for $n < 0$. Since Rot S^1 commutes with the Hardy space projection on $L^2(S^1, V)$, it is canonically quantised. Let R_θ be the corresponding representation on $\mathscr{F}_V$. Then $R_\theta = e^{iD\theta}$ where D is self-adjoint. If r_θ is the action of Rot S^1 on $L^2(S^1, V)$ given by $(r_\theta f)(z) = f(e^{i\theta}z)$, then $r_\theta = e^{id}$ where $d = -i\frac{d}{d\theta}$ (we always regard functions on S^1 as functions either of $z \in \mathbb{T}$ or of $\theta \in [0, 2\pi]$). Now $R_\theta a(f)R_\theta^* = a(r_\theta f)$. Hence $R_\theta v(m)R_\theta^* = e^{-im\theta}v(m)$, so that R_θ acts on the basis vector $e_{i_1}(n_1)e_{i_2}(n_2)\cdots e_{i_p}(n_p)e_{j_1}(m_1)^*e_{j_2}(m_2)^*\cdots e_{j_q}(m_q)^*\Omega$ as multiplication by $e^{iM\theta}$ where $M = \sum m_j - \sum n_i$. Since $R_\theta = e^{iD\theta}$, it follows that D acts on this basis vector as multiplication by M, i.e. this vector has energy $M = \sum m_j - \sum n_i$. In particular $D\Omega = 0$ and we can check that $[D, v(n)] = -nv(n)$. Thus if f is a trigonometric power series with values in V, we have $[D, a(f)] = a(df)$. Note that if T is a linear operator on $\mathscr{F}_V^0$ commuting with the $e_i(a)$'s and $e_i(a)^*$'s, then $T = \lambda I$ for $\lambda \in \mathbb{C}$: for, as in section 3, Ω is the unique vector such that $e_i(n)^*\Omega = 0$ $(n \geq 0)$, $e_i(n)\Omega = 0$ $(n > 0)$ and Ω is cyclic.

Theorem. *Let* $E_{ij}(n) = \sum_{m>0} e_i(n-m)e_j(-m)^* - \sum_{m\geq 0} e_j(m)^*e_i(m+n)$, *and define* $X(n) = \sum a_{ij}E_{ij}(n)$ *for* $X = \sum a_{ij}E_{ij} \in \mathrm{Lie}\ U(V) \subset \mathrm{End}(V)$. *Then, as operators on* H^0, *we have*

(a) $[X(m), a(f)] = a(X_m \cdot f)$ *if* f *is a trigonometric polynomial with values in* V; *equivalently* $[X(n), v(m)] = (Xv)(n+m)$.

(b) $[D, X(m)] = -mX(m)$.

(c) $[X(n), Y(m)] = [X, Y](n+m) + n(X, Y)\delta_{n+m,0}I$ *where* $(X, Y) = -\mathrm{Tr}(XY) = \mathrm{Tr}(XY^*)$ *for* $X, Y \in \mathrm{Lie}\ U(V)$.

Proof. (a) Observe that $[e_i(a)^*e_j(b), e_k(c)] = -\delta_{ac}\delta_{ik}e_j(b)$ and $[e_j(b)\ e_i(a)^*, e_k(c)] = \delta_{ac}\delta_{ik}e_j(b)$. Moreover if $i \neq j$, then $e_i(a)$ anticommutes with both $e_j(b)$ and $e_j(b)^*$. Using these identities, it is easy to check that $E_{ij}(n)$ satisfies

the commutation relations (a) with respect to the $e_i(n)$'s. Note that $X(n)\Omega = 0$ for $n \geq 0$ since $e_i(n)\Omega = 0$ for $n \geq 0$, $e_i(n)^*\Omega = 0$ for $n < 0$ and (\,formally) $X(n)^* = -X(-n)$ for $X \in \text{Lie } U(V)$.
(b) Since $[D, e_i(m)] = -me_i(m)$ and $[D, e_i(m)^*] = me_i(m)^*$, it follows that $[D, X(m)] = -mX(m)$.
(c) From (a) we find that $T = [X, (m), Y(n)] - [X, Y](m+n)$ commutes with all $e_i(a)$'s and hence also all $e_i(a)^*$'s by the adjointness property. Hence $[X(m), Y(n)] = [X, Y](m+n) + \lambda(X, Y)(m, n)I$, where $\lambda(X, Y)(m, n)$ is a scalar, bilinear in X and Y. Now from (b), $[X(m), Y(n)] - [X, Y](m+n)$ lowers the energy by $-m-n$, so that $\lambda(X, Y)(m, n) = 0$ unless $m + n = 0$. To compute the value of λ when $m = -n$, we note that we may assume that $m \geq 0$, since $\lambda(X, Y)(m, n)^* = \lambda(Y, X)(-n, -m)$ by the adjoint relations. Taking vacuum expectations, we get

$$\begin{aligned}\lambda(X, Y)(-m, m) = ([X(-m), Y(m)]\Omega, \Omega) = (X(-m)\Omega,\\ Y(-m)\Omega) = -m\text{Tr}(XY) = m(X, Y).\end{aligned}$$

In fact if $X = \sum a_{ij}E_{ij}$ and $Y = \sum b_{ij}E_{ij}$, we have

$$\begin{aligned}(X(-m)\Omega, Y(-m)\Omega) &= \sum_{ijpq}\sum_{r,s=0}^{m-1}(a_{ij}e_j(r)^*e_i(r-m)\Omega, b_{pq}e_q(s)^*e_p(s-m)\Omega)\\ &= m\sum a_{ij}\overline{b_{ij}} = m(X, Y),\end{aligned}$$

since the terms $e_i(a)^*e_j(b)\Omega$ with $a \geq 0$ and $b < 0$ are orthonormal.

8. The exponentiation theorem

We wish to show that the Lie algebra action just defined on $\mathscr{F}_V$ exponentiates to give the fundamental representation of $LSU(N) \rtimes \text{Rot } S^1$. We have already discussed the action of Rot S^1, which is canonically quantised. So we now must show that if x is an element of $L^0\mathrm{g}$ and X is the corresponding operator constructed above, then $\pi \exp x$ and $\exp X$ have the same image in $PU(\mathscr{F})$. To see that this completely determines π on LG, we need the following result on products of exponentials.

Exponential lemma. *Every element of LG is a product of exponentials in $L\mathrm{g} = C^\infty(S^1, \mathrm{g})$. Products of exponentials in $L^0\mathrm{g}$ are dense in LG.*

Proof. If $g \in LG \subset C(S^1, M_N(\mathbb{C}))$ satisfies $\|g - I\|_\infty < 1$, then $\log g = \log(I - (I - g))$ lies in $C^\infty(S^1, \mathrm{g}) = L\mathrm{g}$. Thus exp $L\mathrm{g}$ contains an open neighbourhood of I in LG. Since LG is connected, exp $L\mathrm{g}$ must generate LG, as required.

The bilinear formulas for the Lie algebra operators X immediately imply Sobolev type estimates for the infinitesimal action of $L^0\mathrm{g}$ on finite energy vectors. We define the Sobolev norms by $\|\xi\|_s = \|(I+D)^s\xi\|$ for $s \in \mathbb{R}$, usually a half-integer. Recall that if A is a skew-adjoint operator, the smooth vectors for A are the subspace $C^\infty(A) = \bigcap \mathscr{D}(A^n)$ and for any $\xi \in C^\infty(A)$ we have $e^{At}\xi = \sum_{i=0}^n \frac{t^k}{k!} A^k\xi + O(t^{n+1})$.

Exponentiation Theorem. *Let $H = \mathscr{F}_V$ be the level one fermionic representation of $LSU(V)$ and let H^0 be the subspace of finite energy vectors.*

(1) *For $x \in L^0\mathrm{g}$, there is a constant K depending on s and x such that $\|X\cdot\xi\|_s \leq K\|\xi\|_{s+1}$ for $\xi \in H^0$, $X = \pi(x)$.*

(2) *For each $x \in L^0\mathrm{g}$, the corresponding operator X is essentially skew-adjoint on H^0 and leaves H^0 invariant.*

(3) *Each vector in H^0 is a C^∞ vector for any $x \in L^0\mathrm{g}$.*

(4) *For $x \in L^0\mathrm{g}$, the unitary* $\exp(X)$ *agrees up to a scalar with* $\pi(\exp(x))$.

Proof. (1) It clearly suffices to prove the estimates in the lemma for $X = E_{ij}(n)$ and ξ of fixed energy, say $D\xi = \mu\xi$. Then $E_{ij}(n)\xi = \sum_{m>0} e_i(n-m)e_j(-m)^*\xi - \sum_{m\geq 0} e_j(m)^* e_i(m+n)\xi$. So $\|E_{ij}(n)\xi\| \leq 2(|n| + \mu)\|\xi\|$, since at most $2(|n|+\mu)$ of the terms in the sums can be non-zero and each has norm bounded by $\|\xi\|$. Hence for $s \geq 0$,

$$\begin{aligned}\|E_{ij}(n)\xi\|_s &\leq (1+|n|+\mu)^s\|E_{ij}(n)\xi\| \leq 2(1+|n|+\mu)^s(|n|+\mu)\\ &\leq 2(1+|n|)^{s+1}(1+\mu)^{s+1}\|\xi\| \leq 2(1+|n|)^{s+1}\|\xi\|_{s+1}.\end{aligned}$$

(2) Clearly any $X \in L^0\mathrm{g}$ acts on H^0. We need the Glimm-Jaffe-Nelson commutator theorem see [11, 31] or [42]: if D is the energy operator on H^0 and $X: H^0 \to H^0$ is formally skew-adjoint with $X(D+I)^{-1}$, $(D+I)^{-1}X$ and $(D+I)^{-1/2}[X,D](D+I)^{-1/2}$ bounded, then the closure of X is skew-adjoint. The Sobolev estimates show that these conditions hold for D and X, since $[D,X]$ is actually in $L^0\mathrm{g}$.

(3) Since $XH^0 \subset H^0$ and the C^∞ vectors for X are just $\cap\mathscr{D}(X^n)$, it follows that the vectors in H^0 are C^∞ vectors for X.

(4) We prove the commutation relation $e^{tX}a(f)e^{-tX} = a(e^{tx}f)$ for $f \in L^2(S^1)\otimes V$. We start by noting that

$$a(Xf)\xi = Xa(f)\xi - a(f)X\xi$$

for f a trigonometric polynomial with values in V, $X \in L^0\mathrm{g}$ and $\xi \in H^0$. We fix X and f and denote by $C^\infty(X)$ the space of C^∞ vectors for X, i.e. $\cap\mathscr{D}(X^n)$. Now say $\xi \in \mathscr{D}(X)$ and $f \in L^2(S^1, V)$. Take $\xi_n \in H^0$, such that $\xi_n \to \xi$ and $X\xi_n \to X\xi$, and f_n trigonometric polynomials with values in V such that $f_n \to f$. Then $a(f_n)\xi_n \to a(f)\xi$ and $Xa(f_n)\xi_n = a(Xf_n)\xi_n + a(f_n)X\xi_n \to a(Xf)\xi + a(f)X\xi$. Since X is closed, we deduce that $a(f)\xi$ lies in $\mathscr{D}(X)$ and $a(Xf)\xi = Xa(f)\xi - a(f)X\xi$. Successive applications of this identity imply that $a(f)\xi$ lies in $\mathscr{D}(X^n)$ for all n if ξ lies in $C^\infty(X)$, so that $a(f)C^\infty(X) \subset C^\infty(X)$.

Now take $\xi, \eta \in C^\infty(X)$ and consider $F(t) = (e^{-Xt} a(e^{xt} f) e^{Xt} \xi, \eta) = (a(e^{xt} f) e^{Xt} \xi, e^{Xt} \eta)$. Since ξ, η lie in $C^\infty(X)$, we have $e^{X(t+s)} \xi = e^{Xt} \xi + sXe^{Xt} \xi + O(s^2)$ and $e^{X(t+s)} \eta = e^{Xt} \eta + sXe^{Xt} \eta + O(s^2)$. For any f, we have $e^{x(t+s)} f = e^{xt} f + sxe^{xt} f + O(s^2)$ in $L^2(S^1) \otimes V$. Since $\|a(g)\| = \|g\|$, it follows that $a(e^{x(t+s)} f) = a(e^{xt} f) + sa(xe^{xt} f) + O(t^2)$ in the operator norm. Hence we get

$$\begin{aligned} F(t+s) &= (a(e^{xt} f) e^{Xt} \xi, e^{Xt} \eta) + s[(a(e^{xt} f) X e^{Xt} \xi, e^{Xt} \eta) \\ &\quad + (a(xe^{xt} f) e^{Xt} \xi, e^{Xt} \eta) + (a(e^{xt} f) e^{Xt} \xi, X e^{Xt} \eta)] + O(s^2) \\ &= (a(e^{xt} f) e^{Xt} \xi, e^{Xt} \eta) + O(s^2). \end{aligned}$$

since $[X, a(g)] = a(xg)$. Thus $F(t)$ is differentiable with $F'(t) \equiv 0$. Hence $F(t)$ is constant and therefore equal to $F(0)$. This proves that $e^{-tX} a(e^{tx} f) e^{tX} \xi = a(f) \xi$ for $\xi \in H^0 \subset C^\infty(X)$. Hence $a(e^{tx} f) = e^{tX} a(f) e^{-tX}$, as required. Thus e^{tX} implements the Bogoliubov automorphism corresponding to e^{tx}.

Corollary. *Let H be a level ℓ positive energy representation of $LSU(N)$ and let H^0 be the subspace of finite energy vectors.*

(1) *There is a projective representation of $L^0\mathrm{g} \rtimes \mathbb{R}$ on H^0 such that $[D, X(n)] = -nX(n)$, $D^* = D$, $X(n)^* = -X(-n)$ and $[X(m), Y(n)] = [X, Y](n+m) + m\ell\delta_{m+n,0}\,(X, Y)$.*

(2) *For each $x \in L^0\mathrm{g}$, the corresponding operator X is essentially skew-adjoint on H^0 and leaves H^0 invariant.*

(3) *For $x \in L^0\mathrm{g}$, the unitary* $\exp(X)$ *agrees up to a scalar with the corresponding group element in LG.*

(4) *Each vector in H^0 is a C^∞ vector for any X.*

Proof. We observe that the embedding $LSU(N) \subset LU(N\ell)$ gives all representations of $LSU(N)$ at level ℓ. The continuity properties of the action of the larger group and its Lie algebra are immediately inherited by $LSU(N)$. Note that it is clear from the functoriality of the fermionic construction that the restriction of the fermionic representation of $LU(N\ell)$ to $LU(N)$ can be identified with $\mathscr{F}^{\otimes \ell}$ where $\mathscr{F}$ is the (level 1) fermionic representation of $LU(N)$. The other properties follow immediately from the following result, applied to irreducible summands K of $H = \mathscr{F}^{\otimes \ell}$.

Lemma. *Let X be a skew-adjoint operator on H with core H^0 such that $X(H^0) \subseteq H^0$. Let K be a closed subspace such that $P(H^0) \subseteq H^0$, where P is the projection onto K. Let $K^0 = K \cap H^0$. Then $X(K^0) \subseteq K^0$ iff* $\exp(Xt)K = K$ *for all t. In this case K^0 is a core for $X|_K$.*

Proof. Suppose that K is invariant under $\exp(Xt)$. Then $\exp(Xt)\ \xi = \xi + tX\xi + \cdots$ for $\xi \in K^0$ and hence $XK^0 \subseteq K \cap H^0 = K^0$. Conversely, if $X(K^0) \subseteq K^0$, take $\xi \in \mathscr{D}(X)$ and let P be the orthogonal projection onto K.

It will suffice to show that $P\xi \in \mathscr{D}(X)$ and $XP\xi = PX\xi$, for then X commutes with P in the sense of the spectral theorem. Since $P(H^0) \subseteq H^0$, we have $H^0 = H^0 \cap K \bigoplus H^0 \cap K^\perp$. Since X is skew-adjoint and $X(K^0) \subseteq K^0$, it follows that X leaves $H^0 \cap K^\perp$ invariant. Thus $PX = XP$ on H^0. Take $\xi_n \in H^0$ such that $\xi_n \to \xi$ and $X\xi_n \to X\xi$. Then $XP\xi_n = PX\xi_n \to PX\xi$ and $P\xi_n \to \xi$. Since X is closed, $XP\xi = PX\xi$ as required. Finally since $P\xi_n \to P\xi$ and $XP\xi_n \to XP\xi$, it follows that K^0 is a core for $X|_K$.

9. Classification of positive energy representations of level ℓ

Proposition. *Let (π, H) be an irreducible positive energy projective representation of $LG \rtimes \mathrm{Rot}\, S^1$ of level ℓ.*

(1) *The action of $L^0\mathfrak{g} \rtimes \mathbb{R}$ on H^0 is algebraically irreducible.*

(2) *$H(0)$ is irreducible as an $SU(N)$-module.*

(3) *If $H(0) = V_f$, then $f_1 - f_N \leq \ell$.*

(4) (*Existence*) *If $f_1 - f_N \leq \ell$, there is a an irreducible positive energy representation of LG of level ℓ of the above form with $H(0) \cong V_f$ as $SU(N)$-modules.*

(5) (*Uniqueness*) *If H and H' are irreducible positive energy representations of level ℓ of the above form with $H(0) \cong H'(0)$ as $SU(N)$-modules, then H and H' are unitarily equivalent as projective representations of $LG \rtimes \mathrm{Rot}\, S^1$.*

Proof. (1) Recall that H is irreducible as an $LG \rtimes \mathbb{T}$-module iff it is irreducible as an LG-module by the proposition in section 6. Any subspace K of H^0 invariant under $L^0\mathfrak{g} \rtimes \mathbb{R}$ is clearly invariant under Rot S^1. It therefore coincides with the space of finite energy vectors of its closure. By the lemma in section 8, its closure is invariant under all operators $\exp(X)$ for $x \in L^0\mathfrak{g}$. But $\exp(L^0\mathfrak{g})$ generates a dense subgroup of LG, so the closure must be invariant under LG and therefore coincide with the whole of H by irreducibility. Hence $K = H^0$ as required.

(2) Let V be an irreducible $SU(N)$-submodule of H(0). From (1), the $L^0\mathfrak{g} \rtimes \mathbb{R}$-module generated by V is the whole of H^0. Since D fixes V, it follows that the $L^0\mathfrak{g}$-module generated by V is the whole of H^0. The commutation rules show that any monomial in the $X(n)$'s can be written as a sum of monomials of the form $P_-P_0P_+$, where P_- is a monomial in the $X(n)$'s for $n < 0$ (energy raising operators), P_0 is a monomial in the $X(0)$'s (constant energy operators) and P_+ is a monomial in the $X(n)$'s with $n > 0$ (energy lowering operators). Hence H^0 is spanned by products P_-v ($v \in V$). Since the lowest energy subspace of this $L^0\mathfrak{g}$-module is V, $H(0) = V$, so that $H(0)$ is irreducible as a G-module.

(3) Suppose that $H(0) \cong V_f$ and let $v \in H(0)$ be a highest weight vector, so that $(E_{ii}(0) - E_{jj}(0))v = (f_i - f_j)v$ and $E_{ij}(0)v = 0$ if $i < j$. Let $E = E_{N1}(1)$, $F = E_{1N}(-1)$ and $H = [E, F] = E_{NN}(0) - E_{11}(0) + \ell$. Thus $H^* = H$, $E^* = F$, $[H, E] = 2E$ and $[H, F] = -2F$. Moreover $Ev = 0$ and $Hv = \lambda v$ with $\lambda = f_N - f_1 + \ell$. By induction on k, we have $[E, F^{k+1}] = (k+1)F^k(H - kI)$

for $k \geq 0$. Hence $(F^{k+1}v, F^{k+1}v) = (F^*F^{k+1}v, F^kv) = (EF^{k+1}v, F^kv) = (k+1)(\lambda - k)(F^kv, F^kv)$. For these norms to be non-negative for all $k \geq 0$, λ has to be non-negative, so that $f_1 - f_N \leq \ell$ as required.

(4) We have $\mathscr{F}_V^{\otimes\ell}(0) = (\Lambda V)^{\otimes\ell}$. By the results of section 6, the LG-module generated by any irreducible summand V_f of $\mathscr{F}_V(0)$ gives an irreducible positive energy representation H with $H(0) \cong V_f$. So certainly any irreducible summand in $\Lambda V^{\otimes\ell}$ appears as an $H(0)$. From the tensor product rules with the $\lambda^k V$'s, these representations are precisely those with $f_1 - f_N \leq \ell$.

(5) Any monomial A in operators from $L^0\mathfrak{g}$ is a sum of monomials RDL with R a monomial in energy raising operators, D a monomial in constant energy operators and L a monomial in energy lowering operators. As in section 2, if $v, w \in H(0)$ the inner products (A_1v, A_2w) are uniquely determined by v, w and the monomials A_i: for $A_2^*A_1$ is a sum of terms RDL and $(RDLv, w) = (DLv, R^*w)$ with R^* an energy lowering operator. Hence, if H' is another irreducible positive energy representation with $H'(0) \cong H(0)$ by a unitary isomorphism $v \mapsto v'$, $U(Av) = Av'$ defines a unitary map of H^0 onto $(H')^0$ intertwining $L^0\mathfrak{g}$. This induces a unique unitary isomorphism $H \to H'$ which intertwines the one parameter subgroups corresponding to the skew-adjoint elements in $L^0\mathfrak{g}$, since H^0 and H'^0 are cores for the corresponding skew-adjoint operators. But these subgroups generate a dense subgroup of LG, so that U must intertwine the actions of LG, i.e. $\pi'(g) = U\pi(g)U^*$ in $PU(H')$ for $g \in LG$. Thus H and H' are isomorphic as projective representations of LG. From section 6, H and H' are therefore unitarily equivalent as projective representations of $LG \rtimes \mathrm{Rot}\, S^1$.

Corollary. *The irreducible positive energy representations H of LG of level ℓ are uniquely determined by their lowest energy subspace $H(0)$, an irreducible G-module. Only finitely many irreducible representations of G occur at level ℓ: their signatures must satisfy the quantisation condition $f_1 - f_N \leq \ell$. The action of $L^0\mathfrak{g} \rtimes \mathbb{R}$ on H^0 is algebraically irreducible.*

II. Local loop groups and their von Neumann algebras

10. von Neumann algebras

Let H be a Hilbert space. The commutant of $S \subset B(H)$ is defined by $S' = \{T \in B(H) : Tx = xT \text{ for all } x \in S\}$. If $S^* = S$, for example if S is a *-algebra or a subgroup of $U(H)$, then S' is a unital *-algebra, closed in the weak or strong operator topology. Such an algebra is called a von Neumann algebra. von Neumann's double commutant theorem states that S'' coincides with the von Neumann algebra generated by S, i.e. the weak operator closure of the unital *-algebra generated by S. Thus a *-subalgebra $M \subseteq B(H)$ is a von Neumann algebra iff $M = M''$. By the spectral theorem, the spectral projections (or more generally bounded Borel functions) of any self-adjoint or unitary operator in M must also lie in M. This implies in particular that M

is generated both by its projections and its unitaries. Note that, if $M = S'$, the projections in M correspond to subrepresentations for S, i.e. subspaces invariant under S.

The centre of a von Neumann algebra M is given by $Z(M) = M \cap M'$. A von Neumann algebra is said to be a *factor* iff $Z(M) = \mathbb{C}I$. A unitary representation of a group or a *-representation of a *-algebra is said to be a factor representation if the commutant is a factor. If H is a representation with commutant M, then two subrepresentations H_1 and H_2 of H are unitarily equivalent iff the corresponding projections $P_1, P_2 \in M$ are the initial and final projections of a partial isometry $U \in M$, i.e. $U^*U = P_1$ and $UU^* = P_2$. P_1 and P_2 are then said to be equivalent in the sense of Murray and von Neumann [26]. We shall only need the following elementary result, which is an almost immediate consequence of the definitions.

Proposition. *If (π, H) is a factor representation of a set S with $S^* = S$ and (π_1, H_1) and (π_2, H_2) are subrepresentations, then*

(1) *there is a unique *-isomorphism θ of $\pi_1(S)''$ onto $\pi_2(S)''$ such that* $\theta(\pi_1(x)) = \pi_2(x)$ for $x \in S$;

(2) *the intertwiner space $\mathscr{X} = \mathrm{Hom}_S(H_1, H_2)$ satisfies $\overline{\mathscr{X}H_1} = H_2$, so in particular is non-zero*;

(3) *$\theta(a)T = Ta$ for all $a \in \pi_1(S)''$ and $T \in \mathscr{X}$*;

(4) *if $\mathscr{X}_0 \subseteq \mathscr{X}$ with $\overline{\mathscr{X}_0 H_1} = H_2$, then $\theta(a)$ is the unique $b \in \pi_2(S)''$ such that $bT = Ta$ for all $T \in \mathscr{X}_0$.*

Proof. Let $M = \pi(S)''$ and $M_i = \pi_i(S)''$. Then $\overline{M'H_i}$ is invariant under both M and M'. Hence the corresponding projection lies in $M \cap M' = \mathbb{C}$ (since M is a factor). So $\overline{M'H_i} = H$. Let p_i be the projection onto H_i, so that $p_i \in M'$. Clearly $M_i = Mp_i$. Moreover, the map $\theta_i : M \to M_i$, $a \mapsto ap_i$ must be a *-isomorphism: for $ap_i = 0$ implies $aM'H_i = (0)$ and hence $a = 0$. By definition $\theta_i(\pi(x)) = \pi_i(x)$ for $x \in S$. Now set $\theta = \theta_2\theta_1^{-1}$; θ is unique because M_1 is generated by $\pi_1(S)$.

Since $\mathscr{X} = \mathrm{Hom}_S(H_1, H_2) = p_2 M' p_1$, we have $T\theta_1(x) = \theta_2(x)T$ for all $x \in M$. Hence $\theta(a)T = Ta$ for $a \in M_1$ and $T \in \mathrm{Hom}_S(H_1, H_2)$. Moreover $\overline{\mathscr{X}H_1} = \overline{p_2 M' H_2} = \overline{p_2 H} = H_2$. Conversely suppose that $\mathscr{X}_0 \subset \mathrm{Hom}_S(H_1, H_2)$ is a subspace such that $\mathscr{X}_0 H_1$ is dense in H_2 and $a \in M_1$, $b \in B(H_2)$ satisfy $bT = Ta$ for all $T \in \mathscr{X}_0$. Let $c = b - \theta(a)$. Then $c\mathscr{X}_0 = (0)$ and hence $cH_2 = (0)$, so that $c = 0$. Thus $b = \theta(a)$ as required.

11. Abstract modular theory

Let H be a complex Hilbert space, and $K \subset H$ a closed real subspace with $K \cap iK = (0)$ and $K + iK$ dense in H. Let e and f be the projections onto K and iK respectively and set $r = (e + f)/2$, $t = (e - f)/2$. Then $K^\perp$, $iK^\perp$ and iK satisfy the same conditions as K, where $\perp$ is taken with respect to the real inner product $\mathrm{Re}(\xi, \eta)$.

Proposition 1. (1) $0 \le r \le I$, t, r *are self-adjoint*, t *is conjugate-linear*, r *is linear*, *and* t, $I - r$, r *have zero kernels.*

(2) $t^2 = r(I - r)$, $rt = t(I - r)$, $(I - r)t = tr$.

(3) $et = t(I - f)$, $ft = t(I - e)$.

(4) *If* t *has polar decomposition* $t = |t|j = j|t|$, *then* $j^2 = I$, $ej = j(I - f)$ *and* $fj = j(I - e)$.

(5) $jK = iK^\perp$ *and* $(j\xi, \eta) \in \mathbb{R}$ *for* $\xi, \eta \in K$.

(6) *Let* $\delta^{it} = (I - r)^{it} r^{-it}$. *Then* $j\delta^{it} = \delta^{it} j$ *and* $\delta^{it} K = K$.

Proof. (1), (2) and (3) are straightforward. (4) follows from (3), because e and f commute with $t^2 = (e - f)^2/4$, hence with $|t|$, and $|t|$ has zero kernel. (4) implies (5), since $jej = I - f$. Finally since $jrj = I - r$ and j is conjugate-linear, j commutes with δ^{it}. So δ^{it} commutes with j, r, $|t| = \sqrt{r(I - r)}$ and hence t. So δ^{it} commutes with e and f.

Proposition 2 (characterisation of modular operators). (1) (*Kubo-Martin-Schwinger condition*) *For each* $\xi \in K$, *the function* $f(t) = \delta^{it}\xi$ *on* $\mathbb{R}$ *extends* (*uniquely*) *to a continuous bounded function* $f(z)$ *on* $-1/2 \le \operatorname{Im} z \le 0$, *holomorphic in* $-1/2 < \operatorname{Im} z < 0$. *Furthermore* $f(t - i/2) = jf(t)$ *for* $t \in \mathbb{R}$.

(2) (*KMS uniqueness*) *Suppose that* u_t *is a one-parameter unitary group on* H *and* j_1 *is a conjugate-linear involution such that* $u_t K = K$ *and* $j_1 u_t = u_t j_1$. *Suppose that there is a dense subspace* K_1 *of* K *such that for each* $\xi \in K_1$ *the function* $g(t) = u_t\xi$ *extends to a bounded continuous function* $g(z)$ *on the strip* $-1/2 \le \operatorname{Im} z \le 0$ *into* H, *holomorphic in* $-1/2 < \operatorname{Im} z < 0$, *such that* $f(t - i/2) = j_1 f(t)$ *for* $t \in \mathbb{R}$. *Then* $u_t = \delta^{it}$ *and* $j_1 = j$.

Proof. (1) (cf [33]) If $\xi \in K$, then $\xi = p\xi = (r + t)\xi = r^{\frac{1}{2}}(r^{\frac{1}{2}} + (I - r)^{\frac{1}{2}} j)\xi$. Thus $\xi = r^{\frac{1}{2}}\eta$, where $\eta = (r^{\frac{1}{2}} + (I - r)^{\frac{1}{2}} j)\xi$. Set $f(z) = (I - r)^{iz} r^{\frac{1}{2} - iz}\eta$ for $-1/2 \le \operatorname{Im} z \le 0$.

(2) For $\xi \in K_1$, set $h(z) = (g(z), g(\bar{z} - i/2))$. Then h is continuous and bounded on $-1/2 \le \operatorname{Im} z \le 0$, holomorphic on $-1/2 < \operatorname{Im} z < 0$. By uniqueness of analytic extension, $u_t f(z) = f(z + t)$ since they agree for z real. Hence $h(z + t) = h(z)$, so that h is constant on lines parallel to the real axis and hence constant everywhere. Since $h(-i/4) = \|g(-i/4)\|^2 \ge 0$, it follows that $h(0) \ge 0$, i.e. $(j_1\xi, \xi) \ge 0$. Polarising, we get $(j_1\xi, \eta) \in \mathbb{R}$ for all $\xi, \eta \in K$. Since u_t leaves K and iK invariant, it follows that u_t commutes with e and f and hence δ^{it}. Now let $f(z)$ be the function corresponding to ξ and δ^{it}. Define $k(z) = (g(z), jf(z))$ for $-\frac{1}{2} \le \operatorname{Im} z \le 0$. Then $k(t) = (u_t\xi, j\delta^{it}\xi)$ is real for $t \in \mathbb{R}$ and $k(t - i/2) = (j_1 u_t\xi, j^2\delta^{it}\xi) = (j_1 u_t \xi, \delta^{it}\xi)$ is real for $t \in \mathbb{R}$. k is bounded and continuous on $-\frac{1}{2} \le \operatorname{Im} z \le 0$ and holomorphic on $0 < \operatorname{Im} z < \frac{1}{2}$. By Schwartz's reflection principle, k extends to a holomorphic function on $\mathbb{C}$ satisfying $k(z + i) = k(z)$. This extension is bounded and therefore constant by Liouville's theorem. Hence $k(t) = k(0) = k(-i/2)$. Thus $(u_t\delta^{-it}\xi, j\xi) = (\xi, j\xi) = k(-i/2) = (j_1\xi, \xi)$. By polarisation it follows that $u_t = \delta^{it}$ and $j = j_1$, as required.

12. Modular operators and Takesaki devissage for von Neumann algebras

The main application of the modular theory for a closed real subspace is when the subspace arises from a von Neumann algebra with a vector cyclic for the algebra and its commutant. Let $M \subset B(H)$ be a von Neumann algebra and let $\Omega \in H$ (the "vacuum vector") satisfy $\overline{M\Omega} = H = \overline{M'\Omega}$. The condition $\overline{M'\Omega} = H$ is clearly equivalent to the condition that Ω is *separating* for M, i.e. $a\Omega = 0$ iff $a = 0$ for $a \in M$. If in addition M and H are $\mathbb{Z}_2$-graded, then the graded commutant M^q equals $\kappa M' \kappa^{-1}$ where the Klein transformation κ is given by multiplication by 1 on the even part of H and by i on the odd part; in this case we will always require that Ω be even. Let $K = \overline{M_{\rm sa}\Omega}$, a closed real subspace of H.

Lemma 1. *$K + iK$ is dense in H and $K \cap iK = (0)$.*

Proof. $K + iK \supseteq M\Omega = M_{\rm sa}\Omega + iM_{\rm sa}\Omega$, so $K + iK$ is dense. Now $K^\perp \supseteq iM'_{\rm sa}\Omega$, since for $a \in M_{\rm sa}$, $b \in M'_{\rm sa}$, we have $\mathrm{Re}(a\Omega, ib\Omega) = \mathrm{Re} - i\ (ab\Omega, \Omega) = 0$, because $(ab)^* = ab$ implies that $(ab\Omega, \Omega)$ is real. Hence $iK^\perp \supseteq M'_{\rm sa}\Omega$. Thus $K^\perp + iK^\perp \supseteq M'\Omega$, so $K^\perp + iK^\perp$ is dense. So $K \cap iK = (K^\perp + iK^\perp)^\perp = (0)$.

Let Δ^{it} and J be the modular operators on H associated with $K = \overline{M_{\rm sa}\Omega}$. The main theorem of Tomita-Takesaki asserts that $JMJ = M'$ and $\Delta^{it} M \Delta^{-it} = M$. (General proofs can be found in [8] or [33] for example; for hyperfinite von Neumann algebras an elementary proof is given in [42], based on [33] and [16].) Once the theorem is known, the map $x \mapsto Jx^*J$ gives an isomorphism between $M^{\rm op}$ (M with multiplication reversed) and M' and $\sigma_t(x) = \Delta^{it} x \Delta^{-it}$ gives a one-parameter group of automorphisms of M. Our development, however, does not logically require any form of the main theorem of Tomita-Takesaki; instead we verify it directly for fermions and deduce it for subalgebras invariant under the modular group using a crucial result of Takesaki ("Takesaki devissage").

Lemma 2. *If $JMJ \subseteq M'$, then $JMJ = M'$.*

Proof (cf [33]). Clearly $J\Omega = \Omega$. If $A, B \in M'_{\rm sa}$, then $(JB\Omega, A\Omega)$ is real since $A\Omega, B\Omega$ lie in $iK^\perp$ and J is also the modular conjugation operator for $iK^\perp$. Thus $(AJBJ\Omega, \Omega) = (JB\Omega, A\Omega) = (A\Omega, JB\Omega) = (JBJA\Omega, \Omega)$. By complex linearity in A and conjugate-linearity in B, it follows that $(AJBJ\Omega, \Omega) = (JBJA\Omega, \Omega)$ for all $A, B \in M'$. Now take $a, b \in M'$, $x, y \in M$ and set $A = a$ and $B = Jy^*JbJxJ$. Since $JxJ, JyJ \in M'$, B lies in M'. Hence $(JbJax\Omega, y\Omega) = (aJbJx\Omega, y\Omega)$. Since $\overline{M\Omega} = H$, this implies that $aJbJ = JbJa$. Thus $JM'J \subseteq M'' = M$ and so $JMJ = M'$.

Corollary. *If $A \subset B(H)$ is an Abelian von Neumann algebra and Ω a cyclic vector for A, then $\Delta^{it} = I$, $Ja\Omega = a^*\Omega$ and $JaJ = a^*$ for $a \in A$, and $A = JAJ = A'$.*

Proof. Since $A \subset A'$, Ω is separating for A. Thus $Ja\Omega = a^*\Omega$ extends by continuity to an antiunitary. If $a \in A_{\mathrm{sa}}$, the map $f(z) = a$ satisfies the KMS conditions for the trivial group and J, so they must be the modular operators. Since $JAJ = A \subseteq A'$, the last assertion follows from the lemma.

Theorem (Takesaki devissage [37]). *Let $M \subset B(H)$ be a von Neumann algebra, $\Omega \in H$ cyclic for M and M' and Δ^{it}, J the corresponding modular operators. Suppose that $\Delta^{it} M \Delta^{-it} = M$ and $JMJ = M'$. If $N \subset M$ is a von Neumann subalgebra such that $\Delta^{it} N \Delta^{-it} = N$, then*

(a) *Δ^{it} and J restrict to the modular automorphism group Δ_1^{it} and conjugation operator J_1 of N for Ω on the closure H_1 of $N\Omega$.*

(b) *$\Delta_1^{it} N \Delta_1^{-it} = N$ and $J_1 N J_1 = N'$.*

(c) *If e is the projection onto H_1, then $eMe = Ne$ and $N = \{x \in M : xe = ex\}$ (the Jones relations* [18]).

(d) *$H_1 = H$ iff $M = N$.*

(e) *The modular group fixes the centre. In fact $\Delta^{it} x \Delta^{-it} = x$ and $JxJ = x^*$ for $x \in Z(M) = M \cap M'$.*

Proof. (a) By KMS uniqueness, Δ^{it} and J restrict to Δ_1^{it} and J_1 on $H_1 = eH$.

(b) It is clear that $\mathrm{Ad}\Delta_1^{it}$ normalises $Ne = N_1$ on H_1. Now $J_1 N e J_1 = eJNJe \subseteq eJMJe = eM'e \subseteq eN'e = (eN)'$. Thus $J_1 N_1 J_1 \subseteq N_1'$. By Lemma 2, $J_1 N_1 J_1 = N_1'$.

(c) Since $M' \subset N'$ and $M' = JMJ$, this implies that $M \subset JN'J$. Compressing by e we get $eMe \subseteq eJN'Je = JeN'eJ = J_1 eN'eJ_1 = J_1 (N \cdot e)' J_1 = N \cdot e$. But trivially $Ne \subseteq eMe$, so that $eMe = Ne$. Clearly $N \subset \langle e\rangle'$. Now suppose that $x \in M$ commutes with e. Then $xe = ye$ for some $y \in N$. But then $(x - y)e = 0$, so that $(x - y)\Omega = 0$. Since Ω is separating for M, $x = y$ lies in N.

(d) Immediate from (c).

(e) Immediate from (a) and the corollary to Lemma 2.

13. Araki duality and modular theory for Clifford algebras

We develop the abstract results implicit in the work of Araki on the canonical commutation and anticommutation relations [1, 2]. This reduces the computation of the modular operators for Clifford algebras to "one particle states", i.e. to the prequantised Hilbert space. We first recall that the assignment $H \mapsto \Lambda(H)$ defines a *functor* from the additive theory of Hilbert spaces and contractions to the multiplicative theory of Hilbert spaces and contractions. A *contraction* $A : H_1 \to H_2$ between two Hilbert spaces is a bounded linear map with $\|A\| \le 1$. We define $\Lambda(A)$ to be $A^{\otimes k}$ on $\Lambda^k(H_1) \subset H_1^{\otimes k}$. Then $\Lambda(A)$ gives a bounded linear operator from $\Lambda(H_1)$ to $\Lambda(H_2)$ with $\|\Lambda(A)\| \le 1$. Clearly if $\|A\|, \|B\| \le 1$, then $\Lambda(AB) = \Lambda(A)\Lambda(B)$. Also $\Lambda(A)^* = \Lambda(A^*)$, so if A is unitary, then so too is $\Lambda(A)$. Similarly, if $H_1 = H_2 = H$, then if A is self-adjoint or positive, so too is $\Lambda(A)$. In particular if $A = UP$ is the polar decomposition of A with U unitary, then

$\Lambda(A)=\Lambda(U)\Lambda(P)$ is the polar decomposition of $\Lambda(A)$ by uniqueness. Moreover $\Lambda(A^{it})=\Lambda(A)^{it}$ if A is in addition positive (note that $(A^{it})^{\otimes k}=(A^{\otimes k})^{it}$). Similarly every conjugate-linear contraction T induces an operator $\tilde{\Lambda}(T)(\xi_1\wedge\xi_2\wedge\cdots\wedge\xi_n)=T\xi_n\wedge T\xi_{n-1}\wedge\cdots\wedge T\xi_1$. Note that $\tilde{\Lambda}(T)=\kappa^{-1}\Lambda(iT)$, where κ is the Klein transformation. If $T=UP$ is the polar decomposition of T with U a conjugate-linear unitary, then $\tilde{\Lambda}(T)=\tilde{\Lambda}(U)\Lambda(P)$ is the polar decomposition of $\tilde{\Lambda}(T)$. If U is a linear or conjugate-linear unitary, then it is easy to check that $\Lambda(U)a(\xi)\ \Lambda(U)^*=a(U\xi)$ and $\Lambda(U)c(\xi)\Lambda(U)^*=c(U\xi)$.

Let H be a complex Hilbert space and $K\subset H$ a closed real subspace of H such that $K\cap iK=(0)$ and $K+iK$ is dense in H. For $\xi\in H$ let $a(\xi)$ denote exterior multiplication by ξ and let $c(\xi)=a(\xi)+a(\xi)^*$ denote Clifford multiplication. Thus $c(\xi)c(\eta)+c(\eta)c(\xi)=2\mathrm{Re}(\xi,\eta)$. Since the *-algebra generated by the $a(\xi)$'s acts irreducibly on ΛH and since $a(\xi)=(c(\xi)-ic(i\xi))/2$, the $c(\xi)$' s act irreducibly on ΛH.

Lemma. *If $M(K)$ is the von Neumann algebra generated by the $c(\xi)$'s* ($\xi\in K$), *then Ω is cyclic for $M(K)$.*

Proof. Let $H_0=\overline{M(K)\Omega}$ and assume by induction that all forms of degree N or less lie in H_0. Let ω be an N-form and take $f\in K$. Then $f\wedge\omega=c(f)\omega-a(f)^*\omega$, so that $f\wedge\omega\in H_0$. Since $K+iK$ is dense in H and H_0 is a complex subspace of ΛH, it follows that $\xi\wedge\omega\in H_0$ for all $\xi\in H$. Hence H_0 contains all $(N+1)$-forms.

Since Ω is cyclic for $M(K^\perp)$, which lies in the graded commutant of $M(K)$, it follows that Ω is cyclic and separating for $M(K)$. Let R, T, $\Delta^{it}=(I-R)^{it}R^{-it}$ and J be the corresponding modular operators for $M(K)$ and Ω.

Theorem. (i) $J=\tilde{\Lambda}(j)=\kappa^{-1}\Lambda(ij)$, $\Delta^{it}=\Lambda(\delta^{it})$, *where j and δ^{it} are the modular operators for K.*

(ii) *For $\xi\in H$, $\Delta^{it}c(\xi)\Delta^{-it}=c(\delta^{it}\xi)$ and $\kappa Jc(\xi)J\kappa^{-1}=c(ij\xi)$, where κ is the Klein transformation.*

(iii) *$M(K^\perp)$ is the graded commutant of $M(K)$ and $M(K)'=JM(K)J$* (*Araki duality*).

Remark. For another proof, analogous to that of [24] for bosons and the canonical commutation relations, see [42].

Proof (cf [2]). Let δ^{it} and j be the modular operators associated with the closed real subspace $K\subset H$. Let S be the conjugate-linear operator on $\pi_P(\mathrm{Cliff}_{\mathbb{R}}(K))\Omega$ defined by $Sa\Omega=a^*\Omega$ for $a\in M=\pi_P(\mathrm{Cliff}_{\mathbb{R}}(K))$. This is well-defined, because Ω is separating for M. Thus $Sc(\xi_1)\cdots c(\xi_n)\Omega=c(\xi_n)\cdots c(\xi_1)\Omega$ for $\xi_i\in K$. If the ξ_i's are orthogonal, it follows that $S\xi_1\wedge\cdots\wedge\xi_n=\xi_n\wedge\cdots\wedge\xi_1$. Since any finite dimensional subspace of K admits an orthonormal basis, this formula holds by linearity for arbitrary $\xi_1, \ldots, \xi_n \in K$. Since S is conjugate-linear, it follows that for $\xi_i, \eta_i \in K$ we have $S(\xi_1 + i\eta_1) \wedge \cdots \wedge (\xi_n + i\eta_n) = (\xi_n - i\eta_n) \wedge \cdots \wedge (\xi_1 - i\eta_1)$.

Let $J = \tilde{\Lambda}(j) = \kappa^{-1}\Lambda(ij)$ and $\Delta^{it} = \Lambda(\delta^{it})$. Clearly $\Delta^{it}J = \Delta^{it}J$ and Δ^{it} preserves $\overline{M_{\mathrm{sa}}\Omega}$. To check the KMS condition, it suffices to show that for $x \in M\Omega$, the function $F(t) = \Delta^{it}x$ extends to a bounded continuous function on $-\frac{1}{2} \leq \text{ Im } z \leq 0$, holomorphic on the interior, with $F(t - i/2) = JSF(t)$. We may assume that $x = (\xi_1 + i\eta_1) \wedge \cdots \wedge (\xi_n + i\eta_n)$ with $\xi_i, \eta_i \in K$. For each i, let $f_i(z)$ be continuous bounded function on $-\frac{1}{2} \leq \text{ Im } z \leq 0$, holomorphic in the interior, $f_i(t) = \delta^{it}(\xi_i + i\eta_i)$ and $f_i(t - i/2) = j\delta^{it}(\xi_i - i\eta_i)$. Let $F(z) = f_1(z) \wedge \cdots \wedge f_n(z)$. Then $F(z)$ is bounded and continuous on $-\frac{1}{2} \leq \text{ Im } z \leq 0$, holomorphic in the interior, and $F(t) = \Delta^{it}x$. Now $F(t - i/2) = f_1(t - i/2) \wedge \cdots \wedge f_n(t - i/2) = j\delta^{it}(\xi_1 - i\eta_1) \wedge \cdots \wedge j\delta^{it} \quad (\xi_n - i\eta_n) = \tilde{\Lambda}(j)SF(t) = JSF(t)$. Thus $F(t - i/2) = JSF(t)$ as required. This proves (i) and (ii) follows immediately. To prove (iii), note that $ij(K) = K^\perp$, so that $M(K^\perp) = \kappa JM(K)J\kappa^{-1}$ by this covariance relation. But $M(K^\perp) \subseteq M(K)^q = \kappa M(K)'\kappa^{-1}$. Thus $JM(K)J \subseteq M(K)'$, so the result follows from Lemma 2 in Section 12.

14. Prequantised geometric modular theory

In this section we compute the prequantised modular operators corresponding to fermions on the circle by two methods: firstly using a KMS argument due to Jones reminiscent of computations of Bisognano and Wichmann [4]; and then using the fact that a Hilbert space, endowed with two projections in general position, can be written as a direct integral of two-dimensional irreducible components. Let H be the complex Hilbert space $L^2(S^1, V)$ where $V = \mathbb{C}^N$. We give H a new complex structure by defining multiplication by i as $i(2P - I)$, where P is the orthogonal projection onto Hardy space $H^2(S^1, V)$. Let I be the upper semicircle and let $K = L^2(I, V)$, a real closed subspace of H_P. The real orthogonal projection onto K, regarding H as a real inner product space, is given by Q, multiplication by χ_I.

Theorem. (a) $K \cap iK = (0)$ *and* $K + iK$ *is dense in* H_P.

(b) $K^\perp = L^2(I^c, V)$.

(c) $j = -i(2P - I)$ *where* $Ff(z) = z^{-1}f(z^{-1})$ *is the flip, and* $\delta^{it} = u_t$, *where* $(u_t f)(z) = (z \sinh \pi t + \cosh \pi t)^{-1} f(z \cosh \pi t + \sinh \pi t / z \sinh \pi t + \cosh \pi t)$.

First proof. (a) It suffices to show that P and Q are in general position. Now conjugation by r_π takes Q onto $I - Q$ and fixes P while conjugation by the flip $Vf(z) = z^{-1}f(z^{-1})$ takes Q onto $I - Q$ and P onto $I - P$. Thus it will suffice to show that $PH \cap QH = (0)$. Suppose that the negative Fourier coefficients of $f \in L^2(I, V)$ all vanish. Then so do those of $\psi \star f$ for any $\psi \in C^\infty(S^1)$. But $\psi \star f \in C^\infty(S^1, V)$ is the boundary value of a holomorphic function. If ψ is supported near 1, $\psi \star f$ vanishes in a subinterval of I^c and therefore must vanish identically (since $\psi \star f$ can be extended by reflection across this subinterval). Since $\psi \star f$ and f can be made arbitrarily close in $L^2(S^1, V)$, we must have $f = 0$.

(b) The real orthogonal complement of $L^2(I, V)$ in $L^2(S^1, V)$ is clearly $L^2(I^c, V)$.

(c) Let $K_1 \subset K$ be the dense subset of QH consisting of functions Qp where p is the restriction of a polynomial in $e^{i\theta}$. We must show that the map $f(t) = u_t Qp$ extends to a bounded continuous function $f(z)$ on the closed strip $-1/2 \leq \text{Im } z \leq 0$, holomorphic in the open strip with $f(t - i/2) = jf(t)$ for $t \in \mathbb{R}$. Now $f(t) = Pu_t Qp + (I - P)u_t Qp$. Because of the modified complex structure on $H = PH \oplus (I - P)H$, we have to extend $f_1(t) = Pu_t Qp$ to a holomorphic function with values in PH and $(I - P)u_t Qp$ to an antiholomorphic function with values in $(I - P)H$. Note that if $\theta \in [0, \pi]$ and $-3/4 < \text{Im } z < 1/2$, the function $s_z e^{i\theta} + c_z$ is non-zero, where $s_z = \sinh \pi z$ and $c_z = \cosh \pi z$. For $-3/4 < \text{Im } z < 1/2$, let $p_z(e^{i\theta}) = (s_z e^{i\theta} + c_z)^{-1} p(c_z e^{i\theta} + s_z / s_z e^{i\theta} + c_z)$. Then Qp_z is holomorphic for such z, so $f_1(z) = PQp_z$ gives a holomorphic extension of f_1 to $-3/4 < \text{Im } z < 1/2$. Next note that $f_2(t) = -(I - P)u_t(I - Q)p$, since $(I - P)p = 0$. Set $f_2(z) = -(I - P)(I - Q)p_{\bar{z}}$. This gives an antiholomorphic extension of f_2 to $-3/4 < \text{Im } z < 1/4$, because $s_{\bar{z}} e^{i\theta} + c_{\bar{z}}$ does not vanish for $\theta \in [-\pi, 0]$. Thus $f(z) = f_1(z) + f_2(z)$ is a holomorphic function from $-3/4 < \text{Im } z < 1/2$ into H. It equals $f(t)$ for $t \in \mathbb{R}$. If we show that $f(t - i/2) = jf(t)$, then $f(z)$ will be bounded for $\text{Im } z = 0$ or $-1/2$ and hence, by the maximum modulus principle, on the strip $-1/2 \leq \text{Im } z \leq 0$. Now $jf(t) = -i(2P - I)Ff(t) = -iPQFp_t + i(I - P)(I - Q)Fp_t$. Since $s_{t \pm i/2} = \pm i c_t$ and $c_{t \pm i/2} = \pm i s_t$, we have $p_{t \pm i/2} = \mp iFp_t$. Hence $f_1(t - i/2) = -iPQFp_t$ and $f_2(t - i/2) = i(I - P)(I - Q)Fp_t$, so that $f(t - i/2) = jf(t)$ as required.

Second proof. Let $U : L^2(S^1, V) \to L^2(\mathbb{R}, V)$, $Uf(x) = (x - i)^{-1} f(x + i/x - i)$ be the unitary induced by the Cayley transform. Let $V : L^2(\mathbb{R}, V) \to L^2(\mathbb{R}, V) \oplus L^2(\mathbb{R}, V)$ be the unitary defined by $Vf = (\widehat{f_+}, \widehat{f_-})$, where $\widehat{g}$ denotes the Fourier transform of g and $f_\pm(t) = e^{t/2} f(\pm e^t)$. Let $W = VU : L^2(S^1, V) \to L^2(\mathbb{R}, V) \oplus L^2(\mathbb{R}, V)$. If $e_n(\theta) = e^{in\theta}$, it is easy to check that $We_0 = (g_+, g_-)$ and $We_{-1} = (-g_-, -g_+)$ where $g_\pm(x) = \pi^{\frac{1}{2}}(i \pm 1)e^{\pm \pi x/2}(1 + e^{\pm 2\pi x})^{-1}$.

Clearly WQW^* is the projection onto the first summand $L^2(\mathbb{R}, V)$. Now $Uu_t U^* = v_{2\pi t}$, where $(v_s f)(x) = e^{s/2} f(e^s x)$; and $Vv_s V^* = m(e_s)$, where $e_s(t) = e^{ist}$ and $m(e_s)$ is the corresponding multiplication operator (acting diagonally). Hence $Wu_t W^* = m(e_{2\pi t})$. These operators generate a copy of $L^\infty(\mathbb{R})$ on $L^2(\mathbb{R})$, which by the corollary to Lemma 2 in section 12 equals its own commutant on $L^2(\mathbb{R})$. On the other hand P commutes with u_t and End V, so that WPW^* lies in the commutant of the $m(e_{2\pi t})$'s and End V. Hence $WPW^* = \begin{pmatrix} m(a) & m(b) \\ m(c) & m(d) \end{pmatrix}$ with $a, b, c, d \in L^\infty(\mathbb{R})$. But $Pe_0 = e_0$ and $Pe_{-1} = 0$. Transporting these equations by W, we get equations for a, b, c, d

which can be solved to yield $a(x) = (1 + e^{2\pi x})^{-1}$, $b(x) = -c(x) = ie^{\pi x}(1 + e^{2\pi x})^{-1}$ and $d(x) = e^{2\pi x}(1 + e^{2\pi x})^{-1}$.

These formulas show that WQW^* and WPW^* are in general position, so (a) follows. (b) is clear, since $L^2(I, V)^\perp = L^2(I^c, V)$. To prove (c), note that $e = Q$ and $f = (2P - I)Q(2P - I)$, so that $r = PQP \oplus P^\perp QP^\perp$ and $I - r = PQ^\perp P \oplus P^\perp QP^\perp$. Remembering that r^{it} and $(I - r)^{it}$ must be defined using the complex structure $i(2P - I)$, we get $(I - r)^{it} r^{-it} = (I - A)^{it} A^{-it}$, where $A = PQP \oplus P^\perp Q^\perp P^\perp = QPQ \oplus Q^\perp P^\perp Q^\perp$. Hence $WAW^* = m(a)$ and $W\delta^{it}W^* = m((1 - a)^{it} a^{-it}) = m(e_{2\pi t}) = Wu_t W^*$, so that $\delta^{it} = u_t$. Finally $t = (e - f)/2 = (2P - I)(QP - PQ)$. Now $W(QP - PQ)W^* = W(QPQ^\perp - Q^\perp PQ)$ $W^* = \begin{pmatrix} 0 & m(b) \\ m(b) & 0 \end{pmatrix}$ so that $j = -i(2P - I)F_1$ where $WF_1W^* = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix}$. Now $UFU^* = F'$, where $(F'f)(x) = -f(-x)$, so that $WFW^* = VF'V^* = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix}$. Hence $F_1 = F$, as required.

15. Haag-Araki duality and geometric modular theory for fermions on the circle

Let $H = L^2(S^1) \otimes V$ with $V = \mathbb{C}^N$ and let P be the orthogonal projection onto the Hardy space $H^2(S^1) \otimes V$. Let π_P denote the corresponding irreducible representation of Cliff(H) on fermionic Fock space $\mathscr{F}_V$. For any interval $J \subset S^1$, let $M(J) \subset B(\mathscr{F}_V)$ be the von Neumann algebra generated by the operators $\pi_P(a(f))$ with $f \in L^2(J, V)$. Our main result was obtained jointly with Jones [19, 42]; it follows almost immediately from the previous sections.

Theorem. *Let I denote the upper semicircle with complement $I^c = S^1 \backslash \overline{I}$.*

(a) *The vacuum vector Ω is cyclic and separating for $M(I)$.*

(b) (*Haag-Araki duality*) *$M(I^c)$ is the graded commutant of $M(I)$ and $JM(I)J = M(I)'$, where J is the modular conjugation with respect to Ω.*

(c) (*Geometric modular group*) *Let $I \subset S^1$ be the upper semi-circle. The modular group Δ^{it} of $M(I)$ with respect to the vacuum vector Ω is implemented by u_t, where $(u_t f)(z) = (z \sinh \pi t + \cosh \pi t)^{-1} f(z \cosh \pi t + \sinh \pi t / z \sinh \pi t + \cosh \pi t)$ is the Möbius flow fixing the endpoints of I. In particular $\Delta^{it}\pi_P (a(f))\Delta^{-it} = \pi_P(a(u_t f))$ for $f \in H$.*

(d) (*Geometric modular conjugation*) *If κ is the Klein transformation, then the antiunitary κJ is implemented by F, where $Ff(z) = z^{-1} f(z^{-1})$ is the flip. In particular $J\pi_P(a(f))J = \kappa^{-1}\pi_P(a(Ff))\kappa$ for $f \in H$.*

Remark. Analogous results hold when I is replaced by an arbitrary interval J. This follows immediately by transport of structure using the canonically quantised action of $SU(1, 1)$.

Proof. If $H_P = PH \oplus \overline{P^\perp H}$ (H with multiplication by i given by $i(2P - I)$), then $\mathscr{F}_V = \Lambda H_P$ and $\pi_p(a(f)) = a(Pf) + a(\overline{P^\perp f})^*$ on ΛH_P for $f \in H$. Hence $\pi_P(a(f) + a(f)^*) = c(Pf) + c(\overline{P^\perp f}) = c(f)$ for $f \in H$. Now $M(I)$ coincides with the von Neumann algebra generated by $\pi_P(a(f) + a(f)^*)$ for

$f \in L^2(I, V)$. It therefore may be identified with the von Neumann algebra generated by the $c(f)$ with $f \in K = L^2(I, V)$, a closed real subspace of H_P. From Section 13, the vacuum vector Ω is cyclic for $M(I)$ and $JM(I)J = M(I)' = \kappa^{-1}M(I^c)\kappa$, since $L^2(I, V)^\perp = L^2(I^c, V)$. From Section 14, we see that Δ^{it} is the canonical quantisation of u_t and the antiunitary κJ is the canonical quantisation of F. Finally the relations $\Delta^{it}c(f)\Delta^{-it} = c(u_t f)$ and $\kappa Jc(f)J\kappa^{-1} = c(Ff)$ for $f \in H_P$ immediately imply that $\Delta^{it}\pi_P(a(f))\Delta^{-it} = \pi_P(a(u_t f))$ and $J\pi_P(a(f))J = \kappa^{-1}\pi_P(a(Ff))\kappa$ for $f \in H$.

16. Ergodicity of the modular group

Proposition. *The action $\Lambda(u_t)^{\otimes k}$ of $\mathbb{R}$ on $(\Lambda H_P)^{\otimes k}$ is ergodic, i.e. has no fixed vectors apart from multiples of the vacuum vector $\Omega^{\otimes k}$.*

Proof. First note that the action u_t of $\mathbb{R}$ on $L^2(\mathbb{T})$ is unitarily equivalent to the direct sum of two copies of the left regular representation. In fact the unitary equivalence between $L^2(\mathbb{T})$ and $L^2(\mathbb{R})$ induced by the Cayley transform $Uf(x) = (x-i)^{-1}f(x+i/x-i)$ carries u_t onto the scaling action $v_{2\pi t}$ of $\mathbb{R}$ on $L^2(\mathbb{R})$, where $(v_s f)(x) = e^{s/2}f(e^s x)$. For $f \in L^2(\mathbb{R})$ define $f_\pm \in L^2(\mathbb{R})$ by $f_\pm(t) = e^{t/2}f(\pm e^t)$ and set $W(f) = (f_+, f_-)$. Thus W is an unitary between $L^2(\mathbb{R})$ and $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$. This unitary carries the scaling action of $\mathbb{R}$ onto the direct sum of two copies of the regular representation.

Thus $L^2(\mathbb{T}) \cong L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ as a representation of $\mathbb{R}$. Now $H = L^2(\mathbb{T}, V)$ is a direct sum of copies of $L^2(\mathbb{T})$. On the other hand $\overline{L^2(\mathbb{R})} \cong L^2(\mathbb{R})$ (by conjugation), it follows that both H and $\overline{H}$ are subrepresentations of a direct sum of copies of $L^2(\mathbb{R})$. But $H_P = PH \oplus (I-P)H$ is a subrepresentation of $H \oplus \overline{H}$, so that H_P is unitarily equivalent to a subrepresentation of $L^2(\mathbb{R}) \otimes \mathbb{C}^n$ for some n.

Thus the action of $\mathbb{R}$ on $(\Lambda H_P)^{\otimes k} = \Lambda(H_P \otimes \mathbb{C}^k)$ is unitarily equivalent to a subrepresentation of $\mathbb{R}$ on ΛH_1, where $H_1 = L^2(\mathbb{R}) \otimes \mathbb{C}^m$ for some $m \geq 2$. It therefore suffices to check that $\mathbb{R}$ has no fixed vectors in $\lambda^k H_1$ for $k \geq 1$, since the action of $\mathbb{R}$ preserves degree.

Now $\lambda^k H_1 \subset H_1^{\otimes k}$. On the other hand if $t \mapsto \pi(t)$ is any unitary representation of $\mathbb{R}$ on H and $\lambda(t)$ is the left regular representation on $L^2(\mathbb{R})$, then $\lambda \otimes \pi$ and $\lambda \otimes I$ are unitarily equivalent: the unitary V, defined by $Vf(x) = \pi(x)f(x)$ for $f \in L^2(\mathbb{R}, H) = L^2(\mathbb{R}) \otimes H$, gives an intertwiner. It follows that $H_1^{\otimes k}$ is unitarily equivalent to a direct sum of copies of the left regular representation. Hence $\lambda^k H_1$ is unitarily equivalent to a subrepresentation of a direct sum of copies of the left regular representation. Since the Fourier transform on $L^2(\mathbb{R})$ transforms $\lambda(t)$ into multiplication by $e_t(x) = e^{itx}$, no non-zero vectors in $L^2(\mathbb{R})$ are fixed by λ. Hence there are no non-zero vectors in $\lambda^k H_1$ fixed by $\mathbb{R}$ for $k \geq 1$, as claimed.

Corollary. *The modular group acts ergodically on the local algebra $M(I) = \pi_P(\mathrm{Cliff}(L^2(I, V)))''$, i.e. it fixes only the scalar operators. In particular $M(I)$ must be a factor* [*in fact a type* III_1 *factor*].

Proof. Suppose that $x \in M(I)$ is fixed by the modular group. Then $x\Omega$ is fixed by the modular group, so that $x\Omega = \lambda\Omega$ for $\lambda \in \mathbb{C}$. Since Ω is separating for $M(I)$, this forces $x = \lambda I$. Since the modular group fixes the centre, $M(I)$ must be a factor.

17. Consequences of modular theory for local loop groups

Using only Haag-Araki duality for fermions and Takesaki devissage, we establish several important properties of the von Neumann algebras generated by local loop groups in positive energy representations. These include Haag duality in the vacuum representation, local equivalence, the fact that local algebras are factors and a crucial irreducibility property for local loop groups. This irreducibility result will be deduced from a von Neumann density result, itself a consequence of a generalisation of Haag duality; it can also be deduced from a careful study of the topology on the loop group induced by its positive energy representations.

Let L_IG be the local loop group consisting of loops concentrated in I, i.e. loops equal to 1 off I, and let $\mathscr{L}_IG$ be the corresponding subgroup of $\mathscr{L}G$. We need to know in what sense these subgroups generate LG.

Covering lemma. *If $S^1 = \bigcup_{k=1}^n I_k$, then LG is generated by the subgroups $L_{I_k}G$.*

Proof. By the exponential lemma we just have to prove that every exponential $\exp(X)$ lies in the group generated by $L_{I_k}G$. Let $(\psi_k) \subset C^\infty(S^1)$ be a smooth partition of the identity subordinate to (I_k). Then $X = \sum \psi_k \cdot X$, so that $\exp(X) = \exp(\psi_1 \cdot X) \cdots \exp(\psi_n \cdot X)$ with $\exp(\psi_k \cdot X) \in L_{I_k}G$.

Let $\pi : LSU(N) \to PU(\mathscr{F}_V)$ be the basic representation of $LSU(N)$, so that $\pi(g)\pi_P(a(f))\pi(g)^* = \pi_P(a(g \cdot f))$ for $f \in L^2(S^1, V)$ and $g \in LSU(N)$. Let π_i be an irreducible positive energy representation of level ℓ. Haag-Araki duality and the fermionic construction of π_i imply that operators in $\pi_i(L_IG)$ and $\pi_i(L_{I^c}G)$, defined up to a phase, actually commute ("locality"):

Proposition (locality). *For any positive energy representation π_i, we have $\pi_i(g)\pi_i(h)\pi_i(g)^*\pi_i(h)^* = I$ for $g \in \mathscr{L}_ISU(N)$ and $h \in \mathscr{L}_{I^c}SU(N)$.*

Proof. As above let $M(I) \subset B(\mathscr{F}_V)$ be the von Neumann algebra generated by fermions $a(f)$ with $f \in L^2(I, V)$. Since $\pi(g)$ commutes with $M(I^c)$ and is even, it must lie in $M(I)$ by Haag-Araki duality. Similarly $\pi(h)$ lies in $M(I)$. Since they are both even operators they must therefore commute. Clearly this result holds also with $\pi^{\otimes\ell}$ in place of π and passes to any subrepresentation π_i of $\pi^{\otimes\ell}$.

The embedding of $LSU(N)$ in $LSU(N\ell)$ gives a projective representation Π on $\mathscr{F}_W$ where $W = (\mathbb{C}^N)^{\otimes\ell}$. Now $\mathscr{F}_W$ is can naturally be identified with $\mathscr{F}_V^{\otimes\ell}$ and under this identification $\Pi = \pi^{\otimes\ell}$. Let $M = \pi_P(\mathrm{Cliff}(L^2(I, W)))''$ and let $N = \pi^{\otimes\ell}(\mathscr{L}_ISU(N))'' = \Pi(\mathscr{L}_ISU(N))''$, so that $N \subset M$. The operators u_t and F lie in $SU_{\pm}(1,1)$ so are compatible with the central extension $\mathscr{L}G$ introduced in section 5. It follows immediately that N is invariant under the modular group of M. In order to identify $\overline{N\Omega}$ we need a preliminary result.

Reeh-Schlieder theorem. *Let π be an irreducible positive energy projective representation of LG on H and let v be a finite energy vector (i.e. an eigenvector for rotations). Then the linear span of $\pi(\mathscr{L}_I G)v$ is dense in H.*

Proof (cf [32]). It suffices to show that if $\eta \in H$ satisfies $(\pi(g)v, \eta) = 0$ for all $g \in \mathscr{L}_I G$, then $\eta = 0$. We start by using the positive energy condition to show that this identity holds for all $g \in LG$. For $z_1, \dots, z_n \in \mathbb{T}$ and $g_1, \dots, g_n \in \mathscr{L}_J G$, where $J \subset\subset I$, consider $F(z_1, \dots, z_n) = (R_{z_1}\pi(g_1)\, R_{z_2}\pi(g_2) \cdots R_{z_n}\pi(g_n)v, \eta)$. This vanishes if all the z_i's are sufficiently close to 1. Now freeze $z_1, \dots, z_{n-1}$. As a function of z_n, the positive energy condition implies that the function F extends to a continuous function on the closed unit disc, holomorphic in the interior and vanishing on the unit circle near 1. By the Schwarz reflection principle, F must vanish identically in z_n. Now freeze all values of z_i except z_{n-1}. The same argument shows that F vanishes for all values of z_{n-1}, and so on. After n steps, we see that F vanishes for all values of z_i on the unit circle. Thus $(\pi(g)v, \eta) = 0$ for all g in the group generated by $\mathscr{L}_J G$ and its rotations, i.e. the whole group $\mathscr{L}G$. Therefore, since π is irreducible, $\eta = 0$ as required.

We may now apply Takesaki devissage with the following consequences.

Theorem A (factoriality). $N = \pi^{\otimes \ell}(\mathscr{L}_I G)''$, *and hence each isomorphic* $\pi_i(\mathscr{L}_I G)''$, *is a factor.*

Proof. By Takesaki devissage, N has ergodic modular group and therefore must be a factor. If p_i is a projection in $\pi^{\otimes \ell}(LG)' \subset \pi^{\otimes \ell}(\mathscr{L}_I G)'$ corresponding to the irreducible positive energy representation H_i, then $\pi_i(\mathscr{L}_I G)''$ is isomorphic to $\pi^{\otimes \ell}(\mathscr{L}_I G)'' p_i \cong N$ and is therefore also a factor.

Theorem B (local equivalence). *For every positive energy representation π_i of level ℓ, there is a unique *-isomorphism $\pi_i : \pi_0(\mathscr{L}_I G)'' \to \pi_i(\mathscr{L}_I G)''$ sending $\pi_0(g)$ to $\pi_i(g)$ for all $g \in \mathscr{L}_I G$. If $\mathscr{X} = \mathrm{Hom}_{\mathscr{L}_I G}(H_0, H_i)$, then $\mathscr{X}\Omega$ is dense in H_i and $\pi_i(a)T = Ta$ for all $T \in \mathscr{X}$ and $a \in \pi_0(\mathscr{L}_I G)''$. If $\mathscr{X}_0$ is a subspace of $\mathscr{X}$ with $\mathscr{X}_0 H_0$ dense in H_i, then $\pi_i(a)$ is the unique operator $b \in B(H_i)$ such $bT = Ta$ for all $T \in \mathscr{X}_0$.*

Proof. This is immediate from the proposition in Section 10, since π_0 and π_i are subrepresentations of the factor representation $\pi^{\otimes \ell} \otimes I$. Since $\mathscr{X} = \mathscr{X}\pi_0(\mathscr{L}_I G)$ and Ω is cyclic for $\pi_0(\mathscr{L}_I G)$, it follows that $\overline{\mathscr{X}\Omega} = \overline{\mathscr{X}H_0} = H_i$.

Remarks. Note that, if p_i, p_j are projections onto copies of H_i, H_j in $\mathscr{F}_W$, explicit intertwiners $H_j \to H_i$ are given by compressed fermi fields $p_i a(f) p_j$ with f supported in I^c; these are essentially the smeared vector primary fields that we study in Chapter IV. Theorem B is a weaker version of the much stronger result that the restrictions of π_0 and π_i to $\mathscr{L}_I G$ are unitarily equivalent. This follows because $\pi^{\otimes \ell}$ restricts to a type III factor representation of $\mathscr{L}_I G$ (because the modular group is ergodic). Thus any non-zero subrepresentations are unitarily equivalent. Local equivalence may also be proved more directly using an argument of Borchers [6] to show that the local algebras are "properly infinite" instead of type III (see [42] and [43]).

Theorem C (Haag duality). *If π_0 is the vacuum representation at level ℓ, then $\pi_0(\mathscr{L}_I G)'' = \pi_0(\mathscr{L}_{I^c} G)'$. The corresponding modular operators are geometric. Analogous results hold when I is replaced by an arbitrary interval.*

Remark. Locality leads immediately to the canonical so-called "Jones-Wassermann" inclusion $\pi_i(\mathscr{L}_I G)'' \subseteq \pi_i(\mathscr{L}_{I^c} G)'$ [19, 41]. This inclusion *measures the failure of Haag duality* in non-vacuum representations.

Proof. By the Reeh-Schlieder theorem, the vacuum vector is cyclic for $\pi_0(\mathscr{L}_I G)''$, and hence $\pi_0(\mathscr{L}_I G)'$ (since it contains $\pi_0(\mathscr{L}_{I^c} G)''$) . Let e be the projection onto $\overline{N\Omega}$. Then $N \to Ne$, $x \mapsto xe$ is an isomorphism. Clearly Ne may be identified with $\pi_0(\mathscr{L}_I G)''$. Its commutant is $JNJe$, so $\pi_0(\mathscr{L}_{I^c} G)''$. The identification of the modular operators is immediate. Now $SU(1,1) = SU_+(1,1)$ acts on the vacuum representation fixing the vacuum vector and carries I onto any other interval of the circle. Since the modular operators lie in $SU_\pm(1,1)$, the results for an arbitrary interval follow by transport of structure.

Theorem D (generalised Haag duality). *Let e be the projection onto the vacuum subrepresentation of $\pi^{\otimes \ell}$. Then $\pi_P(\mathrm{Cliff}(L^2(I,W)))'' \bigcap (\mathbb{C}e)' = \pi^{\otimes \ell} (\mathscr{L}_I G)''$. Moreover $\pi^{\otimes \ell}(\mathscr{L}_I G)''$ is the subalgebra of the "observable algebra" $\pi^{\otimes \ell}(LG)''$ commuting with all fields $\pi_P(a(f))$ with f localised in I^c.*

Proof. The first assertion is just the second of the Jones relations $N = \{x \in M : ex = xe\}$ and therefore a consequence of Takesaki devissage. To prove the second, note that

$$\begin{aligned}\pi^{\otimes \ell}(\mathscr{L}_I G)'' \subseteq \pi_P(\mathrm{Cliff}(L^2(I,W)))'' \bigcap \pi^{\otimes \ell}(LG)'' &\subseteq \pi_P(\mathrm{Cliff}(L^2(I,W)))'' \\ &\bigcap (\mathbb{C}e)' = \pi^{\otimes \ell}(\mathscr{L}_I G)''.\end{aligned}$$

Thus we obtain $\pi^{\otimes \ell}(\mathscr{L}_I G)'' = \pi_P(\mathrm{Cliff}(L^2(I,W)))'' \bigcap \pi^{\otimes \ell}(LG)''$. But $\pi_P(\mathrm{Cliff}(L^2(I,W)))''$ is equal to the graded commutant of $\pi_P(\mathrm{Cliff}(L^2(I^c,W)))$. Since all operators in $\pi^{\otimes \ell}(LG)''$ are even, it follows that $\pi_P(\mathrm{Cliff}(L^2(I^c,W)))' \bigcap \pi^{\otimes \ell}(LG)'' = \pi^{\otimes \ell}(\mathscr{L}_I G)''$, as required.

Theorem E (von Neumann density). *Let I_1 and I_2 be touching intervals obtained by removing a point from the proper interval I. Then if π is a positive* energy representation of LG (*not necessarily irreducible*), *we have* $\pi(\mathscr{L}_{I_1}G)'' \vee \pi(\mathscr{L}_{I_2}G)'' = \pi(\mathscr{L}_I G)''$ ("*irrelevance of points*").

Proof. By local equivalence, there is an isomorphism π between $\pi_0(\mathscr{L}_I G)''$ and $\pi(\mathscr{L}_I G)''$ taking $\pi_0(g)$ onto $\pi(g)$ for $g \in \mathscr{L}_I G$. Thus π carries $\pi_0(\mathscr{L}_{I_1}G)'' \vee \pi_0(\mathscr{L}_{I_2}G)''$ onto $\pi(\mathscr{L}_{I_1}G)'' \vee \pi(\mathscr{L}_{I_2}G)''$. It therefore suffices to prove the result for the vacuum representation π_0. Let $J_1 = I_1^c$ and $J_2 = I_2^c$. Now for $k = 1, 2$ we have $\pi^{\otimes\ell}(\mathscr{L}_{J_k}G)'' = \pi_P(\mathrm{Cliff}(L^2(I_k, W)))' \cap (\mathbb{C}e)'$. So

$$\begin{aligned}\pi^{\otimes\ell}(\mathscr{L}_{J_1}G)'' \cap \pi^{\otimes\ell}(\mathscr{L}_{J_2}G)'' &= \pi_P(\mathrm{Cliff}(L^2(I_1, W)))' \cap \pi_P(\mathrm{Cliff}(L^2(I_2, W)))' \cap (\mathbb{C}e)' \\ &= \pi_P(\mathrm{Cliff}(L^2(I, W)))' \cap (\mathbb{C}e)' = \pi^{\otimes\ell}(\mathscr{L}_{I^c}G)''.\end{aligned}$$

Here we have used Theorem C and the equality $L^2(I, W) = L^2(I_1, W) \oplus L^2(I_2, W)$. Taking commutants, we get $\pi^{\otimes\ell}(\mathscr{L}_{J_1}G)' \vee \pi^{\otimes\ell}(\mathscr{L}_{J_2}G)' = \pi^{\otimes\ell}(\mathscr{L}_{I^c}G)'$. Compressing by e, this yields $\pi_0(\mathscr{L}_{J_1}G)' \vee \pi_0(\mathscr{L}_{J_2}G)' = \pi_0(\mathscr{L}_{I^c}G)'$. Using Haag duality in the vacuum representation to identify these commutants, we get $\pi_0(\mathscr{L}_{I_1}G)'' \vee \pi_0(\mathscr{L}_{I_2}G)'' = \pi_0(\mathscr{L}_I G)''$, as required.

Theorem F (irreducibility). *Let A be finite subset of S^1 and let $L^A G$ be the subgroup of LG consisting of loops trivial to all orders at points of A. Let $\mathscr{L}^A G$ be the corresponding subgroup of $\mathscr{L}G$. If π is a positive energy representation of LG* (*not necessarily irreducible*), *we have $\pi(L^A G)'' = \pi(LG)''$. In particular the irreducible positive energy representations of LG stay irreducible and inequivalent when restricted to $L^A G$.*

Proof. Clearly $\mathscr{L}^A G = \mathscr{L}_{I_1}G \cdot \cdots \cdot \mathscr{L}_{I_n}G$, if $S^1 \backslash A$ is the disjoint union of the consecutive intervals $I_1, \ldots, I_n$. Let J_k be the interval obtained by adding the common endpoint to $I_k \cup I_{k+1}$ (we set $I_{n+1} = I_1$). By von Neumann density, $\pi(\mathscr{L}_{I_k}G)'' \vee \pi(\mathscr{L}_{I_{k+1}}G)'' = \pi(\mathscr{L}_{J_k}G)''$. Hence $\pi(\mathscr{L}^A G)'' = \bigvee \pi(\mathscr{L}_{J_k}G)''$. But the subgroups $\mathscr{L}_{J_k}G$ generate $\mathscr{L}G$ algebraically. Hence $\pi(\mathscr{L}^A G)'' = \pi(\mathscr{L}G)''$. Taking commutants, we get $\pi(\mathscr{L}^A G)' = \pi(\mathscr{L}G)'$. By Schur's lemma, this implies that the irreducible positive energy representations of LG stay irreducible and inequivalent when restricted to $L^A G$.

Remark. Direct proofs of Haag duality (Theorem C) have been discovered since the announcement in [19] that do not use Takesaki devissage from fermions. Theorems A, B and F can also be proved without using Takesaki devissage. In fact Jones and I proved in [42] that the topology on $\mathscr{L}G$ induced by pulling back the strong operator topology on $U(\mathscr{F}_P)$ makes $\mathscr{L}^A G$ dense in $\mathscr{L}G$. Since any level ℓ representation π is continuous for this topology, it follows that $\pi(\mathscr{L}^A G)$ is dense in $\pi(\mathscr{L}G)$ in the strong operator topology. So $\pi(\mathscr{L}^A G)'' = \pi(\mathscr{L}G)''$ and Theorem F follows. The reader is warned that several incorrect proofs of these results have appeared in published articles.

III. The basic ordinary differential equation

18. The basic ODE and the transport problem

Consider the ODE

$$\frac{df}{dz} = \frac{Pf}{z} + \frac{Qf}{1-z} \tag{1}$$

where $f(z)$ takes values in $V = \mathbb{C}^N$ and $P, Q \in$ End V. Suppose that P has distinct eigenvalues λ_i with corresponding eigenvectors ξ_i, none of which differ by positive integers, and Q is a non-zero multiple of a rank one idempotent in general position with respect to P. Thus $Q^2 = \delta Q$, $\mathrm{Tr}(Q) = \delta$ with $\delta \neq 0$, so that $Q(x) = \phi(x)v$ for $v \in V$, $\phi \in V^*$ with $\phi(v) = \delta$. "General position" means that $v = \sum \delta_i \xi_i$ with $\delta_i \neq 0$ for all i and $\phi(\xi_i) \neq 0$ for all i; the eigenvectors can therefore be normalised so that $\phi(\xi_i) = 1$. Let $R = Q - P$ and suppose that R satisfies the same conditions as P with respect to Q. Let $(\zeta_j, -\mu_j)$ be the normalised eigenvectors and eigenvalues of R. Let $f_i(z) = \sum \xi_{i,n} z^{\lambda_i + n}$ be the formal power series solutions of (1) expanded about 0 with $\xi_{i,0} = \xi_i$. The $f_i(z)$'s are defined and converge in $\{z : |z| < 1, z \notin [0,1)\}$. If $g(z) = f(z^{-1})$, then

$$\frac{dg}{dz} = \frac{Rg}{z} + \frac{Qg}{1-z}, \tag{2}$$

so we can look for formal power series solutions $h_j(z) = \sum \zeta_{j,n} z^{\mu_j - n}$ of (1) expanded about ∞ with $\zeta_{j,0} = \zeta_j$. The $h_j(z)$'s are defined and converge in $\{z : |z| > 1, z \notin [1,\infty)\}$. The solutions $f_i(z)$ and $h_j(z)$ extend analytically to single-valued holomorphic functions on $\mathbb{C} \backslash [0,\infty)$.

Problem. *Compute the transport coefficients c_{ij} for which $f_i(z) = \sum c_{ij} h_j(z)$ for $z \in \mathbb{C} \backslash [0,\infty)$.*

This problem will be solved by finding a rational canonical form for the matrices P, Q, R which links the ODE with the generalised hypergeometric equation, first studied by Thomae. It can be seen directly that the projected solutions $(1-z)\phi(f_i(z))$ can be represented by multiple Euler integrals. This allows one coefficient of the transport matrix (c_{ij}) to be computed when the λ_i's and μ_j's are real and δ is negative. The rational canonical form shows that the transport matrices are holomorphic functions of the λ_i's and μ_j's alone, symmetric in an obvious sense. So the computation of the c_{ij}'s follows by analytic continuation and symmetry from the particular solution:

Theorem. *The coefficients of the transport matrix are given by the formula*

$$c_{ij} = e^{i\pi(\lambda_i - \mu_j)} \frac{\prod_{k \neq i} \Gamma(\lambda_i - \lambda_k + 1) \prod_{\ell \neq j} \Gamma(\mu_j - \mu_\ell)}{\prod_{\ell \neq j} \Gamma(\lambda_i - \mu_\ell + 1) \prod_{k \neq i} \Gamma(\mu_j - \lambda_k)}.$$

For applications it will be convenient to have a slightly generalised version of this result. Let B be a matrix of the form $-\alpha I + \beta Q$ ($\beta \neq 0$) where Q is a rank one idempotent. Let A be a matrix such that both A and $B - A$ are in general position with respect to Q and have distinct eigenvalues not differing by integers (so distinct). Around 0 the ODE

$$\frac{df}{dz} = \frac{Af}{z} + \frac{Bf}{1-z} \tag{3}$$

has a canonical basis of solutions $f_i(z) = \xi_i z^{\lambda_i} + \xi_{i,1} z^{\lambda_i+1} + \cdots$, where $A\xi_i = \lambda_i \xi_i$ and $\phi(\xi_i) = 1$ if $Q(\xi) = \phi(\xi)v$. Similarly around ∞, the ODE has a canonical basis of solutions $h_j(z) = \zeta_i z^{\mu_i} + \zeta_{i,1} z^{\mu_i - 1} + \cdots$ where $(A - B)\zeta_i = \mu_i \zeta_i$ and $\phi(\zeta_i) = 1$.

Corollary. *In* $\mathbb{C}\backslash[0,\infty)$ *we have* $f_i(z) = \sum c_{ij} h_j(z)$, *where*

$$c_{ij} = e^{i\pi(\lambda_i - \mu_j)} \frac{\prod_{k\neq i} \Gamma(\lambda_i - \lambda_k + 1) \prod_{\ell \neq j} \Gamma(\mu_j - \mu_\ell)}{\prod_{\ell \neq j} \Gamma(\lambda_i - \mu_\ell + \alpha + 1) \prod_{k \neq i} \Gamma(\mu_j - \lambda_k - \alpha)}.$$

Proof. By a gauge transformation $f(z) \mapsto (1-z)^\gamma f(z)$, the ODE (3) is changed into the ODE considered before. It is then trivial to check that the transport relation for that ODE implies the stated transport relation for (3).

19. Analytic transformation of the ODE (*cf* [*17*])

Consider the ODE $f'(z) = A(t,z)f(z)$ where $A(t,z) = \sum_{n\geq 0} A_n(t) z^{n-1}$ with each matrix $A_n(t) \in$ End V a polynomial (or holomorphic function) in $t \in W = \mathbb{C}^m$ and $A(t,z)$ is convergent in $0 < |z| < R$ for all $t \in \mathbb{C}^m$.

Proposition. *Let* $U = \{t \in \mathbb{C}^m : A_0(t)$ *has no eigenvalues differing by positive integers*$\}$. *For* $t \in U$, *there is a unique gauge transformation* $g(t,z) \in GL(V)$, *holomorphic on* $U \times \{z : |z| < R\}$, *such that* $g(t,z)^{-1} A(t,z) g(t,z) - g(t,z)^{-1} \partial g(t,z)/\partial z = A_0(t)/z$.

Proof. If we write $g(t,z) = \sum_{n\geq 0} g_n(t) z^n$ with $g_0(t) = I$, then the $g_n(t)$'s are given by the recurrence relation

$$n g_n(t) = n(n - \text{ad } A_0(t))^{-1} \sum_{m=1}^{n} A_m(t) g_{n-m}(t).$$

Let $\overline{B}$ be a closed ball in U. Then $\sup_n \|n(n - \mathrm{ad}\, A_0(t))^{-1}\|$ is bounded by $M < \infty$ on $\overline{B}$. So $\|g_n(t)\|$ is bounded on $\overline{B}$ by the solutions f_n of the recurrence relation

$$nf_n = \sum_{m=1}^{n} b_m f_{n-m},$$

where $b_m = M \sup_{t\in\overline{B}} \|A_m(t)\|$ and $\sum_{m\geq 1} b_m z^m$ is convergent in $|z| < R$. But then $f(z) = \sum_{n\geq 0} f_n z^n$ is the formal power series solution of $zf'(z) = (\sum_{m\geq 1} b_m z^m) f(z)$ with $f(0) = 1$, i.e. $f'(z) = b(z)f(z)$ where $b(z) = \sum_{m\geq 0} b_{m+1} z^m$. This has the unique solution $f(z) = \exp \int_0^z b(w)dw$ so that in particular $f(z) = \sum f_n z^n$ is convergent in $|z| < R$. Since $\|g_n(t)\| \leq f_n$, it follows that $\sum g_n(t) z^n$ converges uniformly on $\{(t,z) : t \in \overline{B}, |z| \leq r\}$ for any $r < R$. Since $t \mapsto g_n(t)$ is holomorphic in t, for fixed z, $g(z,t)$ is the uniform limit on compacta of holomorphic functions in t. Since the uniform limit on compacta of holomorphic functions is holomorphic, it follows that $t \mapsto g(t,z)$ is holomorphic on U for fixed z.

To show that $g(t,z)$ is invertible for fixed t, note that $\partial_z g = Ag - gA_0/z$. Replacing A by $-A^t$, we find f such that $\partial_z f = -fA + A_0 f/z$. Hence $\partial_z(fg) = [A_0, fg]/z$. The only formal power series solution h of this equation with $h(0) = I$ is $h \equiv I$. Hence $fg \equiv I$ as required.

Remarks. This argument applies also when $A_0(t) = 0$. Clearly we may apply the proposition to the basic ODE. The argument with $A_0(t) = 0$ near points $z \neq 0, 1$ shows that the gauge transformation $g(z)$ extends to a holomorphic map $\mathbb{C}\backslash[1,\infty) \to GL(N,\mathbb{C})$ such that $g(z)^{-1}A(z)g(z) - g(z)^{-1}g'(z) = A_0/z$ for $z \notin [1,\infty)$. The gauge transformation reduces the basic ODE about 0 to the ODE $f'(z) = z^{-1}A_0 f(z)$ which has solutions $z^{A_0}v = \exp(A_0 \log z)v$ defined in $\mathbb{C}\backslash[0,\infty)$ say. Applying the gauge transformation, it follows that any formal power series solution of the original ODE is automatically convergent in $|z| < 1$ and extends to a single-valued holomorphic function on $\mathbb{C}\backslash[0,\infty)$.

20. Algebraic transformation of the ODE

Let P be a matrix with distinct eigenvalues λ_i and corresponding eigenvectors v_i. Let Q be proportional to a rank one idempotent on V so that $Q(x) = \phi(x)v$ with $\phi \in V^*, v \in V$ and $\phi(v) = \delta \neq 0$. We assume that P is in general position with respect to Q. This means that the eigenvectors ξ_i satisfy $\phi(v_i) \neq 0$ and that $v = \sum \alpha_i \xi_i$ with $\alpha_i \neq 0$ for all i. The next result gives a rational canonical form for the matrices P, Q and R.

Proposition (Rational Canonical Form). *If P has distinct eigenvalues and Q is a non-zero multiple of a rank one idempotent in general position with respect to P, there is a (non-orthonormal!) basis of V such that*

$$P=\begin{pmatrix} 0 & 1 & 0 & & & 0 \\ 0 & 0 & 1 & & & 0 \\ 0 & 0 & 0 & 1 & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & & & & & 1 \\ a_1 & a_2 & & & & a_N \end{pmatrix}, \quad Q=\begin{pmatrix} 0 & 0 & 0 & & & 0 \\ & & & & & \\ & & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & & \\ b_1 & b_2 & & & & b_N \end{pmatrix},$$

$$-R=P-Q=\begin{pmatrix} 0 & 1 & 0 & & & 0 \\ 0 & 0 & 1 & & & 0 \\ 0 & 0 & 0 & 1 & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & & & & & 1 \\ c_1 & c_2 & & & & c_N \end{pmatrix},$$

where $b_N = \mathrm{Tr}(Q) \neq 0$ and $c_i = a_i - b_i$. Conversely if P and Q are of the above form and the roots of $a(t) = t^N - \sum a_i t^{i-1}$ (the characteristic polynomial of P) are distinct, then P and Q are in general position iff $b(t) = \sum b_i t^{i-1}$ and $a(t)$ have no common roots iff $c(t) = a(t) - b(t)$ and $a(t)$ have no common roots. (Here $c(t)$ is the characteristic polynomial of $P - Q$.)

Remark. This gives a unique canonical form for $P, Q, R = Q - P$ with equivalence given by conjugation by matrices in $GL(N, \mathbb{C})$: for $a(t)$ and $c(t)$ are the characteristic polynomials of P and $P - Q$, so that the constants a_i, b_j are invariants (since $b(t) = a(t) - c(t)$). Moreover the orbit space of the pairs (P, R) under the action by conjugation of $GL(N, \mathbb{C})$ can naturally be identified with the space of rational canonical forms.

Proof. Let $Q(x) = \phi(x)v$, with $\phi(v) \neq 0$. Since Q and P are in general position, the elements $\phi, \phi \circ P, \cdots, \phi \circ P^{N-1}$ form a basis of V^*. In particular there is a unique solution w of $\phi(w) = \phi(Pw) = \cdots = \phi(P^{N-2}w) = 0$, $\phi(P^{N-1}w) = 1$. The set $w, Pw, \ldots, P^{N-1}w$ must be linearly independent, because otherwise $P^{N-1}w$ would have to be a linear combination of $w, Pw, \ldots, P^{N-2}w$ contradicting $\phi(P^{N-1}w) = 1$. Thus $(P^j w)$ is a basis of V. Clearly P and Q have the stated form with respect to this basis. Furthermore $b_N = \mathrm{Tr}(Q)$.

We next must check that if P and Q have the stated form, then no eigenvector $u \neq 0$ of P can satisfy $Qu = 0$ and no eigenvector ψ of P^t can satisfy $Q^t\psi = 0$. For ψ, the condition $Q^t\psi = 0$ means that $\psi = (x_1, x_2, \ldots, x_{N-1}, 0)$ with $x_i \in \mathbb{C}$. The condition $P^t\psi = \lambda\psi$ forces $x_1 = \lambda x_2$, $x_2 = \lambda x_3$, $\ldots$, $x_{N-1} = 0$. Hence $x_i = 0$ for all i and $\psi = 0$. Now suppose that $Pu = \lambda u$ and $Qu = 0$. Then it is easily verified that u is proportional to $(1, \lambda, \lambda^2, \ldots, \lambda^{N-1})^t$. Thus $Qu = (0, 0, \ldots, 0, b(\lambda))^t$, so that $Qu \neq 0$ iff $b(\lambda) \neq 0$. Finally the characteristic polynomial of R is $c(t) = a(t) - b(t)$. Clearly $a(t)$ and $b(t)$ have no common roots iff $c(t)$ and $b(t)$ have no common roots, so the last assertion follows.

21. Symmetry and analyticity properties of transport matrices

Proposition. *The transport matrix c_{ij} from 0 to ∞ of the basic ODE depends only on the eigenvalues λ_i of P and μ_j of $P - Q$. This dependence is holomorphic. Moreover the coefficients c_{ij}, indexed by the eigenvalues λ_i and μ_j, have the symmetry property $c_{ij}(\lambda_1, \ldots, \lambda_N, \mu_1, \ldots, \mu_N) = c_{\sigma i, \tau j}(\lambda_{\sigma 1}, \ldots, \lambda_{\sigma N}, \mu_{\tau 1}, \ldots, \mu_{\tau N})$ for $\sigma, \tau \in S_N$.*

Proof. We can conjugate by a matrix in $GL(N, \mathbb{C})$ so that P, Q and R are in rational canonical form. The transport matrix from 0 to ∞ is invariantly defined, so does not change under such a conjugation. Thus the assertions are invariant under conjugation, so it suffices to prove them when P, Q, R are in rational canonical form. Setting $g(z) = f(z/(z-1))$, where $f(z)$ is a solution of the basic ODE, we get the ODE

$$\frac{dg}{dz} = \frac{Pg}{z} + \frac{Rg}{z-1} \tag{4}$$

where $R = Q - P$. Thus we have to compute the transport matrices for (4) from 0 to 1 where the solutions at 0 are labelled by the eigenvalues λ_i of P and at 1 by the eigenvalues of μ_j of $-R$. We shall consider variations of P, Q, and R within rational canonical form. P and R can be specified by prescribing the eigenvalues (λ_i) of P and (μ_j) of $-R$. This completely determines the a_i's and c_i's and hence the b_i's. The λ_i's and μ_j's should be distinct and no two λ_i's or μ_j's should differ by a positive integer. We also impose the linear constraint that $\sum \lambda_i - \mu_i \neq 0$. Thus we obtain an open path-connected subset U_0 of the $2N$-dimensional linear space $W = \{(\lambda, \mu)\} = \mathbb{C}^{2N}$. Applying the proposition in section 19 with $t = (\lambda, \mu) \in W$ and $A(t, z) = z^{-1}P + (z-1)^{-1}R$, we deduce that the gauge transformations $g(t, z)$, $h(t, z)$ transforming $A(t, z)$ into $z^{-1}P$ and $(z-1)^{-1}R$ respectively depend holomorphically on $t \in U$ for a fixed $z \in (0, 1)$. We already saw in section 20 that the normalised eigenvectors of P and R are given by

$$\xi_i(t) = b(\lambda_i)^{-1}(1, \lambda_i, \lambda_i^2, \ldots, \lambda_i^{N-1})^t \quad \zeta_j(t) = b(\mu_j)^{-1}(1, \mu_j, \mu_j^2, \ldots, \mu_j^{N-1})^t.$$

Thus the normalised solutions at 0 are $z^{\lambda_i} g(t, z)\xi_i(t)$ and the normalised solutions at 1 are given by $(z-1)^{\mu_j} h(t, z)\zeta_j(t)$. So the transport matrix $c_{ij}(t)$ (independent of z) is specified by the equation

$$z^{\lambda_i} g(t, z)\xi_i(t) = \sum c_{ij}(t)(z-1)^{-\mu_j} h(t, z)\zeta_j(t)$$

for $|z - 1/2| < 1/2$. Fix such a value of z (say $z = 1/2$) and let $(\psi_j(t))$ be the dual basis to $(\zeta_j(t))$. Clearly $\psi_j(t)$ is a rational function of (λ, μ) so is holomorphic on U. Moreover

$$c_{ij}(t) = (z-1)^{\mu_j} z^{\lambda_i} (\psi_j(t), h(t,z)^{-1} g(t,z) \xi_i(t)).$$

This equation shows that $c_{ij}(t)$ depends holomorphically on $t \in U_0$ and has the stated symmetry properties.

22. Projected power series solutions

Let $\lambda = \lambda_i$ be an eigenvalue of P and consider the corresponding (formal) power series solution $f_i(z) = \sum \xi_{i,n} z^{\lambda_i + n}$ of the basic ODE. Dropping the index i for clarity, we have

$$zf'(z) = Pf + Q(z + z^2 + z^3 + \cdots) f,$$

with $f(z) = \sum \xi_n z^{\lambda+n}$ and $P\xi_0 = \lambda \xi_0$. Substituting in the formal power series and dividing out by z^λ, we get

$$\sum_{n \geq 0} (n + \lambda) \xi_n z^n = \sum_{n \geq 0} P \xi_n z^n + Q(z + z^2 + z^3 + \cdots) \sum_{n \geq 0} \xi_n z^n.$$

Thus for $n \geq 1$ we get

$$(n + \lambda - P)\xi_n = Q(\xi_0 + \cdots + \xi_{n-1})$$

and hence

$$Q\xi_n = Q(n + \lambda - P)^{-1} Q(\xi_0 + \cdots + \xi_{n-1}).$$

Let $Q(\xi_0 + \cdots + \xi_n) = \alpha_n v$, where $\alpha_n \in \mathbb{C}$. Thus we obtain the recurrence relation $\alpha_n - \alpha_{n-1} = \chi(\lambda + n)\alpha_n$, so that $\alpha_n = \chi_P(\lambda + n)\alpha_{n-1}$, where the rational function $\chi_P(t)$ is defined by $Q + Q(tI - P)^{-1} Q = \chi_P(t) Q$. Thus, reintroducing the index i, we have

$$\alpha_{i,n} = \alpha_{i,0} \prod_{m=1}^{n} \chi_P(\lambda_i + m), \tag{5}$$

where $\alpha_{i,0} = \phi(\xi_i)$. We now must compute $\chi_P(t)$. Bearing in mind that equation (2) gives the corresponding power series expansions about ∞, we define $\chi_R(t)$ by $Q + Q(tI - R)^{-1} Q = \chi_R(t) Q$.

Inversion lemma. $\chi_R(t) = \chi_P(-t)^{-1}$.

Proof. Let A be an invertible matrix with $QA^{-1}Q = (1 - \alpha)Q$, where $\alpha \neq 0$. Expanding $(A - Q)^{-1} = (I - A^{-1}Q)^{-1}A^{-1}$, we find that $Q(A - Q)^{-1}$ $Q = (\alpha^{-1} - 1)Q$. Hence

$$\chi_R(t)Q = Q + Q(t - R)^{-1}Q = Q + Q(t + P - Q)^{-1}Q = \alpha^{-1}Q,$$

if $Q(t + P)^{-1}Q = (1 - \alpha)Q$. But $Q(t + P)^{-1}Q = -Q(-t - P)^{-1}$ $Q = (1 - \chi_P$ $(-t))Q$, so that $\alpha = \chi_P(-t)$ and hence $\chi_R(t) = \alpha^{-1} = \chi_P(-t)^{-1}$ as required.

Corollary. $\chi_p(t) = \prod(t - \mu_i)/\prod(t - \lambda_j)$ *where the* μ_j*'s are the eigenvalues of* $P - Q$.

Proof. $X_P(t)$ has the form $p(t)/\prod(t - \lambda_i)$, where $p(t)$ is a monic polynomial of degree N. Similarly $X_R(t)$ has the form $q(t)/\prod(t + \mu_i)$ where the μ_i's are the eigenvalues of $-R = P - Q$. Since $X_R(t) = X_P(-t)^{-1}$, we see that $p(t) = \prod(t - \mu_i)$ and $q(t) = \prod(t + \lambda_i)$, as required.

Corollary. $\sum \lambda_i - \sum \mu_i = \delta$.

Proof. This follows by taking the trace of the identity $P + R = Q$.

From (5) and the formula for $\chi_P(t)$, we have for $n \geq 1$

$$\alpha_{i,n} = \alpha_{i,0} \prod_{j=1}^{N} \prod_{m=1}^{n} \frac{m + \lambda_i - \mu_j}{m + \lambda_i - \lambda_j},$$

where $\alpha_{i,0} = \phi(\xi_i)$.

23. Euler-Thomae integral representation of projected solutions (*cf* [*38*, *47*])

We assume here that the eigenvalues λ_i of P are real with $\lambda_1 > \lambda_2 > \cdots > \lambda_N$; that the eigenvalues μ_i of $P - Q$ are real with $\mu_1 > \mu_2 > \cdots > \mu_N$; and that $\lambda_1 + 1 > \mu_j > \lambda_1$ for all j. In particular this implies that $\delta = \mathrm{Tr}(Q)$ must be *negative*. We start by obtaining an integral representation of the projected solutions $(1 - z)\phi(f_i(z))$ around 0. Recalling that the eigenvectors ξ_i and ζ_i of P and $P - Q$ are normalised so that $\phi(\xi_i) = 1 = \phi(\zeta_i)$, where $Q(x) = \phi(x)v = \phi(x)\eta$, we have already shown that

$$(1 - z)^{-1}z^{-\lambda_i}\phi(f_i(z)) = \sum_{n\geq 0} \alpha_{i,n}z^n = \sum_{n\geq 0} z^n \cdot \prod_{j=1}^{N} \prod_{m=1}^{n} \frac{m + \lambda_i - \mu_j}{m + \lambda_i - \lambda_j}.$$

Using the formula $(a)_n \equiv a(a + 1) \cdots (a + n - 1) = \Gamma(a + n)/\Gamma(a)$, we get

$$(1 - z)^{-1}z^{-\lambda_i}\phi(f_1(z)) = \sum_{n\geq 0} \frac{(\lambda_1 - \mu_1 + 1)_n}{n!} \prod_{j\neq 1} \frac{\Gamma(\lambda_1 - \mu_j + n + 1)\Gamma(\lambda_1 - \lambda_j + 1)}{\Gamma(\lambda_1 - \mu_j + 1)\Gamma(\lambda_1 - \lambda_j + n + 1)}.$$

Using the beta function identity $\Gamma(a)\Gamma(b)/\Gamma(a+b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt$ for $a, b > 0$, we obtain

$$\begin{aligned}\phi(f_1(z)) =& (1-z)z^{\lambda_1}K\int_0^1\int_0^1\cdots\int_0^1 (1-zt_2\cdots t_N)^{\mu_1-\lambda_1-1}\\ &\prod_{j\neq 1} t_j^{\lambda_1-\mu_j}(1-t_j)^{\mu_j-\lambda_j-1}\, dt_j,\end{aligned} \tag{6}$$

where

$$K = \prod_{j\neq 1}\frac{\Gamma(\lambda_1-\lambda_j+1)}{\Gamma(\lambda_1-\mu_j+1)\Gamma(\mu_j-\lambda_j)}.$$

(The inequalities $\mu_i > \lambda_i$ and $\lambda_1 - \mu_j > -1$ guarantee that this summation by integrals is valid.) Note that this Euler type integral representation is also valid for z real and negative, since it is analytic in z where defined. The solutions about ∞ have a Laurent expansion (for $|z|$ large) $g_j(z) = \zeta_j z^{\mu_j} + \zeta_{j,1}z^{\mu_j-1} + \cdots$ where ζ_j are the eigenvectors of $P-Q$ with $(P-Q)\zeta_j = \mu_j\zeta_j$. Hence the projected solution $\phi(g_j(z))$ satisfies $\phi(g_j(z)) \sim (\zeta_j, \eta)z^{\mu_j}$ because of the normalisation $\phi(\zeta_j) = 1$. In particular if x is large and negative $\phi(g_j(x)) \sim |x|^{\mu_j}e^{\pi i\mu_j}$. Let c_{ij} be the transport matrix connecting the solutions at 0 and ∞, so that $f_1(z) = \sum c_{1j}g_j(z)$. Since Q and P are in general position, we lose no information by writing the above equation as $\phi(f_1(z)) = \sum c_{1j}\phi(g_j(z))$. Since μ_1 is the largest of the μ_j's, we find that for x large and negative,

$$\phi(f_1(x)) \sim c_{11}|x|^{\mu_1}e^{i\pi\mu_1}. \tag{7}$$

On the other hand by (6) we have for $x << 0$

$$\phi(f_1(x)) \sim Ke^{i\pi\lambda_1}|x|^{\mu_1}\prod_{j\neq i}\int_0^1 t_j^{\mu_1-\mu_j-1}(1-t_j)^{\mu_j-\lambda_j-1}dt_j. \tag{8}$$

Comparing (7) and (8), we obtain

$$\begin{aligned}c_{11} &= e^{i\pi(\lambda_1-\mu_1)}K\prod_{j\neq 1}\int_0^1 t_j^{\mu_1-\mu_j-1}(1-t_j)^{\mu_j-\lambda_j-1}dt_j\\ &= Ke^{i\pi(\lambda_1-\mu_1)}\prod_{j\neq 1}\frac{\Gamma(\mu_1-\mu_j)\Gamma(\mu_j-\lambda_j)}{\Gamma(\mu_1-\lambda_j)}.\end{aligned}$$

Substituting in the value of K, we get the fundamental formula:

$$c_{11} = e^{i\pi(\lambda_1-\mu_1)}\prod_{j\neq 1}\frac{\Gamma(\lambda_1-\lambda_j+1)\Gamma(\mu_1-\mu_j)}{\Gamma(\lambda_1-\mu_j+1)\Gamma(\mu_1-\lambda_j)}. \tag{9}$$

24. Computation of transport matrices

Theorem. *The transport matrix c_{ij} from the solutions at* 0 *to the solutions at* ∞ *of the basic ODE is given by*

$$c_{ij} = e^{i\pi(\lambda_i - \mu_j)} \frac{\prod_{k \neq i} \Gamma(\lambda_i - \lambda_k + 1) \prod_{\ell \neq j} \Gamma(\mu_j - \mu_\ell)}{\prod_{\ell \neq j} \Gamma(\lambda_i - \mu_\ell + 1) \prod_{k \neq i} \Gamma(\mu_j - \lambda_k)}.$$

Proof. We obtained this formula in section 23 for c_{11} when λ_i, μ_j took on special values. On the other hand c_{11} and the right hand side are analytic functions of λ_i, μ_j. The special values sweep out an open subset of the real part of the parameter space U_0, so by analytic continuation we must have equality for all parameters in U_0. The formula for c_{ij} now follows immediately from the symmetry property of the c_{ij}'s.

IV. Vector and dual vector primary fields

25. Existence and uniqueness of vector and dual vector primary fields

Let V be an irreducible representation of $SU(N)$. Then $\mathscr{V} = C^\infty(S^1, V)$ has an action of $LG \rtimes \mathrm{Rot}\, S^1$ with LG acting by multiplication and $\mathrm{Rot}\, S^1$ by rotation, $r_\alpha f(\theta) = f(\theta + \alpha)$. There is corresponding infinitesimal action of $L^0\mathfrak{g} \rtimes \mathbb{R}$ which leaves invariant the finite energy subspace $\mathscr{V}^0$. We may write $\mathscr{V}^0 = \sum \mathscr{V}(n)$ where $\mathscr{V}(n) = z^{-n} \otimes V$. Set $v_n = z^n v$ for $v \in V$. Thus $dv_n = -nv_n$ (so that $d = -id/d\theta$) and $X_n v_m = (Xv)_{m+n}$. Let H_i and H_j be irreducible positive energy representations at level ℓ. A map $\phi : \mathscr{V}^0 \otimes H_j^0 \to H_i^0$ commuting with the action of $L^0\mathfrak{g} \rtimes \mathrm{Rot}\, S^1$ is called a *primary field with charge V*. For $v \in V$ we define $\phi(v, n) = \phi(v_n) : H_i^0 \to H_j^0$: these are called the *modes* of ϕ. The intertwining property of ϕ is expressed in terms of the modes through the commutation relations:

$$[X(n), \phi(v, m)] = \phi(X \cdot v, m + n), \quad [D, \phi(v, m)] = -m\phi(v, m).$$

Uniqueness Theorem. *If $\phi : \mathscr{V}^0 \otimes H_j^0 \to H_i^0$ is a primary field, then ϕ restricts to a G-invariant map ϕ_0 of $\mathscr{V}(0) \otimes H_j(0) = V \otimes H_j(0)$ into $H_i(0)$. Moreover ϕ is uniquely determined by ϕ_0, the initial term of ϕ.*

Proof. $\mathscr{V}(0) \otimes H_j(0)$ is fixed by $\mathrm{Rot}\, S^1$ and hence so is its image under ϕ. It therefore must lie in $H_i(0)$. Since ϕ is G-equivariant (or equivalently $\mathfrak{g}$-equivariant), the restriction of ϕ is G-equivariant. To prove uniqueness, we must show that if the initial term ϕ_0 vanishes then so too does ϕ. It clearly suffices to show that $(\phi(\xi \otimes f), \eta) = 0$ for all $\xi \in H_j^0$, $f \in \mathscr{V}^0$ and $\eta \in H_i^0$. By assumption this is true for $\xi \in H_j(0)$, $v \in \mathscr{V}(0)$ and $\eta \in H_i(0)$. By $\mathrm{Rot}\, S^1$-invariance, this is also true if $v \in \mathscr{V}(n)$ for $n \neq 0$ and hence for any $v \in \mathscr{V}^0$.

Now we assume by induction on n that $(\phi(a_n a_{n-1} \cdots a_1 \xi \otimes v), \eta) = 0$ whenever $\xi \in H_j(0)$, $\eta \in H_i(0)$, $v \in \mathscr{V}^0$ and $a_k = X_k(m_k)$ with $m_k < 0$. Then

$$\begin{aligned}(\phi(a_{n+1} a_n \cdots a_1 \xi \otimes v), \eta) = &- (\phi(a_n \cdots a_1 \xi \otimes a_{n+1} v), \eta) \\ &+ (\phi(a_n \cdots a_1 \xi \otimes v), a_{n+1}^* \eta),\end{aligned}$$

and both terms vanish, the first by induction and the second because

$$a_{n+1}^* \eta = X_{n+1}(m_{n+1})^* \eta = -X_{n+1}(-m_{n+1}) \eta = 0.$$

Finally we prove by induction on n that $(\phi(\xi \otimes v), b_n \cdots b_1 \eta) = 0$ for all $\xi \in H_j^0$, $v \in \mathscr{V}^0$, $\eta \in H_i(0)$ and $b_k = X_k(m_k)$ with $m_k < 0$. In fact

$$(\phi(\xi \otimes v), b_{n+1} b_n \cdots b_1 \eta) = (\phi(b_{n+1}^* \xi \otimes v + \xi \otimes b_{n+1}^* v), b_n \cdots b_1 \eta),$$

which vanishes by induction.

Adjoints of primary fields. Let $\phi(v, n) : H_j^0 \to H_i^0$ be a primary field of charge V. Thus $\phi(v, n)$ takes $H_j(m)$ into $H_i(m-n)$ and satisfies $[X(m), \phi(v, n)] = \phi(X \cdot v, n+m)$, $[D, \phi(v, n)] = -n\phi(v, n)$. Hence the adjoint operator $\phi(v, n)^*$ carries $H_i(m)$ into $H_j(m+n)$. Let $\psi(v^*, n) = \phi(v, -n)^*$ where $v^* \in V^*$ is defined using the inner product: $v^*(w) = (w, v)$. Thus $\psi(v^*, n) : H_i(m) \to H_j(m-n)$, so that $\psi(v^*, n)$ takes H_i^0 into H_j^0. Taking adjoints in the above equation, we get $[D, \psi(v^*, n)] = -n\psi(v^*, n)$ and $[X(m), \psi(v^*, n)] = \psi(X \cdot v^*, n+m)$. Thus $\psi(v^*, z)$ is a primary field of charge V^* called the adjoint of $\phi(v, z)$. Note that the initial terms of ψ and ϕ are related by the simple formula $\psi(v^*, 0) = \phi(v, 0)^*$. Moreover for $\xi \in H_j^0$, $\eta \in H_i^0$ we have $(\phi(v, n)\xi, \eta) = (\xi, \psi(v^*, -n)\eta)$.

Fermionic initial terms. Let $V = V_\square = \mathbb{C}^N$ and $W = V_\square \otimes \mathbb{C}^\ell$. The irreducible summands of $\Lambda W = (\Lambda V)^{\otimes \ell}$ are precisely the permissible lowest energy spaces at level ℓ. Note that ΛW can naturally be identified with the lowest energy subspace of $\mathscr{F}_W = \mathscr{F}_V^{\otimes \ell}$.

Lemma. *Each non-zero intertwiner $T \in \mathrm{Hom}_G(V_\square \otimes V_f, V_g)$ arises by taking the composition of the exterior multiplication map $S : W \otimes \Lambda(W) \to \Lambda(W)$ with projections onto irreducible summands of the three factors, i.e. $T = p_g S(p_\square \otimes p_f)$.*

Proof. Let $e_f = e_1^{\otimes f_1 - f_2} \otimes (e_1 \wedge e_2)^{\otimes f_2 - f_3} \otimes \cdots \otimes (e_1 \wedge \cdots \wedge e_{N-1})^{\otimes f_{N-1} - f_N} \otimes I^{\otimes \ell - f_1 + f_N}$ be the highest weight vector for a copy of V_f in $(\Lambda V)^{\otimes \ell}$. Let $g_i = f_i$ if $i \neq k$ and $g_k = f_k + 1$ so that g is a permissible signature obtained by adding one box to f. Clearly the corresponding highest weight vector e_g is obtained by exterior multiplication by e_k in the $f_1 - f_k$ copy of ΛV in $(\Lambda V)^{\otimes \ell}$. Let $S : W \otimes \Lambda(W) \to \Lambda(W)$ be the map $w \otimes x \mapsto w \wedge x$. Let $p_\square$ be the projection onto the $f_1 - f_k$ copy of V in $W = V \otimes \mathbb{C}^\ell$. Then, up to a sign,

$S(p_\Box \otimes I) : V \otimes (\Lambda V)^{\otimes \ell} \to (\Lambda V)^{\otimes \ell}$ is the operation of exterior multiplication by elements of V on the $f_1 - f_k$ copy of ΛV. Let p_f, p_g be the projections onto the irreducible modules V_f, V_g generated by e_f and e_g. Then $T = p_g S(p_\Box \otimes p_f) : V \otimes V_f \to V_g$ satisfies $T(e_k \otimes e_f) = \pm e_g$. Hence T is nonzero. Since S and the three projections are $SU(N)$-equivariant, it follows that T is also, as required.

Construction of all vector primary fields. *Any $SU(N)$-intertwiner $\phi(0) : V_\Box \otimes H_j(0) \to H_i(0)$ is the initial term of a vector primary field. All vector primary fields arise as compressions of fermions so satisfy $\|\phi(f)\| \leq A\|f\|_2$ for $f \in C^\infty(S^1, V_\Box)$. The map $f \mapsto \phi(f)$ extends continuously to $L^2(S^1, V)$ and satisfies the global covariance relation $\pi_j(g)\phi(f)\pi_i(g)^* = \phi(g \cdot f)$ for $g \in \mathscr{L}G \rtimes \mathrm{Rot}\, S^1$.*

Proof. By the result on initial terms, it is possible to find an $SU(N)$-equivariant map $V \to W$, $v \mapsto \overline{v}$ and projections p_i and p_j onto $SU(N)$-submodules of ΛW isomorphic to V_i and V_j such that $p_i a(\overline{v}_0) p_j : V_j \to V_i$ is the given initial term. But V_i and V_j generate LG modules H_i and H_j with corresponding projections P_i and P_j. The required primary field is $\phi_{ij}(v, n) = P_i a(\overline{v}_n) P_j$ which clearly has all the stated properties.

Dual vector primary fields. Since the adjoint of a vector primary field is a dual vector primary field, we immediately deduce the following result.

Theorem. *Any $SU(N)$-intertwiner $\phi(0) : V_{\overline{\Box}} \otimes H_j(0) \to H_i(0)$ is the initial term of a dual vector primary field. All vector dual primary fields arise as compressions of adjoints of fermions so satisfy $\|\phi(f)\| \leq A\|f\|_2$ for $f \in C^\infty(S^1, V_{\overline{\Box}})$. The map $f \mapsto \phi(f)$ extends continuously to $L^2(S^1, V_{\overline{\Box}})$ and satisfies the global covariance relation $\pi_j(g)\phi(f)\pi_i(g)^* = \phi(g \cdot f)$ for $g \in \mathscr{L}G \rtimes \mathrm{Rot}\, S^1$.*

26. Transport equations for four-point functions and braiding of primary fields

We now establish the braiding properties of primary fields. We divide the circle up into two complementary open intervals I, I^c with I the upper semicircle, I^c the lower semicircle say. Let f, g be test functions with f supported in I and g in I^c, so that $f \in C_c^\infty(I)$ and $g \in C_c^\infty(I^c)$. In general the braiding relations for primary fields will have the following form

$$\phi_{ik}^U(u, f)\phi_{kj}^V(v, g) = \sum c_{k,h} \phi_{ih}^V(v, e_{\mu_{kh}} \cdot g)\phi_{hj}^U(u, e_{-\mu_{kh}} \cdot f),$$

where the braiding matrix (c_{kh}) and the phase corrections μ_{kh} also depend on i, k, h and j. For $f \in C_c^\infty(S^1 \backslash \{1\})$, the expression $e_\mu f$ is defined (unambiguously) by cutting the circle at 1, so that $e_\mu \cdot f(e^{i\theta}) = e^{i\mu\theta} f(e^{i\theta})$ for $\theta \in (0, 2\pi)$. To prove the braiding relation we introduce the formal power series

$$F_k(z) = \sum_{n\geq 0} z^n(\phi_{ik}^U(u,n)\phi_{kj}^V(v,-n)\xi,\eta),\ \ G_h(z) = \sum_{n\geq 0} z^n(\phi_{ih}^V(v,n)\phi_{hj}^U(u,-n)\xi,\eta),$$

where ξ and η range over lowest energy vectors. These power series are called (reduced) four-point functions and take values in $\mathrm{Hom}_G(U\otimes V\otimes V_j, V_i)$. Since the modes $\phi_{ij}^U(n)$ and $\phi_{pq}^V(n)$ are uniformly bounded in norm, they define holomorphic functions for $|z|<1$. We start by showing how the matrix coefficients of products of primary fields can be recovered from four-point functions.

Proposition 1. *Let $F_k(z) = \sum_{n\geq 0}(\phi_{ik}^U(u,n)\phi_{kj}^V(v,-n)\xi,\eta)z^n = \sum F_n z^n$, convergent in $|z|<1$. If $f\in C_c^\infty(I)$, $g\in C_c^\infty(I^c)$ and $\tilde{f}(e^{i\theta}) = f(e^{-i\theta})$, then*

$$(\phi_{ik}^U(u,f)\phi_{kj}^V(v,g)\xi,\eta) = \lim_{r\uparrow 1}\frac{1}{2\pi}\int_0^{2\pi}\tilde{f}\star g(e^{i\theta})F_k(re^{i\theta})\ d\theta.$$

Proof. If $f(z) = \sum f_n z^n$ and $g(z) = g_n z^n$, then

$$\begin{aligned}(\phi_{ik}^U(u,f)\phi_{kj}^V(v,g)\xi,\eta) &= \sum_{n\geq 0} f_n g_{-n}(\phi_{ik}^U(u,n)\phi_{kj}^V(v,-n)\xi,\eta)\\ &= \lim_{r\uparrow 1}\frac{1}{2\pi}\int_0^{2\pi}\tilde{f}\star g(e^{i\theta})F_k(re^{i\theta})\ d\theta.\end{aligned}$$

Corollary. *Suppose that $f\in C_c^\infty(I)$, $g\in C_c^\infty(I^c)$ and suppose further that $F_k(z)$ extends to a continuous function on $S^1\backslash\{1\}$. Then*

$$(\phi_{ik}^U(u,f)\phi_{kj}^V(v,g)\xi,\eta) = \frac{1}{2\pi}\int_{0+}^{2\pi-}\tilde{f}\star g(e^{i\theta})F_k(e^{i\theta})\ d\theta.$$

Proof. The assumptions on f and g imply that the support of $\tilde{f}\star g(e^{i\theta})$ is contained in $[\delta, 2\pi-\delta]$ for some $\delta>0$, so the result follows.

The next result explains how to translate from transport equations for four point functions to braiding relations for smeared primary fields. It is the analogue of the Bargmann-Hall-Wightman theorem in axiomatic quantum field theory [20, 36].

Proposition 2. *Suppose that U and V are the vector representation or its dual. Let*

$$F_k(z) = \sum(\phi_{ik}^U(u,n)\phi_{kj}^V(v,-n)\xi,\eta)z^n,\ \ G_h(z) = \sum(\phi_{ih}^V(v,n)\phi_{hj}^U(u,-n)\xi,\eta)z^n,$$

where ξ and η are lowest energy vectors. If $F_k(z)$, $G_h(z^{-1})$ extend to continuous functions on $S^1\backslash\{1\}$ with

$$F_k(e^{i\theta}) = \sum c_{kh} e^{i\mu_{kh}\theta} G_h(e^{-i\theta}),$$

where $\mu_{kh} \in \mathbb{R}$, *then for* $f \in C_c^\infty(0, \pi)$, $g \in C_c^\infty(\pi, , 2\pi)$ *we have*

$$(\phi_{ik}^U(u,f)\phi_{kj}^V(v,g)\xi, \eta) = \sum c_{kh}(\phi_{ih}^V(v, e_{\mu_{kh}} \cdot g)\phi_{hj}^U(u, e_{-\mu_{kh}} \cdot f)\xi, \eta),$$

where $e_\mu(e^{i\theta}) = e^{i\mu\theta}$ *for* $\theta \in (0, 2\pi)$.

Proof. For $\theta \in (0, 2\pi)$ we have $F_k(e^{i\theta}) = \sum c_{kh} e^{i\mu_{kh}\theta} G_h(e^{-i\theta})$. Substituting in the equation of the corollary and changing variables from θ to $2\pi - \theta$, we obtain

$$(\phi_{ik}^U(u,f)\phi_{kj}^V(v,g)\xi, \eta) = \sum c_{kh} \frac{1}{2\pi} \int_{0+}^{2\pi-} e^{2i\mu_{kh}\pi} e^{-i\mu_{kh}\theta} \tilde{g} \star f(e^{i\theta}) G_k(e^{i\theta})\, d\theta.$$

It can be checked directly that $e_{-\mu} \cdot (\tilde{g} \star f) = e^{-2\pi i\mu} \widetilde{e_\mu \cdot g} \star (e_{-\mu} \cdot f)$ (the corresponding identity is trivial for point measures supported in $(0, \pi)$ and $(\pi, 2\pi)$ and follows in general by weak continuity); this implies the braiding relation.

A standard argument with lowering and raising operators allows us to extend this braiding relation to arbitrary finite energy vectors ξ and η and hence arbitrary vectors.

Proposition 3. *If*

$$(\phi_{ik}^U(u,f)\phi_{kj}^V(v,g)\xi, \eta) = \sum c_{kh}(\phi_{ih}^V(v, e_{\mu_{kh}} \cdot g)\phi_{hj}^U(u, e_{-\mu_{kh}} \cdot f)\xi, \eta),$$

for ξ, η *lowest energy vectors, then the relation holds for all vectors* ξ, η.

Proof. By bilinearity and continuity, it will suffice to prove the braiding relation for finite energy vectors ξ, η. Suppose that η is a lowest energy vector. We start by proving that the braiding relations holds for ξ, η by induction on the energy of ξ. When ξ has lowest energy, the relation is true by assumption. Now suppose that the relation holds for ξ_1, η. Let us prove it for ξ, η where $\xi = X(-n)\xi_1$, where $n > 0$. Then

$$\begin{aligned}
(\phi_{ik}^U(u,f)\phi_{kj}^V(v,g)\xi, \eta) &= (\phi_{ik}^U(u,f)\phi_{kj}^V(v,g)X(-n)\xi_1, \eta) \\
&= -(\phi_{ik}^U(u,f)\phi_{kj}^V(Xv, e_{-n} \cdot g)\xi_1, \eta) - (\phi_{ik}^U(Xu, e_{-n} \cdot f)\phi_{kj}^V(v,g)\xi_1, \eta) \\
&= -\sum_h c_{kh}(\phi_{ih}^V(Xv, e_{\mu_{kh}} e_{-n} g)\phi_{hj}^U(u, e_{-\mu_{kh}} f)\xi_1, \eta) \\
&\qquad - \sum_h c_{kh}(\phi_{ih}^V(v, e_{\mu_{kh}} g)\phi_{hj}^U(u, e_{-\mu_{kh}} e_{-n} f)\xi_1, \eta) \\
&= \sum_h c_{kh}(\phi_{ih}^V(v, e_{\mu_{kh}} g)\phi_{hj}^U(u, e_{-\mu_{kh}} f)\xi, \eta).
\end{aligned}$$

This proves the braiding relation for all ξ and all lowest energy vectors η. A similar inductive argument shows the braiding relation holds for all ξ and all η.

Corollary 1. *If f and g are supported in $S^1\backslash\{1\}$ and the support of g is anticlockwise after the support of f, then*

$$\phi_{ik}^U(u,f)\phi_{kj}^V(v,g) = \sum c_{kh}\phi_{ih}^V(v, e_{\mu_{kh}}\cdot g)\phi_{hj}^U(u, e_{-\mu_{kh}}\cdot f).$$

Proof. This result follows immediately from the proposition, using a partition of unity and rotating if necessary so that neither the support of f nor g pass 1.

Corollary 2. *If f and g are supported in $S^1\backslash\{1\}$ and the support of g is anticlockwise after the support of f, then*

$$\phi_{ik}^U(u,f)\phi_{kj}^V(v,g) = \sum d_{kh}\phi_{ih}^V(v, e_{\mu_{kh}}\cdot g)\phi_{hj}^U(u, e_{-\mu_{kh}}\cdot f),$$

where $d_{kh} = e^{2\pi i\mu_{kh}}c_{kh}$.

Proof. This follows by applying a rotation of 180° in the proposition and then repeating the reasoning in the proof of corollary 1.

27. Sugawara's formula

Let H be a positive energy irreducible representation at level ℓ and let (X_i) be an orthonormal basis of g. *Let L_0 be the operator defined on H^0 by*

$$L_0 = \frac{1}{N+\ell}\left(-\sum_i \frac{1}{2}X_i(0)X_i(0) - \sum_{n>0}\sum_i X_i(-n)X_i(n)\right).$$

Then $L_0 = D + \Delta/2(N+\ell)$ if $-\sum_i X_i(0)X_i(0)$ acts on $H(0)$ as multiplication by Δ.

Remark. Note that the operator $C = -\sum X_iX_i = \sum E_{ij}E_{ji} - (\sum E_{ii})^2/N$ acts in V_f as the constant

$$\Delta_f = \left[\sum f_i^2 + f_i(N-2i+1)\right] - \left(\sum f_i\right)^2\Big/N.$$

In particular, for the adjoint representation on g ($f_1 = 1$, $f_2 = f_3 = \cdots = f_{N-1} = 0$, $f_N = -1$) we have $\Delta = 2N$.

Proof (cf [30]). Since $\sum_i X_i(a)X_i(b)$ is independent of the orthonormal basis (X_i), it commutes with G and hence each $X(0)$ for $X\in$ g. Thus $\sum_i [X,X_i](a)X_i(b) + X_i(a)[X,X_i](b) = 0$ for all a,b. If $A = \sum_i \frac{1}{2}X_i(0)X_i(0) + \sum_{n>0}X_i(-n)X_i(n)$, then using the above relation we get

$$
\begin{aligned}
[X(1), A] &= N\ell X(1) + \sum_i \frac{1}{2}([X, X_i](1)X_i(0) + X_i(0)[X, X_i](1)) \\
&\quad + \sum_n [X, X_i](-n+1)X_i(n) + X_i(-n)[X, X_i](n+1) \\
&= N\ell X(1) + \frac{1}{2}\sum_i [[X, X_i](1), X_i(0)] = N\ell X(1) + \frac{1}{2}\sum_i [[X, X_i], X_i](0),
\end{aligned}
$$

since $([X, X_i], X_i) = 0$ by invariance of $(\cdot, \cdot)$. Hence $[X(1), A] = (N+\ell)X(1)$, since $-\sum_i \mathrm{ad}\,(X_i)^2 = 2N$. Now formally $X(1)^* = -X(-1)$ and $A^* = A$, so taking adjoints we get $[X(-1), A] = -(N+\ell)X(-1)$, so that $(N+\ell)D + A$ commutes with all $X(\pm 1)$'s. Since $[\mathrm{g}, \mathrm{g}] = \mathrm{g}$, these generate $L^0\mathrm{g}$, and hence $(N+\ell)D + A = \lambda I$ for some $\lambda \in \mathbb{C}$. Evaluating on $H(0)$, we get $\lambda = -\Delta/2$.

28. The Knizhnik-Zamolodchikov ODE (cf [23])

Let $\phi(a, n) : H_j^0 \to H_k^0$ and $\phi(b, m) : H_k^0 \to H_i^0$ be primary fields of charges V_2 and V_3 respectively. Let a_{nm} be the matrix coefficient $a_{nm} = (\phi(v_2, n)\phi(v_3, m)v_4, v_1)$, where $V_4 = H_j(0)$ and $V_1 = H_i(0)$. Since $Dv_4 = 0 = Dv_1$ and $[D, \phi(v_2, n)] = -n\phi(v_2, n)$, $[D, \phi(v_3, m)] = -m\phi(v_3, m)$, it follows immediately that $a_{n,m} = 0$ unless $n + m = 0$. Moreover $\phi(a, m)v = 0$ if $n > 0$, so that $a_{nm} = 0$ if $m > 0$. We define four commuting actions of $SU(N)$ on $\mathrm{Hom}(V_2 \otimes V_3 \otimes V_4, V_1)$ by $\pi_1(g)T = gT$, $\pi_2(g)T = T(g^{-1} \otimes I \otimes I)$, $\pi_3(g)T = T(I \otimes g^{-1} \otimes I)$ and $\pi_4(g)T = T(I \otimes I \otimes g^{-1})$. Thus $\pi_1(g)\pi_2(g)\,\pi_3(g)\pi_4(g)T = T$ if T is G-equivariant.

Now let (X_i) be an orthonormal basis of g and define operators Ω_{ij} on $W = \mathrm{Hom}_G(V_2 \otimes V_3 \otimes V_4, V_1)$ as $-\sum \pi_i(X_k)\pi_j(X_k)$. Thus $\Omega_{ij} = \Omega_{ji}$. Moreover, if i, j, k are distinct, then $\Omega_{ij} + \Omega_{jk} + \Omega_{ki} = h$ on W, where h is a constant. In fact, if m is the missing index,

$$
\begin{aligned}
\Omega_{ij} + \Omega_{jk} + \Omega_{ki} &= -\frac{1}{2}\Big[\sum_p (\pi_i(X_p) + \pi_j(X_p) + \pi_k(X_p))^2 - \pi_i(X_p)^2 \\
&\qquad - \pi_j(X_p)^2 - \pi_k(X_p)^2\Big] \otimes I \\
&= -\frac{1}{2}\Big[\sum (-\pi_m(X_p))^2 + \Delta_i + \Delta_j + \Delta_k\Big] \\
&= (\Delta_m - \Delta_i - \Delta_j - \Delta_k)/2,
\end{aligned}
$$

since g acts trivially on W.

Theorem. *The formal power series* $f(v, z) = \sum_{n \geq 0} (\phi(v_2, n)\phi(v_3, -n)v_4, v_1)z^n$, *taking values in W, satisfies the Knizhnik-Zamolodchikov ODE*

$$
(N+\ell)\frac{df}{dz} = \left(\frac{\Omega_{34} - (\Delta_k - \Delta_3 - \Delta_4)/2}{z} + \frac{\Omega_{23}}{z-1}\right) f(z).
$$

Proof. This is proved by inserting D in the 4-point function $f(z)$ and comparing it with the Sugawara formula $D = L_0 - h$. In fact $zf'(z) = \sum_{n\geq 0}(\phi(v_2,n)D\phi(v_3,-n)v_4,v_1)z^n$, since $[D,\phi(v,m)] = -m\phi(v,m)$ and $Dv_4 = 0$. Now on H_k^0 we have $D = L_0 - h$ where $h = \Delta_k/2(N+\ell)$, so that

$$zf'(z) = -h\cdot f(z) - (N+\ell)^{-1}\sum_{n\geq 0,i}\Big[\sum_{m>0}(\phi(v_2,n)X_i(-m)X_i(m)\phi(v_3,-n)v_4,v_1)z^n + \frac{1}{2}(\phi(v_2,n)X_i(0)X_i(0)\phi(v_3,-n)v_4,v_1)z^n\Big].$$

Now $[X(n),\phi(v,m)] = \phi(X\cdot v,n+m)$, so that $\phi(v_2,n)X_i(m) = X_i(m)\phi(v_2,n) -\phi(X_i\cdot v_2,n+m)$ and $X_i(m)\phi(v_3,n) = \phi(v_3,n)X_i(m) + \phi(X_i\cdot v_3,n+m)$. Substituting in these expressions, we get

$$\begin{aligned}
zf'(z) &= -h\cdot f(z) + (N+\ell)^{-1}\sum_{n\leq 0,i}\sum_{m>0}(\phi(X_iv_2,n-m)\phi(X_iv_3,-n+m)v_4,v_1)z^n \\
&\quad - (2(N+\ell))^{-1}\sum_{n\geq 0,i}((X_i(0)\phi(v_2,n) - \phi(X_iv_2,n))(\phi(v_3,-n)X_i(0) \\
&\quad + \phi(X_iv_3,-n))v_4,v_1)z^n \\
&= (N+\ell)^{-1}(-\Delta_k/2 - \frac{1}{2}\Omega_{23}\frac{z}{1-z} - \frac{1}{2}(\Omega_{23}+\Omega_{13}+\Omega_{14}+\Omega_{24}))f(z) \\
&= (N+\ell)^{-1}(\Omega_{34} - \frac{1}{2}(\Delta_k - \Delta_3 - \Delta_4) + \Omega_{23}\frac{z}{z-1})f(z).
\end{aligned}$$

29. *Braiding relations between vector and dual vector primary fields*

Consider the four-point functions $F_k(z) = \sum_{n\geq 0}(\phi_{ik}^U(u,-n)\phi_{kj}^V(v,n)\xi,\eta)z^n$ and $G_h(z) = \sum_{n\geq 0}(\phi_{ih}^V(v,-n)\phi_{hj}^U(u,n)\xi,\eta)z^n$, where the charges U and V are either $\mathbb{C}^N$ or its dual. Thus any V_k appears with multiplicity one in the tensor product $V\otimes V_j$ or $U\otimes V_j$, and all but possibly one of these summands will be permissible at level ℓ.

Proposition. (a) *$f_k(z) = z^{\lambda_k}F_k(z)$ satisfies the KZ ODE*

$$(N+\ell)\frac{df}{dz} = \frac{\Omega_{vj}}{z}f(z) + \frac{\Omega_{uv}}{z-1}f(z),$$

where $\lambda_k = (\Delta_k - \Delta_v - \Delta_j)/2(N+\ell)$ is the eigenvalue of $(N+\ell)^{-1}\Omega_{vj}$ corresponding to the summand $V_k\subset V\otimes V_j$.

(b) *$g_h(z) = z^{\mu_h}G_h(z^{-1})$ satisfies the same ODE, where $\mu_h = (\Delta_i - \Delta_v - \Delta_h)/2(N+\ell)$ is the eigenvalue of $(N+\ell)^{-1}(\Omega_{vj}+\Omega_{uv})$ corresponding to the summand $V_h\subset U\otimes V_j$.*

Proof. (a) Since

$$\Omega_{vj} = -\sum \pi_v(X_i)\pi_j(X_i) = -\frac{1}{2}\sum(\pi_v(X)+\pi_j(X))^2 + \frac{1}{2}\sum \pi_q(X_i)^2 + \frac{1}{2}\sum \pi_j(X_i)^2,$$

$(N+\ell)^{-1}\Omega_{vj}$ acts as the scalar $\lambda_k = (\Delta_k - \Delta_v - \Delta_j)/2(N+\ell)$ on the subspace $V_k \subset V \otimes V_j$. Thus the result follows from the previous section.

(b) Similarly $v_h = \Delta_h - \Delta_u - \Delta_j)/2(N+\ell)$ eigenvalue of $(N+\ell)^{-1}\Omega_{uj}$ corresponding to the summand V_h of $U \otimes V_j$. Let $\mu = (\Delta_i - \Delta_u - \Delta_v - \Delta_j)/2(N+\ell)$. It is easy to verify that $h(z) = z^{\mu - v_h} G_h(z^{-1})$ satisfies the same ODE, since $(N+\ell)^{-1}(\Omega_{uv} + \Omega_{vj} + \Omega_{ju}) = \mu$ on $\mathrm{Hom}_G(U \otimes V \otimes V_j, V_i)$. Here $\mu_h = \mu - v_h = (\Delta_i - \Delta_v - \Delta_h)/2(N+\ell)$ is the eigenvalue of $(N+\ell)^{-1}(\Omega_{vj} + \Omega_{uv})$ corresponding to the summand $V_h \subset U \otimes V_j$.

Thus the solutions $f_k(z)$ form part of a complete set of solutions about 0 of the KZ ODE; and the solutions $g_h(z)$ form part of a solution set about ∞ of the same ODE. They may only form part, because one of the summands V_k or V_h, and hence eigenvalues λ_k or μ_h, might correspond to a representation not permissible at level ℓ; there can be at most one such summand. Let $f_k(z)$ and $g_h(z)$ denote the two complete sets of solutions, regardless of whether the eigenvalues λ_k or μ_h are permissible. They define holomorphic functions in $\mathbb{C}\backslash[0,\infty)$. Let c_{kh} be the transport matrix relating the solutions at 0 to the solutions around ∞, so that $f_k(z) = \sum c_{kh} g_h(z)$ for $z \in \mathbb{C}\backslash[0,\infty)$. Thus $F_k(z) = \sum c_{kh} z^{\mu_{kh}} G_h(z^{-1})$, for $z \in \mathbb{C}\backslash[0,\infty)$ where $\mu_{kh} = \mu_h - \lambda_k = (\Delta_i + \Delta_j - \Delta_h - \Delta_k)/2(N+\ell)$. Whenever an F_k or G_h does not correspond to a product of primary fields (because V_k or V_h is not permissible at level ℓ), we will find that the corresponding transport coefficient c_{kh} is zero. (This is not accidental. As explained in [43], there is an algebraic boundary condition which picks out the solutions that arise as four-point functions.) All the examples we will consider will be those for which the theory of the previous chapter is applicable.

Theorem A (generalised hypergeometric braiding). *Let* $F \in L^2(I, V)$ *and* $G \in L^2(J, V^*)$ *where* I *and* J *are intervals in* $S^1\backslash\{1\}$ *with* J *anticlockwise after* I. *Then*

$$\phi_{gf}^{\square}(F)\phi_{fg}^{\overline{\square}}(G) = \sum v_{fh}\phi_{gh}^{\overline{\square}}(e_{\mu_{fh}}G)\phi_{hg}^{\square}(e_{-\mu_{fh}}F)$$

with $v_{fh} \neq 0$, if $h > g$ *and* $\mu_{fh} = (2\Delta_g - \Delta_f - \Delta_h)/2(N+\ell)$.

Proof. The KZ ODE reads

$$(N+\ell)\frac{df}{dz} = \frac{\Omega_{\overline{\square}f} f(z)}{z} + \frac{\Omega_{\square\overline{\square}} f(z)}{z-1},$$

where $f(z)$ takes values in $W = \mathrm{Hom}_G(V_\square \otimes V_{\overline{\square}} \otimes V_g, V_g)$. Now the eigenvalue of $\Omega_{\square\overline{\square}}$ corresponding to the trivial representation is

$(0-\Delta_{\square}-\Delta_{\overline{\square}})/2=N^{-1}-N$ and has multiplicity one, while that corresponding to the adjoint representation is $(\Delta_{\mathrm{Ad}}-\Delta_{\square}-\Delta_{\overline{\square}})/2=N^{-1}$ with multiplicity at most $N-1$. Thus $\Omega_{\square\overline{\square}}=N^{-1}-NQ$, if Q is the rank one projection in W corresponding to the trivial representation. So

$$-(N+\ell)^{-1}\Omega_{\square\overline{\square}}=\frac{N}{N+\ell}Q-\frac{1}{N(N+\ell)}.$$

Thus $\alpha=1/N(N+\ell)$ and $\beta=N/(N+\ell)$ (in the notation of section 18).

We next check that $A=(N+\ell)^{-1}\Omega_{\overline{\square}f}$ and Q are in general position. In fact if we identify W with $\mathrm{End}_G(V_g\otimes V_{\overline{\square}})$, then the inner product becomes $\mathrm{Tr}(xy^*)$. The identity operator I is the generator of the range of Q with $Q(x)$ proportional to $\mathrm{Tr}(x)$. The eigenvectors of A are just given by the orthogonal projections e_g onto the irreducible summands V_g of $V_f\otimes V_{\square}$. Since $\mathrm{Tr}(e_g)>0$, it follows that A and Q are in general position.

The eigenvalues of A are given by $\lambda_f=(\Delta_f-\Delta_{\overline{\square}}-\Delta_g)/2(N+\ell)$, so that $|\lambda_f-\lambda_{f_1}|=|\Delta_f-\Delta_{f_1}|/2(N+\ell)$. This has the form $|g_i-g_j-i+j|/(N+\ell)$ for $i\neq j$, if f and f_1 are obtained by removing boxes from the ith and jth rows of g. Since g_i+N-i is strictly increasing and $g_1-g_N\leq\ell$, the maximum possible difference is $|g_N-g_1-N+1|/(N+\ell)=1-(N+\ell)^{-1}<1$. Hence $0<|\lambda_f-\lambda_{f_1}|<1$ if $f\neq f_1$. Similarly $\mu_h=(\Delta_g-\Delta_h-\Delta_{\overline{\square}})/2(N+\ell)$ and the difference $|\mu_h-\mu_{h_1}|$ has the form $|g_i-g_j-i+j|/(N+\ell)$ for $i\neq j$, if h and h_1 are obtained by adding boxes to the ith and jth rows of g. Hence $0<|\mu_h-\mu_{h_1}|<1$ if $h\neq h_1$.

Caveat. The indexing sets for the f_j' and h_k's are distinct, even though they have the same cardinality. This is easy to see if one draws f as a Young diagram. The f_j's correspond to corners pointing north-west while the h_k's correspond to corners pointing south-east.

The anomaly μ_{fh} is given by the stated formula by our preamble, so it only remains to check that permitted terms c_{fh} are non-zero and forbidden terms zero. In fact the numerator is always non-zero because $\Gamma(x)\neq 0$ for all $x\notin -\mathbb{N}$. Thus the only way c_{fh} can vanish is if one of the arguments of Γ in the denominator $\prod_{\ell\neq j}\Gamma(\lambda_i-\mu_\ell+\alpha+1)\prod_{k\neq i}\Gamma(\mu_j-\lambda_k-\alpha)$ is a non-positive integer. Now $\mu_h=(\Delta_g-\Delta_h-\Delta_{\overline{\square}})/2(N+\ell)$ and $\lambda_f=(\Delta_f-\Delta_{\overline{\square}}-\Delta_g)/2(N+\ell)$. Suppose that h is obtained by adding a box to the ith row of g and f is obtained by removing a box from the jth row of g. Then $\lambda_f-\mu_h=(N+\ell)^{-1}(g_j-g_i+1+i-j-N^{-1})$. Thus

$$\lambda_f-\mu_h+\alpha=(N+\ell)^{-1}(g_j-g_i+1+i-j).$$

This has modulus less than 1 unless $i=1$, $j=N$ and $g_1-g_N=\ell$, when it gives -1. It is then easy to see that if f or h is non-permissible, the corresponding coefficient vanishes and otherwise it is non-zero.

The next example of braiding could have been done using the classical theory of the hypergeometric function [17, 47]; however, since the equation is in matrix form and some knowledge of Young's orthogonal form is required to translate this matrix equation into the hypergeometric equation, it is much simpler to use the matrix and eigenvalue techniques.

Theorem B (hypergeometric braiding). *Let $F \in L^2(I, V)$ and $G \in L^2(J, V)$ where I and J are intervals in $S^1 \backslash \{1\}$ with J anticlockwise after I. Then $\phi_{hg}^{\square}(F)\phi_{gf}^{\square}(G) = \sum \mu_{gg_1} \phi_{hg_1}^{\square}(e_{\alpha_{gg_1}} G)\phi_{g_1 f}^{\square}(e_{-\alpha_{gg_1}} F)$ with $\mu_{gg_1} \neq 0$, if $h > g$, $g_1 > f$ and $\alpha_{gg_1} = (\Delta_h + \Delta_f - \Delta_g - \Delta_{g_1})/2(N + \ell)$.*

Proof. In this case $W = \mathrm{Hom}_G(V_\square \otimes V_\square \otimes V_f, V_h)$ has dimension 2. The eigenvalues of $(N+\ell)^{-1}\Omega_{\square\square}$ correspond to the summands $V_{\square\square}$ and $V_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$. We have $\lambda_{\square\square} = (\Delta_{\square\square} - 2\Delta_\square)/2(N+\ell) = (N-1)/N(N+\ell)$ and $\lambda_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} = (\Delta_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} - 2\Delta_\square)/2(N+\ell) = (-N-1)/N(N+\ell)$. If Q is the projection corresponding to $V_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$ and $\beta Q - \alpha I = -(N+\ell)^{-1}\Omega_{\square\square}$, then $\beta = 2/N(N+\ell)$ and $\alpha = (N-1)/N(N+\ell)$.

We have $\lambda_g = (\Delta_g - \Delta_f - \Delta_\square)/2(N+\ell)$ and $\mu_g = (\Delta_h - \Delta_g - \Delta_\square)/2(N+\ell)$. Thus $|\lambda_g - \lambda_{g_1}| = |\mu_g - \mu_{g_1}| = |\Delta_g - \Delta_{g_1}|/2(N+\ell) = |f_i - i - f_j + j|/(N+\ell)$, if g and g_1 are obtained by adding boxes to f in the ith and jth rows. As above, it follows that $|\lambda_g - \lambda_{g_1}| = |\mu_g - \mu_{g_1}| < 1$.

We next check that the operators $A = (N+\ell)^{-1}\Omega_{\square f}$ and Q are in general position. The operator $\Omega_{\square\square}$ is a linear combination of the identity operator id and σ, where $\sigma(T) = T(S \otimes I)$ and S is the flip on $V_\square \otimes V_\square$. The operators T_i in W which diagonalise $\Omega_{\square f}$ are obtained by composing intertwiners $V_\square \otimes V_f \to V_{g_i}$ and $V_\square \otimes V_{g_i} \to V_h$. These intertwiners are specified by their action on vectors $e_i \otimes v$ where (e_i) is a basis of $V_\square$ and v is a highest weight vector. If g_1 and g_2 are obtained by adding boxes to f in rows i and j with i, j, it is easy to see that $T_2(e_i \otimes e_j \otimes v_f)$ is a non-zero highest weight vector in V_h while $\sigma(T_2)(e_i \otimes e_j \otimes v_f) = T_2(e_j \otimes e_i \otimes v_f) = 0$. So T_2 is not an eigenvector of σ. This proves that A and Q are in general position.

The anomaly α_{gg_1} is as stated by our preamble, so it only remains to check that permitted terms c_{gg_1} are non-zero and forbidden terms zero. As above, a term can vanish iff one of the arguments in the denominator $\Gamma(\lambda_g - \mu_{g_1'} + \alpha + 1)\Gamma(\mu_{g_1'} - \lambda_{g'} - \alpha)$ is a non-positive integer (where g' denotes the other diagram to g between f and h). Now $\lambda_g - \mu_{g_1} = (\Delta_g + \Delta_{g_1} - \Delta_f - \Delta_h)/2(N+\ell)$. Hence $\lambda_g - \mu_{g'} = 1/N(N+\ell)$, so that $\lambda_g - \mu_{g'} + \alpha + 1 = 1 + (N+\ell)^{-1}$ and $\mu_{g'} - \lambda_g - \alpha = -(N+\ell)^{-1}$. This shows that, if g is permissible, none of the arguments is a non-positive integer and hence that $c_{gg} \neq 0$. On the other hand $\lambda_g - \mu_g = (f_i - i - f_j + j)/(N+\ell) + 1/N(N+\ell)$, if g is obtained by adding a box to the ith row of f. Thus $\lambda_g - \mu_g + \alpha + 1 = 1 + (f_i - i - f_j + j + 1)/(N+\ell)$, which can never be a non-positive integer, while

$$\mu_{g'} - \lambda_{g'} - \alpha = (f_i - i - f_j + j - 1)/(N+\ell).$$

This has modulus less than 1 unless $i = N$, $j = 1$ and $f_1 - f_N = \ell$, when it gives -1. This is the critical case where g is permissible (it is obtained by adding a box to the last row of f) while g' is inadmissible (it is obtained by adding a box to the first row of f). In this case therefore $c_{gg'} = 0$ while in all other cases the coefficient is non-zero.

Theorem C (Abelian braiding). *Let $F \in L^2(I, V)$ and $G \in L^2(I^c, V^*)$. Let $g \neq g_1$ be signatures, permissible at level ℓ, obtained by adding one box to f. Then $\phi_{gf}^{\square}(F)\phi_{fg_1}^{\overline{\square}}(G) = \varepsilon\phi_{gh}^{\overline{\square}}(e_\mu G)\phi_{hg_1}^{\square}(e_{-\mu}F)$ with $\varepsilon \neq 0$ and $\mu = (\Delta_g + \Delta_{g_1} - \Delta_f - \Delta_h)/2$.*

Proof. The corresponding ODE takes values in the one-dimensional space $\mathrm{Hom}_G(V_\square \otimes V_{\overline{\square}} \otimes V_{g_1}, V_g)$ so ε must be non-zero and μ is as stated by our preamble.

Theorem D (Abelian braiding). *Suppose that g is the unique signature such that $h > g > f$, so that h is obtained either by adding two boxes in the same row of f (symmetric case $+$) or in the same column (antisymmetric case $-$). Let $F \in L^2(I, V)$ and $G \in L^2(J, V)$ where I and J are intervals in $S^1\backslash\{1\}$ with J anticlockwise after I. Then there are non-zero constants $\delta_+ \neq \delta_-$ depending only on the case such that*

$$\phi_{hg}^{\square}(F)\phi_{gf}^{\square}(G) = \delta_\pm\phi_{hg}^{\square}(e_\alpha G)\phi_{gf}^{\square}(e_{-\alpha}F)$$

with $\delta_\pm \neq 0$ and $\alpha = (\Delta_h + \Delta_f - 2\Delta_g)/2$. In fact $\delta_\pm = e^{i\pi v_\pm}$ where $v_\pm = (\pm N - 1)/N(N + \ell)$.

Proof. We use the same reasoning as in the proof of Theorem C. The ODE is now a scalar equation $f' = (\lambda_g z^{-1} + v_\pm(z-1)^{-1})f$. The v_+ and v_- are the eigenvalues of $(N+\ell)^{-1}\Omega_{\square\square}$ corresponding to the summands $V_{\square\square}$ and $V_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$ respectively. So $v_\pm = (\pm N - 1)/N(N + \ell)$. The normalised solution near 0 of the ODE is $z^{\lambda_g}(1-z)^{v_\pm}$ while near ∞ it is $z^{\lambda_g + v_\pm}(1 - z^{-1})^{v_\pm}$. Taking $z = -x$, with x real and positive, it follows immediately that the transport coefficient is $e^{i\pi v_\pm}$.

Summary of braiding properties. If we define $a_{gf}^{\square} = \phi_{gf}^{\square}(e_{-\alpha}F)$ where $\alpha = (\Delta_g - \Delta_f - \Delta_\square)/2(N+\ell)$ and $a_{fg}^{\overline{\square}} = \phi_{fg}^{\overline{\square}}(e_\alpha F^*)$, then the adjoint relation between these two primary fields implies that $(a_{gf}^{\square})^* = a_{fg}^{\overline{\square}}$. Incorporating the anomalies e_μ into the smeared primary fields in this way, the braiding properties established above for vector and dual vector primary fields may be stated in the following form.

Theorem. *Let (a_{ij}), (b_{ij}) denote vector primary fields smeared in intervals I and J in $S^1\backslash\{1\}$ with J anticlockwise after I.*

(a) $a_{gf}b_{g_1f}^* = \sum v_h b_{hg}^* a_{hg_1}$ *with $v_h \neq 0$, if $h > g, g_1 > f$.*
(b) $a_{gf}b_{fh} = \sum \mu_{f1} b_{gf_1} a_{f_1h}$ *with $\mu_{f_1} \neq 0$ if $h < f_1 < g$.*
(c) $a_{gf}b_{g_1f}^* = \varepsilon b_{hg}^* a_{hg_1}$ *with $\varepsilon \neq 0$*

(d) $a_{hg}b_{gf} = \delta_{\pm} b_{hg} a_{gf}$ *where* $\delta_+ \neq \delta_-$ *are non-zero, with* $+$ *if* h *is obtained from* f *by adding two boxes in the same row and* $-$ *if they are in the same column.*

Note that (c) and (d) may be regarded as degenerate versions of (a) and (b) respectively so may be combined. Rotating through 180° as before, or taking adjoints and simply rewriting the above equations, we obtain our final result. (For simplicity we have suppressed the resulting phase changes in the coefficients.)

Corollary. *Let* (a_{ij}), (b_{ij}) *denote vector primary fields smeared in intervals* I *and* J *in* $S^1\backslash\{1\}$ *with* J *anticlockwise after* I.
(a) $b_{gf}a^*_{g_1f} = \sum \nu_h a^*_{hg} b_{hg_1}$ *with* $\nu_h \neq 0$, *if* $h > g, g_1 >$ *is permissible.*
(b) $b_{gf}a_{fh} = \sum \mu_{f_1} a_{gf_1} b_{f_1h}$ *with* $\mu_{f_1} \neq 0$ *if* $h < f_1 < g$.
(c) $b_{gf}a^*_{g_1f} = \varepsilon a^*_{hg} b_{hg_1}$ *with* $\varepsilon \neq 0$.
(d) $b_{hg}a_{gf} = \delta_{\pm}^{-1} a_{hg} b_{gf}$ *with* $\delta_+ \neq \delta_-$ *non-zero.*

V. Connes fusion of positive energy representations

30. Definition and elementary properties of Connes fusion for positive energy representations

In [42] and [43] we gave a fairly extensive treatment of Connes' tensor product operation on bimodules over von Neumann algebras. It was then applied to define a fusion operation on positive energy representations of $\mathscr{L}G$. Here we give a simplified direct treatment of fusion with more emphasis on loop groups than von Neumann algebras. Let X and Y be positive energy representations of LG at level ℓ. To define their fusion, we consider intertwiners (or fields) $x \in \mathscr{X} = \mathrm{Hom}_{\mathscr{L}_{I^c}G}(H_0, X)$, $y \in \mathscr{Y} = \mathrm{Hom}_{\mathscr{L}_I G}(H_0, Y)$ instead of the vectors (or states) $\xi = x\Omega$ and $\eta = y\Omega$ they create from the vacuum. We define an inner product on the algebraic tensor product $\mathscr{X} \otimes \mathscr{Y}$ by the *four-point formula* $\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle = (x_2^* x_1 y_2^* y_1 \Omega, \Omega)$.

Lemma. *The four-point formula defines an* (*pre-*) *inner product on* $\mathscr{X} \otimes \mathscr{Y}$. *The Hilbert space completion* $H = X \boxtimes Y$ *naturally admits a continuous unitary representation* π *of* $\mathscr{L}^{\pm 1}G = \mathscr{L}_I G \cdot \mathscr{L}_{I^c} G$ *of level* ℓ.

Proof. If $z = \sum x_i \otimes y_i \in \mathscr{X} \otimes \mathscr{Y}$, then $\langle z, z \rangle = \sum (x_i^* x_j y_i^* y_j \Omega, \Omega)$. Now $x_{ij} = x_i^* x_j$ lies in $M = \pi_0(\mathscr{L}_{I^c}G)' = \pi_0(\mathscr{L}_I G)''$. The operator $X = (x_{ij}) \in M_n(M)$ is non-negative, so has the form $X = A^*A$ for some $A = (a_{ij}) \in M_n(M)$. Similarly, if $y_{ij} = y_i^* y_j \in M'$, then $Y = (y_{ij}) \in M_n(M')$ can be written $Y = B^*B$ for some $B = (b_{ij}) \in M_n(M')$. Hence

$$\langle z, z \rangle = \sum_{p,q,i,j} (a^*_{pi} a_{pj} b^*_{qi} b_{qj} \Omega, \Omega) = \sum_{p,q} \left\| \sum_i a_{pi} b_{qi} \Omega \right\|^2 \geq 0.$$

We next check that $\mathscr{L}_I G \cdot \mathscr{L}_{I^c} G$ acts continuously on $\mathscr{X} \otimes \mathscr{Y}$, preserving the inner product. The action of $g \cdot h$ on $x \otimes y$ is given by $(g \cdot h)(x \otimes y) = gx \otimes hy$. It clearly preserves the inner product, so the group action passes to the Hilbert space completion. Note that since we have defined things on the level of central extensions, we have to check that $\zeta \in \mathbb{T} = \mathscr{L}_I G \cap \mathscr{L}_{I^c} G$ acts by the correct scalar. This is immediate. Finally we must show that the matrix coefficients for vectors in $\mathscr{X} \otimes \mathscr{Y}$ are continuous on $\mathscr{L}_I G \cdot \mathscr{L}_{I^c} G$. But

$$\langle gx_1 \otimes hy_1, x_2 \otimes y_2 \rangle = (x_2^* g x_1 y_2^* h y_1 \Omega, \Omega) = (x_1 y_2^* h y_1 \Omega, g^* x_2 \Omega).$$

Since the maps $\mathscr{L}_I G \to X$, $g \mapsto g^* x_2 \Omega$ and $\mathscr{L}_{I^c} G \to Y$, $h \mapsto hy_1 \Omega$ are continuous, the matrix coefficient above is continuous.

Lemma. *There are canonical unitary isomorphisms $H_0 \boxtimes X \cong X \cong X \boxtimes H_0$.*

Proof. If $Y = H_0$, the unitary $X \boxtimes H_0 \to X$ is given by $x \otimes y \mapsto xy\Omega$ and the unitary $H_0 \boxtimes X \to X$ is given by $y \otimes x \mapsto xy\Omega$.

Lemma. *If J is another interval of the circle and the above construction is accomplished using the local loop groups $\mathscr{L}_J G$ and $\mathscr{L}_{J^c} G$ to give a Hilbert space K with a level ℓ unitary representation σ of $\mathscr{L}_J G \cdot \mathscr{L}_{J^c} G$, then if $\phi \in SU(1,1)$ carries I onto J, there is a natural unitary $U_\phi : H \to K$ that $U_\phi(\pi(g)) U_\phi^* = \sigma(g \circ \phi^{-1})$.*

Proof. Take $\phi \in SU(1,1)$ such that $\phi(I) = J$. If $x \in \mathscr{X}_I = \mathrm{Hom}_{\mathscr{L}_{I^c} G}(H_0, X)$ and $y \in \mathscr{Y}_I = \mathrm{Hom}_{\mathscr{L}_I G}(H_0, Y)$. Choose once and for all unitary representatives $\pi_X(\phi)$ and $\pi_Y(\phi)$ (there is no choice for $\pi_0(\phi)$). Define $x' = \pi_X(\phi) x \pi_0(\phi)^*$ and $y' = \pi_Y(\phi) y \pi_0(\phi)^*$. The assignments $x \mapsto x'$, $y \mapsto y'$ give isomorphisms $\mathscr{X}_I \to \mathscr{X}_J$, $\mathscr{Y}_I \to \mathscr{Y}_J$ which preserve the inner products since $\pi_0(\phi)\Omega = \Omega$. Since $\pi_X(\phi)\pi_X(g)\pi_X(\phi)^* = \pi_X(g \cdot \phi^{-1})$ and $\pi_y(\phi)\pi_Y(g)\pi_Y(\phi)^* = \pi_Y(g \cdot \phi^{-1})$ for $\phi \in SU(1,1)$ and $g \in \mathscr{L}G$, the map $U_\phi : x \otimes y \mapsto x' \otimes y'$ extends to a unitary between $X \boxtimes_I Y$ and $X \boxtimes_J Y$ such that $U_\phi \pi_I(g) U_\phi^* = \pi_J(g \cdot \phi^{-1})$ for $g \in \mathscr{L}_I G \cdot \mathscr{L}_{I^c} G$.

Hilbert space continuity lemma. *The natural map $\mathscr{X} \otimes \mathscr{Y} \to X \boxtimes Y$ extends canonically to continuous maps $X \otimes \mathscr{Y} \to X \boxtimes Y$ and $\mathscr{X} \otimes Y \to X \boxtimes Y$. In fact $\| \sum x_i \otimes y_i \|^2 \le \| \sum x_i x_i^* \| \sum \| y_i \Omega \|^2$ and $\| \sum x_i \otimes y_i \|^2 \le \| \sum y_i y_i^* \| \sum \| x_i \Omega \|^2$.*

Proof (cf [25]). If $z = \sum x_i \otimes y_i \in \mathscr{X} \otimes \mathscr{Y}$, then $\sum ((x_i^* x_j) y_i^* y_j \Omega, \Omega) = \sum y_i^* \pi_Y(x_i^* x_j) y_j \Omega, \Omega)$, since $S_{ij} = x_i^* x_j$ lies in $\pi_0(\mathscr{L}_{I^c} G)' = \pi_0(\mathscr{L}_I G)''$. Let $\eta_j = y_j \Omega$ and $\eta = (\eta_1, \ldots, \eta_n) \in H_0^n$. Then

$$\left\| \sum x_i \otimes y_i \right\|^2 = (\pi_Y(S)\eta, \eta) \le \|S\| \|\eta\|^2 = \left\| \sum x_i x_i^* \right\| \sum \|y_i \Omega\|^2.$$

Here we used the fact that $S = x^* x$ where x is the column vector with entries x_i; this gives $\|S\| = \|x^* x\| = \|x x^*\| = \| \sum x_i x_i^* \|$. Similarly we can prove that $\| \sum x_i \otimes y_i \|^2 \le \| \sum y_i y_i^* \| \sum \| x_i \Omega \|^2$.

Corollary (associativity of fusion). *There is a natural unitary isomorphism* $X \boxtimes (Y \boxtimes Z) \to (X \boxtimes Y) \boxtimes Z$.

Proof. The assignment $(x \otimes y) \otimes z \to x \otimes (y \otimes z)$ makes sense by the lemma and clearly implements the unitary equivalence of bimodules.

31. Connes fusion with the vector representation

In the previous chapter we proved that if (a_{ij}), (b_{ij}) are vector primary fields smeared in intervals I and J in $S^1\backslash\{1\}$ with J anticlockwise after I, then:

(a) $b_{gf}a^*_{g_1f} = \sum \nu_h a^*_{hg} b_{hg_1}$ with $\nu_h \neq 0$ if $h > g, g_1$ is permissible.
(b) $b_{gf}a_{fh} = \sum \mu_{f_1} a_{gf_1} b_{f_1h}$ with $\mu_{f_1} \neq 0$ if $h < f_1 < g$.

We use these braiding relations to establish the main technical result required in the computation of $H_\square \boxtimes H_f$. This answers the following natural question. The operator $a^*_{\square 0}a_{\square 0}$ on H_0 commutes with $\mathscr{L}_{I^c}G$, so lies in $\pi_0(\mathscr{L}_IG)''$. Thus, by local equivalence, we have the right to ask what its image is under the natural isomorphism $\pi_f : \pi_0(\mathscr{L}_IG)'' \to \pi_f(\mathscr{L}_IG)''$.

Theorem (transport formula). $\pi_f(a^*_{\square 0}a_{\square 0}) = \sum \lambda_g a^*_{gf}a_{gf}$ *with* $\lambda_g > 0$.

Remark. It is possible, using induction or the braiding computations in [43], to obtain the precise values of the coefficients. Since the precise numerical values are not important for us, we have preferred a proof which makes it manifest why the right hand side must have the stated form with strictly positive coefficients λ_g.

Proof. (1) We proceed by induction on $|f| = \sum f_i$. Suppose that $\pi_f(a^*_{\square 0}a_{\square 0}) = \sum \lambda_g a^*_{gf}a_{gf}$ and $\pi_f(b^*_{\square 0}b_{\square 0}) = \sum \lambda_g b^*_{gf}b_{gf}$ with $\lambda_g > 0$. Polarising the second identity, we get $\pi_f(b^*_{\square 0}b'_{\square 0}) = \sum \lambda_g b^*_{gf}b'_{gf}$. In particular if $x \in \mathscr{L}_JG$, we may take $b'_{ij} = \pi_i(x)b_{ij}\pi_j(x)^*$ and thus obtain

$$\pi_f(b^*_{\square 0}\pi_\square(x)b_{\square 0}\pi_0(x)^*) = \sum \lambda_g b^*_{gf}\pi_g(x)b_{gf}\pi_f(x)^*.$$

Since $\pi_f(\pi_0(x)^*) = \pi_f(x)^*$, we may cancel $\pi_f(x)$ on both sides to get

$$\pi_f(b^*_{\square 0}\pi_\square(x)b_{\square 0}) = \sum \lambda_g b^*_{gf}\pi_g(x)b_{gf}.$$

(2) Take $x \in \mathscr{L}_JG$. By the braiding relations and (1), we have

$$\begin{aligned} a^*_{gf}\pi_g(b^*_{\square 0}\pi_\square(x)b_{\square 0})a_{gf} &= \pi_f(b^*_{\square 0}\pi_\square(x)b_{\square 0})a^*_{gf}a_{gf} \\ &= \sum_{g_1}\sum_{h,k} \lambda_{g_1}\nu_h\mu_k b^*_{g_1f}a^*_{hg_1}a_{hk}\pi_k(x)b_{kf}. \end{aligned}$$

If $x_i \in \mathscr{L}_JG$, let $Y = (y_{ij})$ be the operator-valued matrix with entries $y_{ij} = a^*_{gf}\pi_g(b^*_{\square 0}\pi_\square(x_i^{-1}x_j)b_{\square 0})a_{gf}$. Then Y is positive, so that $\sum(y_{ij}\xi_j, \xi_i) \geq 0$

for $\xi_i \in H_f$. Substituting the expression on the left hand side above, this gives

$$\sum_{i,j} \sum_{g_1} \lambda_{g_1} (b_{g_1 f}^* \pi_{g_1}(x_i^{-1}) \Big(\sum v_h \mu_k a_{hg_1}^* a_{hk} \Big) \pi_k(x_i) b_{kf} \xi_j, \xi_i) \geq 0.$$

On the other hand, von Neumann density implies that $\pi(\mathscr{L}_J G \cdot \mathscr{L}_{J^c} G)'' = \pi(\mathscr{L} G)''$ for any positive energy representation at level ℓ. This implies that vectors of the form $\eta = (\eta_k)$, where $\eta_k = \pi_k(x) b_{kf} \xi$ with $\xi \in H_f$ and $x \in \mathscr{L}_J G$, span a dense subset of $\bigoplus H_k$. But from the above equation we have $\sum \lambda_{g_1} v_h \mu_k (a_{hk} \eta_k, a_{hg_1} \eta_{g_1}) \geq 0$, and this inequality holds for all choices of η_k. In particular, taking all but one η_{g_1} equal to zero, we get $\lambda_{g_1} v_h \mu_{g_1} > 0$. Thus in the expression $b_{g_1 f} a_{gf}^* a_{gf} = \sum_{h,k} v_h \mu_k a_{hg_1}^* a_{hk} b_{kf}$, we have $v_h \mu_{g_1} > 0$.

(3) Now for $x \in \mathscr{L}_J G$, we have

$$\begin{aligned} b_{g_1 f}^* \pi_{g_1}(a_{\square 0}^* a_{\square 0}) \pi_{g_1}(x) b_{g_1 f} &= b_{g_1 f}^* \pi_{g_1}(x) b_{g_1 f} \sum \lambda_g a_{gf}^* a_{gf} \\ &= \sum \lambda_g v_h \mu_k b_{g_1 f}^* a_{hg_1}^* \pi_h(x) a_{hk} b_{kf}. \end{aligned}$$

If $x_i \in \mathscr{L}_J G$, let $Z = (z_{ij})$ be the operator-valued matrix with entries $z_{ij} = b_{g_1 f}^* \pi_{g_1}(a_{\square 0}^* a_{\square 0}) \pi_{g_1}(x_i^{-1} x_j) b_{g_1 f}$. Then Z is positive, so that if $\xi_i \in H_f$, $\sum (z_{ij} \xi_j, \xi_i) \geq 0$. Let $\eta = (\eta_k)$ where $\eta_k = \sum \pi_k(x_i) b_{kf} \xi_i$. As above, von Neumann density implies the vectors η are dense in $\bigoplus H_k$. Moreover we have

$$\sum \lambda_g v_h \mu_k (a_{hk} \eta_k, a_{hg_1} \eta_{g_1}) = (\pi_{g_1}(a_{\square 0}^* a_{\square 0}) \eta_{g_1}, \eta_{g_1}).$$

Since this is true for all η_k's, all the terms with $k \neq g_1$ must give a zero contribution and

$$(\pi_{g_1}(a_{\square 0}^* a_{\square 0}) \eta_{g_1}, \eta_{g_1}) = \sum \lambda_g v_h \mu_{g_1} (a_{hg_1} \eta_{g_1}, a_{hg_1} \eta_{g_1}).$$

But we already saw that $v_h \mu_{g_1} > 0$ and hence $\pi_{g_1}(a_{\square 0}^* a_{\square 0}) = \sum \Lambda_h a_{hg_1}^* a_{hg_1}$, with $\Lambda_h > 0$, as required.

Corollary. *If H_f is any irreducible positive energy representation of level ℓ, then as positive energy bimodules we have*

$$H_\square \boxtimes H_f \cong \bigoplus H_g,$$

where g runs over all permissible Young diagrams that can be obtained by adding a box to f. Moreover the action of $\mathscr{L}_I G \cdot \mathscr{L}_{I^c} G$ on $H_\square \boxtimes H_f$ extends uniquely to an action of $\mathscr{L} G \rtimes$ Rot S^1.

Proof. Let $\mathscr{X}_0 \subset \mathrm{Hom}_{\mathscr{L}_{I^c}G}(H_0, H_\square)$ be the linear span of intertwiners $x = \pi_\square(h)a_{\square 0}$, where $h \in \mathscr{L}_I G$ and a is a vector primary field supported in I. Since $x\Omega = (\pi_\square(h)a_{\square 0}\pi_0(h)^*)\pi_0(h)\Omega$, it follows from the Reeh-Schlieder theorem that $\mathscr{X}_0\Omega$ is dense in $\mathscr{X}_0 H_0$. But then the von Neumann density argument (for example) implies that $\mathscr{X}_0\Omega$ is dense in $H_\square$. If $x = \sum \pi_\square(h^{(j)})a^{(j)}_{\square 0} \in \mathscr{X}_0$, set $x_{gf} = \sum \pi_g(h^{(j)})a^{(j)}_{gf}$. Let $y \in \mathrm{Hom}_{\mathscr{L}_I G}(H_0, H_f)$. By the transport formula

$$(x^*xy^*y\Omega, \Omega) = (y^*\pi_f(x^*x)y\Omega, \Omega) = \sum_g \lambda_g(x^*_{gf}x_{gf}y\Omega, y\Omega) = \sum_g \lambda_g \|x_{gf}y\Omega\|^2.$$

This formula shows that x_{gf} is independent of the expression for x. More importantly, by polarising we get an isometry U of the closure of $\mathscr{X}_0 \otimes \mathscr{Y}$ in $H_\square \boxtimes H_f$ into $\bigoplus H_g$, sending $x \otimes y$ to $\bigoplus \lambda_g^{1/2} x_{gf} y\Omega$. By the Hilbert space continuity lemma, $\mathscr{X}_0 \otimes \mathscr{Y}$ is dense in $H_\square \boxtimes H_f$. Since each of the maps x_{gf} can be non-zero, Schur's lemma implies that U is surjective and hence a unitary. The action of $\mathscr{L}^{\pm 1}G$ extends uniquely to $\mathscr{L}G$ by Schur's lemma. The extension to Rot S^1 is uniquely determined by the fact that Rot S^1 has to fix the lowest energy subspaces of each irreducible summand of $H_f \boxtimes H_\square$.

32. Connes fusion with exterior powers of the vector representation

We now extend the methods of the previous section to the exterior powers $\lambda^k V = V_k$. We shall simply write $[k]$ for the corresponding signature, i.e. k rows with one box in each. For $a \in L^2(I, V)$, we shall write $\phi_{gf}(a)$ for $\phi^\square_{gf}(e_{-\alpha_{gf}}a)$, where $\alpha_{gf} = (\Delta_g - \Delta_f - \Delta_\square)/2(N + \ell)$ is the phase anomaly introduced in Section 29. For any path P of length k, $f_0 < f_1 < \cdots < f_k$ with f_i permissible, we define $a_P = \phi_{f_k f_{k-1}}(a_k) \cdots \phi_{f_1 f_0}(a_1)$ for $a_i \in L^2(I, V)$. In particular we let P_0 be the path $0 < [1] < [2] < \cdots < [k]$.

Theorem. *If a_i, b_i are test functions in $L^2(I, V)$, then*

$$\pi_f(b^*_{P_0}a_{P_0}) = \sum_{g >_k f} \left(\sum_{P:f\to g} \lambda_P(g) b_P\right)^* \left(\sum_{P:f\to g} \lambda_P(g) a_P\right),$$

where P ranges over all paths $f_0 = f < f_1 < \cdots < f_k = g$ with each f_i permissible and where for fixed g either $\lambda(g) = 0$ or $\lambda_P(g) \neq 0$ for all P.

Proof. (1) *The linear span of vectors $\bigoplus_{f_k > f_{k-1} > \cdots > f_1 > f} \phi_{f_k f_{k-1}}(a_k)$ $\phi_{f_{k-1} f_{k-2}}$ $(a_{k-1}) \cdots \phi_{f_1 f}(a_1)\xi$ with $a_j \in L^2(I_j, V)$ (where $I_j \subseteq I$) and $\xi \in H_f$ is dense in $\bigoplus_{f_k > f_{k-1} > \cdots > f_1 > f} H_{f_k}$.*

Proof. We prove the result by induction on k. For $k = 1$, let H denote the closure of this subspace so that H is invariant under $\mathscr{L}^{\pm 1}G$ and hence $\mathscr{L}G$. By Schur's lemma H must coincide with $\bigoplus_{f_1 > f} H_{f_1}$ as required. By induction the linear span of vectors $\bigoplus_{f_{k-1}>\cdots>f_1>f}\phi_{f_{k-1}f_{k-2}}(a_{k-1})\cdots\phi_{f_1f}(a_1)\xi$ with $a_i\in L^2(I,V)$ and $\xi\in H_f$ is dense in $\bigoplus_{f_{k-1}>\cdots>f_1>f}H_{f_{k-1}}$. The proof is completed by noting that if g is fixed and $h_1,\ldots,h_m<g$ (not necessarily distinct) then the vectors $\bigoplus\phi_{gh_i}(a)\xi_i$ with $a\in L^2(I,V)$ and $\xi_i\in H_{h_i}$ span a dense subspace of $H_g\otimes\mathbb{C}^m$. Again the closure of the subspace is $\mathscr{L}G$ invariant and the result follows by Schur's lemma, because the ξ_i's vary independently.

(2) *We have*

$$\pi_f(b_{P_0}^*a_{P_0})=\sum_{g>_kf}\sum_{P,Q:f\to g}\mu_{PQ}(g)b_P^*a_Q,$$

where g ranges over all permissible signatures that can be obtained by adding k boxes to f and P, Q range over all permissible paths $g=f_k>f_{k-1}>\cdots>f_1>f$ *and* $\mu(g)=(\mu_{PQ}(g))$ *is a non-negative matrix.*

Proof. We assume the result by induction on $|f|=\sum f_i$. By polarisation, it is enough to prove the result with $b_j=a_j$ for all j. If $h>f$, let $x_{hf}=\phi_{hf}(c)$ with $c\in L^2(I^c,V)$ and $y=a_{P_0}$. Then for $f'>f$ fixed, $x_{f'f}\pi_f(y^*y)=\pi_{f'}(y^*y)x_{f'f}$. Substituting for $\pi_f(y^*y)$ and using the braiding relations with vector primary fields and their duals, $x_{f'f}\pi_f(y^*y)$ can be rewritten as

$$x_{f'f}\pi_f(y^*y)=\sum_{g'}\sum_{f_1>f}\sum_{P,Q}\mu_{P,Q}(g')a_P^*a_Qx_{f_1f},$$

where g' ranges over signatures obtained by adding k boxes to f', P ranges over paths $f'<h_1<\cdots<h_k=g'$ and Q ranges over paths $f_1<h_1<\cdots<h_k=g'$. By (1), the vectors $\bigoplus_{f_1>f}x_{f_1f}H_f$ span a dense subset of $\bigoplus_{f_1>f}H_{f_1}$. Since $x_{f'f}\pi_f(y^*y)=\pi_{f'}(y^*y)x_{f'f}$, it follows that $\pi_{f'}(y^*y)=\sum_{g'}\sum_{f_1>f}\sum_{P,Q}\mu_{P,Q}(g')a_P^*a_Q$. Since $\pi_{f'}(y^*y)$ lies in $B(H_{f'})$, only terms with $f_1=f'$ appear in the above expression so that

$$\pi_{f'}(y^*y)=\sum_{g'}\sum_{P,Q}\mu_{P,Q}(g')a_P^*a_Q,$$

where P and Q range over paths from f' to g'. Now suppose $z=y_1+\cdots+y_m$ with y_i having the same form as y. Then

$$\pi_{f'}(z^*z)=\sum_{g'}\sum_{P,Q}\mu_{P,Q}(g')\sum_{i,j}a_{P,i}^*a_{Q,j}.$$

But $(\pi_{f'}(z^*z)\xi,\xi)\geq 0$ for $\xi\in H_{f'}$ and the linear span of vectors $\bigoplus_Q a_Q\xi$ is dense in $\bigoplus_Q H_{g'}$. Fixing g', it follows that $\sum\mu_{P,Q}(g')(\xi_P,\xi_Q)\geq 0$ for all choices of ξ_P in $H_{g'}$. Taking all the ξ_P's proportional to a fixed vector in $H_{g'}$, we deduce that $\mu(g')$ must be a non-negative matrix.

(3) *If* $g>k$ f *is permissible, then* $\mu(g)$ *has rank at most one; otherwise* $\mu(g)=0$. If $\mu(g)\neq 0$, *then* $\mu_{PQ}(g)=\overline{\lambda_P(g)}\lambda_Q(g)$ *with* $\lambda_P(g)\neq 0$ *for all P.*

Proof. We have

$$\pi_f(b^*a) = \sum_{g>_k f} \sum_{P,Q:f\to g} \mu_{PQ}(g) b_P a_Q^*,$$

where $a = a_{P_0}$ and $b = b_{P_0}$. We choose a_j to be concentrated in disjoint intervals I_j with I_j preceding I_{j+1} going anticlockwise. Fix i and let a', a'_Q be the intertwiners resulting from swapping a_i and a_{i+1}. Then $a' = \delta_- a$ where $\delta_- \neq 0$ while either $a'_Q = \alpha_Q a_Q + \beta_Q a_{Q_1}$ and $a'_{Q_1} = \gamma_Q a_Q + \delta_Q a_{Q_1}$, with $\alpha_Q, \beta_Q, \gamma_Q, \delta_Q \neq 0$, or $a'_Q = \delta_\pm a_Q$. Here if Q is the path $f < f_1 < \cdots < f_k = g$, then Q_1 is the other possible path $f < f'_1 < \cdots < f'_k = g$ with $f'_j = f_j$ for $j \neq i$. In the second case, δ_+ occurs if f_{i+1} is obtained by adding two boxes to the same row of f_{i-1} while δ_- occurs if they are added to the same column.

Now we still have $\pi_f(b^*a') = \sum_{g>_k f} \sum_{P:f\to g} \mu_{PQ}(g) b_P^* a'_P$. If Q and Q_1 are distinct, it follows that $\delta_- \mu_{PQ} = \alpha_Q \mu_{PQ} + \gamma_Q \mu_{PQ_1}$ and $\delta_- \mu_{PQ} = \beta_Q \mu_{PQ} + \delta_{Q_1} \mu_{PQ_1}$ for all P. In the case where $Q_1 = Q$, we get $\delta_- \mu_{PQ} = \delta_\pm \mu_{PQ}$. Now for a vector (λ_Q), consider the equations $\delta_- \lambda_Q = \alpha_Q \lambda_Q + \gamma_Q \lambda_{Q_1}$ and $\delta_- \lambda_{Q_1} = \beta_Q \lambda_Q + \delta_Q \lambda_{Q_1}$; or $\delta_- \lambda_Q = \delta_\pm \lambda_Q$. These are satisfied when $\lambda_Q = \mu_{PQ}$. We claim that, if $g >_k f$, these equations have up to a scalar multiple at most one non-zero solution, with all entries non-zero, and otherwise only the zero solution. This shows that $\mu(g)$ has rank at most one with the stated form if $g >_k f$ and $\mu(g) = 0$ otherwise.

We shall say that two paths are *adjacent* if one is obtained from the other by changing just one signature. We shall say that two paths Q and Q_1 are *connected* if there is a chain of adjacent paths from Q to Q_1. We will show below that any other path Q_1 from f to g is connected to Q. This shows on the one hand that if a path Q is obtained by successively adding two boxes to the same row, we have $\delta_- \lambda_Q = \delta_+ \lambda_Q$, so that $\lambda_Q = 0$ since $\delta_+ \neq \delta_-$; while on the other hand if Q and Q_1 are adjacent, λ_{Q_1} is uniquely determined by λ_Q and is non-zero if λ_Q is.

We complete the proof by showing by induction on k that any two paths $f = f_0 < f_1 < \cdots < f_k = g$ and $f = f'_0 < f'_1 < \cdots < f'_k = g$ are connected. The result is trivial for $k = 1$. Suppose the result is known for $k - 1$. Given two paths $f = f_0 < f_1 < \cdots < f_k = g$ and $f = f'_0 < f'_1 < \cdots < f'_k = g$, either $f_1 = f'_1$ or $f_1 \neq f'_1$. If $f_1 = f'_1 = h$, the result follows because the paths $h = f_1 < \cdots < f_k = g$ and $h = f'_1 < \cdots < f'_k = g$ must be connected by the induction hypothesis. If $f_1 \neq f'_1$, there is a unique signature f''_2 with $f''_2 > f_1, f'_1$. We can then find a path $f''_2 < f''_3 < \cdots < f''_k = g$. The paths $Q : f < f_1 < f''_2 < \cdots < f''_k = g$ and $Q'_1 : f < f'_1 < f''_2 < \cdots < f''_k = g$ are adjacent. By induction Q is connected to Q' and Q_1 is connected to Q'_1. Hence Q is connected to Q_1, as required.

Corollary. $H_{[k]} \boxtimes H_f = \bigoplus_{g >_k f, \lambda(g) \neq 0} H_g \leq \bigoplus_{g >_k f} H_g$.

Proof. If $h \in \mathscr{L}_I G$, then we have

$$\pi_f(b_{P_0}^* \pi_{[k]}(h) a_{P_0}) = \sum_{g >_k f} \left(\sum_{P : f \to g} \lambda_P b_P \right)^* \pi_g(h) \left(\sum_{P : f \to g} \lambda_P a_P \right).$$

Now the intertwiners $x = \pi_{[k]}(h) a_{P_0}$ span a subspace $\mathscr{X}_0$ of $\mathrm{Hom}_{\mathscr{L}_{I^c}}(H_0, H_{[k]})$. As before the transport formula shows that the assignment $x \otimes y \mapsto \bigoplus_g \sum \lambda_P(g) \pi_g(h) a_P y \Omega$ extends to a linear isometry T of $\mathscr{X}_0 \otimes \mathscr{Y}$ into $\bigoplus_{\lambda(g) \neq 0} H_g$. T intertwines $\mathscr{L}^{\pm 1} G$, so by Schur's lemma extends to an isometry of the closure of $\mathscr{X}_0 \otimes \mathscr{Y}$ in $H_{[k]} \boxtimes H_f$ onto $\bigoplus_{\lambda(g) \neq 0} H_g$. On the other hand, by the argument used in the corollary in the previous section, $\mathscr{X}_0 \Omega$ is dense in $H_{[k]}$. Therefore, by the Hilbert space continuity lemma, the image of $\mathscr{X}_0 \otimes \mathscr{Y}$ is dense in $H_{[k]} \boxtimes H_f$. Hence $H_{[k]} \boxtimes H_f = \bigoplus_{\lambda(g) \neq 0} H_g$, as required.

33. The fusion ring

Our aim now is to show that if H_i and H_j are irreducible positive energy representations, then $H_i \boxtimes H_j = \bigoplus N_{ij}^k H_k$ where the fusion coefficients N_{ij}^k are finite and to be determined.

Lemma (closure under fusion). (1) *Each irreducible positive energy representation H_i appears in some $H_\square^{\boxtimes n}$.*
(2) *The H_i's are closed under Connes fusion.*

Proof. (1) Since $H_f \boxtimes H_\square = \bigoplus H_g$, it follows easily by induction that each H_g is contained in $H_\square^{\boxtimes m}$ for some m.
(2) Since $H_f \subset H_\square^{\boxtimes m}$ for some m and $H_g \subset H_\square^{\boxtimes n}$ for some n, we have $H_f \boxtimes H_g \subset H_\square^{\boxtimes (m+n)}$. By induction $H_\square^{\boxtimes k}$ is a direct sum of irreducible positive energy bimodules. By Schur's lemma any subrepresentation of $H_\square^{\boxtimes (m+n)}$ must be a direct sum of irreducible positive energy bimodules. In particular this applies to $H_f \boxtimes H_g$, as required.

Corollary. *If X and Y are positive energy representations, the action of $\mathscr{L}_I G \cdot \mathscr{L}_{I^c} G$ on $X \boxtimes Y$ extends uniquely to an action of $\mathscr{L} G \rtimes$* Rot S^1.

Proof. The action extends uniquely to $\mathscr{L} G$ by Schur's lemma. The extension to Rot S^1 is uniquely determined by the fact that Rot S^1 has to fix the lowest energy subspaces of each irreducible summand of $X \boxtimes Y$.

Braiding lemma. *The map $B : \mathscr{X} \otimes \mathscr{Y} \to Y \boxtimes X$, $B(x \otimes y) = R_\pi^*[R_\pi(y) R_\pi^* \otimes R_\pi(x) R_\pi^*]$ extends to a unitary of $X \boxtimes Y$ onto $Y \boxtimes X$ intertwining the actions of $\mathscr{L} G$.*

Proof. Note that the B is well-defined, for rotation through π interchanges $\mathscr{L}_I G$ and $\mathscr{L}_{I^c} G$. Hence $R_\pi x R_\pi^*$ lies in $\mathrm{Hom}_{\mathscr{L}_I G}(H_0, X)$ and $R_\pi y R_\pi^*$ lies in $\mathrm{Hom}_{\mathscr{L}_{I^c} G}(H_0, Y)$. So $R_\pi y R_\pi^* \otimes R_\pi x R_\pi^*$ lies in $\mathscr{Y} \otimes \mathscr{X}$. Since $R_\pi \Omega = \Omega$, the map B preserves the inner product. Interchanging the rôles of X and Y, we get an inverse of B which also preserves the inner product. Hence B extends by continuity to a unitary of $X \boxtimes Y$ onto $Y \boxtimes X$. Finally, we check that B has the correct intertwining property. Let $g \in \mathscr{L}_I G$ and $h \in \mathscr{L}_{I^c} G$. Then

$$
\begin{aligned}
B(gx \otimes hy) &= R_\pi^*[R_\pi(hy)R_\pi^* \otimes R_\pi(gx)R_\pi^*] = R_\pi^*[(h \circ r_\pi)(g \circ r_\pi)(R_\pi y R_\pi^* \otimes R_\pi x R_\pi^*)] \\
&= R_\pi^*(h \circ r_\pi)(g \circ r_\pi) R_\pi^* R_\pi [R_\pi y R_\pi^* \otimes R_\pi x R_\pi^*] = gh R_\pi^* [R_\pi y R_\pi^* \otimes R_\pi x R_\pi^*] \\
&= ghB(x \otimes y),
\end{aligned}
$$

as required.

Corollary 1. *$X \boxtimes Y$ is isomorphic to $Y \boxtimes X$ as an $\mathscr{L}G$-module.*

Let $\mathscr{R}$ be the representation ring of formal sums $\sum m_i H_i$ ($m_i \in \mathbb{Z}$) with multiplication extending fusion. $\mathscr{R}$ is called the *fusion ring* (at level ℓ).

Corollary 2. *The fusion ring $\mathscr{R}$ is a commutative ring with an identity.*

Proof. $\mathscr{R}$ is commutative by the braiding lemma and closed under multiplication by the previous lemmas. Multiplication is associative because fusion is.

34. *The general fusion rules* (*Verlinde formulas*)

In order to determine the general coefficients N_{ij}^k in the fusion rules $H_i \boxtimes H_j = \bigoplus N_{ij}^k H_k$, we first have to determine the structure of the fusion ring. Before doing so, we will need some elementary facts on the affine Weyl group. The integer lattice $\Lambda = \mathbb{Z}^N$ acts by translation on $\mathbb{R}^N$. The symmetric group S_N acts on $\mathbb{R}^N$ by permuting the coordinates and normalises Λ, so we get an action of the semidirect product $\Lambda \rtimes S_N$. The subgroup $\Lambda_0 = \{(N+\ell)(m_i) : \sum m_i = 0\} \subset \Lambda$ is invariant under S_N, so we can consider the semidirect product $W = \Lambda_0 \rtimes S_N$. The sign of a permutation defines a homomorphism det of S_N, and hence W, into $\{\pm 1\}$.

Lemma. (a) *$\{(x_i) : |x_i - x_j| \le N + \ell\}$ forms a fundamental domain for the action of Λ_0 on $\mathbb{R}^N$.*

(b) *$D = \{(x_i) : x_1 \ge \cdots \ge x_N, x_1 - x_N \le N + \ell\}$ forms a fundamental domain for the action of $\Lambda_0 \rtimes S_N$ on $\mathbb{R}^N$.*

(c) *A point is in the orbit of the interior of D consists of points iff its stabiliser is trivial. For every other point x there is an transposition $\sigma \in S_N$ such that $\sigma(x) - x$* lies in Λ_0.

Proof. (a) Take $(x_i) \in \mathbb{R}^N$. Write $x_i = a_i + m_i$ with $0 \leq a_i < N + \ell$ and $m_i \in (N+\ell)\mathbb{Z}$. Without loss of generality, we may assume that $a_1 \leq \cdots \leq a_N$. Now (m_i) can be written as the sum of a term $(b_i) = (N+\ell)(M, M, \ldots, M, M-1, M-1, \ldots, M-1)$ and an element (c_i) of Λ_0. Thus $x = a + b + c$ with $c \in \Lambda_0$. It is easy to see that $y = a + b$ satisfies $|y_i - y_j| \leq N + \ell$. (b) follows immediately from (a) since the domain there is invariant under S_N. Finally, since $\mathrm{int}(D) = \{(x_i) : x_1 > \cdots > x_N, x_1 - x_N < N + \ell\}$, it is easy to see that any point in $\mathrm{int}(D)$ has trivial stabiliser. If $x \in \partial D$, then either $x_i = x_{i+1}$ for some i, in which case $\sigma = (i, i+1)$ fixes x; or $x_1 - x_N = N + \ell$, in which case $\sigma = (1, N)$ satisfies $\sigma(x) - x = (-N-\ell, 0, \ldots, 0, N+\ell)$. Thus (c) follows for points in D and therefore in general, since D is a fundamental domain.

Corollary. *Let $\delta = (N-1, N-2, \ldots, 1, 0)$. Then $m \in \mathbb{Z}^N$ has trivial stabiliser in $W = \Lambda_0 \rtimes S_N$ iff $m = \sigma(f + \delta)$ for a unique $\sigma \in W$ and signature $f_1 \geq f_2 \geq \cdots f_N$ with $f_1 - f_N \leq \ell$; m has non-trivial stabiliser iff there is a transposition $\sigma \in S_N$ such that $\sigma(m) - m$ lies in Λ_0.*

Proof. In the first case $m = \sigma(x)$ for $\sigma \in W$ and $x \in \mathbb{R}^N$ with $x_1 > \ldots > x_N$ and $x_1 - x_N < N + \ell$. Since the x_i's must be integers, we can write $x = f + \delta$ with $f_1 \geq \cdots \geq f_N$. Then $f_1 - f_N = x_1 - x_N - (N-1) < \ell + 1$, so that $f_1 - f_N \leq \ell$.

Recall that the character of V_f is given by Weyl's character formula $\chi_f(z) = \det(z_i^{f_j + \delta_j})/\det(z_i^{\delta_j})$. Let S be the space of permitted (normalised) signatures at level ℓ, i.e. $S = \{h : h_1 \geq \cdots \geq h_N, h_1 - h_N \leq \ell, h_N = 0\}$. We now define a ring $\mathscr{S}$ as follows. For $h \in S$, let $D(h) \in SU(N)$ be the diagonal matrix with $D(h)_{kk} = \exp(2\pi i(h_k + N - k - H)/(N+\ell))$ where $H = (\sum h_k + N - k)/N$ and set $\mathscr{T} = \{D(h) : h \in S\}$. We denote the set of functions on $\mathscr{T}$ by $\mathbb{C}^{\mathscr{T}}$. Let $\theta : R(SU(N)) \to \mathbb{C}^{\mathscr{T}}$ be the map of restriction of characters, i.e. $\theta([V]) = \chi_V|_{\mathscr{T}}$. By definition θ is a ring *-homomorphism. Set $\mathscr{S} = \theta(R(SU(N)))$ and let $\theta_f = \theta(V_f)$.

Proposition. (1) *$X_{\sigma(f+\delta)-\delta}|_{\mathscr{T}} = \det(\sigma) X_f|_{\mathscr{T}}$ for $\sigma \in S_N$ and $X_{f+m}|_{\mathscr{T}} = X_f|_{\mathscr{T}}$ for $m \in \Lambda_0$.*

(2) *The θ_f's with f permissible form a $\mathbb{Z}$-basis of $\mathscr{S}$.*

(3) $\ker(\theta)$ *is the ideal in $R(SU(N))$ generated by V_f with $f_1 - f_N = \ell + 1$.*

(4) *If $V_f \otimes V_g = \bigoplus N_{fg}^h V_h$, then $\theta_f \theta_g = \sum N_{fg}^h \det(\sigma_h) \theta_{h'}$ where h ranges over those signatures in the classical rule for which there is a $\sigma_h \in \Lambda_0 \rtimes S_N$ (necessarily unique) such that $h' = \sigma_h(h + \delta) - \delta$ is permissible.*

(5) *If f, h are permissible, then $|\{g_1 : g_1$ permissible, $f < g_1 <_k h\}| = |\{g_2 : g_2$ permissible, $f <_k g_2 < h\}|$.*

Proof. The statements in (1) follow immediately from the form of the $D(h)$'s. The $V_{[k]}$'s generate $R(SU(N))$ and, if $f_1 - f_N = \ell + 1$, it is easy to see that $\chi_f(t) = 0$ for all $t \in \mathscr{T}$: for $f_1 + N - 1 - f_N = N + \ell$ and hence the numerator in $\chi_f(t)$ must vanish. The θ_f's with f permissible are therefore closed under multiplication by $\theta_{[k]}$'s. Since the $\theta_{[k]}$'s generate $\mathscr{S}$, the $\mathbb{Z}$-linear span of the θ_f's with f permissible must equal $\mathscr{S}$. The characters $\chi_{[k]}$ distinguish the points of $\mathscr{T}$ and $\chi_{[0]} = 1$. Hence $\mathscr{S}_{\mathbb{C}}$ is a unital subalgebra of $\mathbb{C}^{\mathscr{T}}$ separating points. So given $x, y \in \mathscr{T}$, we can find $f \in \mathscr{S}_{\mathbb{C}}$ such that $f(x) = 1$ and $f(y) = 0$. Multiplying these together for all $y \neq x$, it follows that $\mathscr{S}_{\mathbb{C}}$ contains δ_x and hence coincides with $\mathbb{C}^{\mathscr{T}}$. So the θ_f's must be linearly independent over $\mathbb{C}$, so a fortiori over $\mathbb{Z}$. This proves (2). Let $I \subset R(SU(N))$ be the ideal generated by the $[V_g]$'s with $g_1 - g_N = \ell + 1$. Since $R(SU(N))$ is generated by the $V_{[k]}$'s and we have the tensor product rule $V_f \otimes V_{[k]} = \bigoplus_{g>_k f} V_g$, it follows that $R(SU(N))/I$ is spanned by the image of the $[V_f]$'s as a $\mathbb{Z}$-module. But $I \subseteq \ker(\theta)$ and the $\theta([V_f])$'s are linearly independent over $\mathbb{Z}$. Hence the images of the $[V_f]$'s give a $\mathbb{Z}$-basis of $R(SU(N))/I$ and therefore $I = \ker(\theta)$, so (3) follows. The assertion in (4) follows from (1) by applying θ and using the corollary to the lemma above. In fact, if $h + \delta$ has non-trivial stabiliser, we can find a transposition $\sigma \in S_N$ such that $\sigma(h+\delta) - h - \delta$ lies in Λ_0. Hence $X_h(t) = -X_{\sigma(h+\delta)-\delta}(t) = -X_h(t)$, so that $\chi_h(t) = X_h(t) = 0$ for all $t \in \mathscr{T}$. When the stabiliser is trivial, we clearly have $\theta_h = \det(\sigma_h)\theta_{h'}$. Finally (5) follows by comparing coefficients of θ_h in $\theta_f \theta_{[k]} \theta_\square = \sum_{g_1 >_k f} \sum_{h > g_1} \theta_h = \sum_{g_2 > f} \sum_{h >_k g_2} \theta_h$.

Theorem. (1) $H_{[k]} \boxtimes H_f = \bigoplus_{g>_k f} H_g$, *where the sum is over permissible* g.

(2) *The* $\mathbb{Z}$*-linear map* $\mathrm{ch} : \mathscr{R} \to \mathscr{S}$ *defined by* $\mathrm{ch}(H_f) = \chi_f|_{\mathscr{T}}$ *is a ring isomorphism.*

(3) *The characters of* $\mathscr{R}$ *are given by* $[H_f] \mapsto \mathrm{ch}(H_f, h) = \chi_f(z)$ *for* $z \in \mathscr{T}$.

(4) *The fusion coefficients* N_{ij}^k*'s can be computed using the multiplication rules for the basis* $\mathrm{ch}(H_f)$ *of* $\mathscr{S}$.

(5) *Each representation* H_f *has a unique conjugate representation* $\overline{H_f}$ *such that* $H_f \boxtimes \overline{H_f}$ *contains* H_0. *In fact* $\overline{H_f} = H_{f'}$, *where* $f_i' = -f_{N-i+1}$, *and* H_0 *appears exactly once in* $H_f \boxtimes H_{f'}$. *The map* $H_f \mapsto \overline{H_f}$ *makes* $\mathscr{R}$ *into an involutive ring and* ch *becomes a *-isomorphism.*

Proof. (1) We know that $H_f \boxtimes H_{[k]} \leq \bigoplus_{g>_k f} H_g$ with equality when $k = 1$. We prove by induction on $|f| = \sum f_j$ that $H_f \boxtimes H_{[k]} = \bigoplus_{g_1 >_k f} H_{g_1}$. It suffices to show that if this holds for f then it holds for all g with $g > f$. Tensoring by $H_\square$ and using part (5) of the preceding proposition, we get

$$\bigoplus_{g>f} H_g \boxtimes H_{[k]} = \bigoplus_{g_1 >_k f} \bigoplus_{h > g_1} H_h = \bigoplus_{g>f} \bigoplus_{h >_k g} H_h.$$

Since $H_g \boxtimes H_{[k]} \leq \bigoplus_{h >_k g} H_h$, we must have equality for all g, completing the induction.

(2) Let ch be the $\mathbb{Z}$-linear isomorphism $\mathrm{ch} : \mathscr{R} \to \mathscr{S}$ extending $\mathrm{ch}(H_f) = \theta_f$. Then by (1), $\mathrm{ch}(H_{[k]} \boxtimes H_f) = \theta_{[k]}\theta_f$. This implies that ch restricts to a ring homomorphism on the subring of $\mathscr{R}$ generated by the $H_{[k]}$'s. On the other hand the $\theta_{[k]}$'s generate $\mathscr{S}$, so the image of this subring is the whole of $\mathscr{S}$.

Since ch is injective, the ring generated by the $H_{[k]}$'s must be the whole of $\mathscr{R}$ and ch is thus a ring homomorphism, as required.
(3) and (4) follow immediately from the isomorphism ch and the fact that $S_{\mathbf{C}} = \mathbf{C}^{\mathscr{T}}$.
(5) We put an inner product on $\mathscr{S}_{\mathbf{C}} = \mathscr{R}_{\mathbf{C}}$ by taking θ_f as an orthonormal basis, so that $(\theta_f, \theta_g) = \delta_{fg}$. We claim that $(\theta_f\theta_g, \theta_h) = (\theta_g, \overline{\theta_f}\theta_h)$ for all θ_f. Note that $\overline{\theta_f} = \theta_{f'}$ where $f_i' = -f_{N-i+1}$. Let θ_f^* be the adjoint of multiplication by θ_f. The multiplication rules for $\theta_{[k]}$ imply that $\theta_{[k]}^* = \overline{\theta_{[k]}}$ for $k = 1, \ldots, N$. Thus the homomorphism $\theta_f \mapsto \overline{\theta_f}^*$ is the identity on a set of generators of $\mathscr{S}$ and therefore on the whole of $\mathscr{S}$, so the claim follows. In particular $(\theta_f\theta_g, \theta_0) = (\theta_g, \overline{\theta_f}) = (\theta_g, \theta_{f'}) = \delta_{gf'}$. Translating to $\mathscr{R}$, we retrieve all the assertions in (5).

The following results are immediate consequences of the theorem and preceding proposition.

Corollary 1 (Verlinde formulas [40, 21]. *If the "classical" tensor product rules for $SU(N)$ are given by $V_f \otimes V_g = \bigoplus N_{fg}^h V_h$, then the "quantum" fusion rules for $LSU(N)$ at level ℓ are given by*

$$H_f \boxtimes H_g = \bigoplus N_{fg}^h \det(\sigma_h) H_{h'},$$

where h ranges over those signatures in the classical rule for which there is a $\sigma_h \in \Lambda_0 \rtimes S_N$ (necessarily unique) such that $h' = \sigma_h(h + \delta) - \delta$ is permissible.

Corollary 2 (Segal-Goodman-Wenzl rule [35, 14]). *The map $V_f \mapsto H_f$ extends to a *-homomorphism of $R(SU(N))$ (the representation ring of $SU(N)$) onto the fusion ring $\mathscr{R}$. The kernel of this homomorphism is the ideal generated by the (non-permissible) representations V_f with $f_1 - f_N = \ell + 1$.*

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