LECTURES
on
CONFORMAL FIELD THEORY

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Introduction

Over the last decade and a half, conformal field theory (CFT) has been one of the main domains of interaction between theoretical physics and mathematics. The present review is designed as an introduction to the subject aimed at mathematicians. Its scope is limited to certain simple aspects of the theory of conformally invariant quantum fields in two space-time dimensions. The two-dimensional CFT experienced an explosive developement following the seminal 1984 paper of Belavin, Polyakov and Zamolodchikov, although many of its concepts were introduced before that date. It still plays a very important role in numerous recent developments concerning higher-dimensional quantum fields. From the mathematical point of view, CFT may be defined as a study of Virasoro algebra (or algebras containing it), of its representations and of their intertwiners. The theory defies, however, such narrowing definitions which obstruct the much wider view that it opens and into which we offer here only some glimpses. In four lectures we discuss:

- conformal free fields,
- axiomatic approach to conformal field theory,
- perturbative analysis of two-dimensional sigma models,
- exact solutions of the Wess-Zumino-Witten and coset theories.

To signal the omissions, whose full list would be much longer, let us point out that almost no mention is made of lattice models whose critical points are described by CFT's, of the perturbative approach to string theory, based on the two-dimensional CFT, of superconformal theories. The modest goal of these lectures is to make the physical literature on CFT, both the original papers and the textbooks (e.g. "Conformal Field Theory" by Di Francesco-Mathieu-Sénéchal, Springer 1996) more accessible to mathematicians.

Lecture 1. Simple functional integrals

Contents:

1. What is quantum field theory?

Field theory deals with maps ϕ between space Σ (the space-time) and space M (the target). These spaces come with additional structure, e.g. they may be Riemannian or pseudo-Riemannian manifolds. The case of Minkowski signature on Σ is the one of field theory proper whereas the Euclidean signature corresponds to static (equilibrium) situations. In many cases, however, (for example for flat Σ) the passage from one signature to the other may be obtained by analytic continuation in the time variable ("Wick rotation" $t \mapsto it$) and both situations may be studied interchangeably, the Euclidean setup being sometimes more convenient.

An important datum of the field theory is the action, a local functional of ϕ. For example, one may consider $S(\phi) = \int_\Sigma |d\phi|^2 dv$ (where the metric structures on Σ and M and a volume on Σ must be used to give sense to the right hand side).

In the classical field theory one studies the extrema of the action functional i.e. maps ϕ_{cl} satisfying

$$\delta S(\phi_{cl}) = 0\,.$$

The extremality condition is a PDE for ϕ_{cl}, e.g. the wave or Laplace equation or the Maxwell, Yang-Mills, or Einstein ones, to mention only the most famous cases. One should bear in mind that non-linear PDE's is a complicated subject where our ignorance exceeds our knowledge.

Following an extremely intuitive reformulation of quantum field theory (QFT) by Feynman, the latter consists in studying functional integrals

$$\int\limits_{Map(\Sigma,M)} F(\phi)\ \mathrm{e}^{-\frac{1}{\hbar}S(\phi)}\ D\phi \tag{1}$$

where $F(\phi)$ is a functional of ϕ (an "insertion") and $D\phi$ stands for a local product $\prod_{x\in\Sigma} d\mu(\phi(x))$ of measures on M. The above expression is formal and one of the aims of these lectures is to show that it may be given sense and even calculated in some simple situations. More generally, however, the functional integral written above should be considered as an approximate expression for structures which live their own lives, different and in some aspects more interesting then the lives of the objects whose symbols appear in the integral. Still, although formal and approximate, the functional integral language proved extremely useful in studying the QFT structures. It also made the relation of QFT to classical field theory quite intuitive: unlike in the case of the latter, all maps ϕ (called often but somewhat abusively classical field configurations or classical fields) give contributions to QFT, each with the probability amplitude $p(\phi) \sim \mathrm{e}^{-\frac{1}{\hbar}S(\phi)}$ (in Minkowski case, these are not probabilities since $S(\phi)$ should be taken imaginary, but never mind). The classical physics corresponds to the stationary phase or saddle point approximation in which all contributions to the integral (1) but those of the stationary points of S are discarded. Such an approximation is justified when the Planck constant $\hbar$ may be treated as very small (as in the usual macro-scale physics but not for example in superfluid helium).

It should be stressed that, in general, the relation of quantum to classical is not one to one (as the formulation (1) could suggest) and not even many to many, except for special situations, e.g. with lots of symmetries on both levels. Such situations are of special interest for mathematicians because they allow to reduce QFT to the more familiar classical structures. The QFT approach allowed in many such cases new insights, recall, for example, the use of topological or quasi-topological field theories as factories of invariants. This motivates the utilitarian interest of mathematicians in QFT. The very difficulty in making sense out of the integrals (1) is at the origin of a deeper source of mathematical interest of QFT, namely in the mathematics of the new structures carried by QFT. The mathematical structures in integrable or conformal two-dimensional field theories or in four-dimensional SUSY gauge theories just emerging provide here the examples (not speaking about mathematics which promises to underlie the panoply of string theory dualities).

It may be good to remind briefly what do physicists use QFT for.

i/. It provides a relativistic theory of interactions of elementary particles. And so Quantum Electrodynamics (QED) describes interactions of electrons, their anti-particles positrons and photons, the Glashow-Weinberg-Salam theory of electro-weak interactions describes at the same time the beta decays and QED, Quantum Chromodynamics (QCD) deals with the strong interactions (quarks forming nucleons). The latter two build what is called the standard model of particle physics.

ii/. QFT in its Euclidean version describes critical phenomena at the 2^{nd} order phase transition points like that in $\mathrm{H_2O}$ at temperature $T_c \cong 374°$C and pressure $p_c \cong 2.2\times 10^7$ pascals ($\sim$ 20atm), or that in Fe at $T_c \cong 770°$C or in the Ising model at its critical temperature. The criticality is characterized by slow decay with the distance of statistical correlations whose asymptotics is described by Euclidean QFT.

iii/. In string theory aiming at unification of gravity with the other interactions (from point i/.) two-dimensional conformal QFT provides the classical (and perturbative) solutions and the the quantum string theory proper (still to be non-perturbatively defined) may be considered as a deformation of quantum field theory where particles are replaced by strings (the typical size of the string being the deformation parameter).

iv/. Finally, many QFT techniques are used in the theory of non-relativistic condensed matter.

2. Euclidean free field and Gaussian functional integrals

In the rest of this lecture we shall describe how one may give sense to functional integral (1) in the simplest case of free field. This is the case where the space of maps $Map(\Sigma, M)$ is a vector space (inheriting the linear structure from that of M) or is a union of affine spaces and where the action functional S is quadratic. The corresponding functional integral is Gaussian plus, in the 2^{nd} case, an easy but interesting decoration (in theta functions). The adjective "free" refers to the absence of particle interactions which is related to linearity of the classical equations.

Let (Σ, γ) be a Riemannian, $(d+1)$-dimensional, oriented, compact manifold and let $M = \mathbf{R}$ (we consider the Euclidean case). The action functional is taken as

$$S(\phi) = \frac{\beta}{4\pi} \int\limits_{\Sigma} (|d\phi|^2 + m^2\phi^2)\, dv \;\equiv\; \tfrac{1}{2}\,(\phi, G^{-1}\phi)_{_{L^2}} \tag{2}$$

where dv is the Riemannian volume and the operator $\frac{\beta}{2\pi}G = (-\Delta + m^2)^{-1}$ is often called the (Euclidean) propagator of free field of mass m.

The simplest functional integral of the type (1) is the one with trivial insertion $F = 1$ giving what is called "statistical sum" or "partition function" and denoted traditionally by Z:

$$Z \;=\; \int\limits_{Map(\Sigma,\mathbf{R})} \mathrm{e}^{-S(\phi)}\; D\phi \;\;=\; \int\limits_{Map(\Sigma,\mathbf{R})} \mathrm{e}^{-\frac{1}{2}(\phi, G^{-1}\phi)}\; D\phi$$

(we have put $\hbar = 1$ for simplicity). The names come from statistical physics: the integral sums the probabilities (probability amplitudes) $p(\phi)$ of all microscopic states ϕ of the system. The space $Map(\Sigma, \mathbf{R})$ may be considered as the Hilbert space with the L^2 scalar product using the metric volume dv on Σ. Were this space finite dimensional, the integral would give

$$Z \;=\; (\det\, G)^{\frac{1}{2}}$$

if we normalized $D\phi = \prod_i \frac{d\phi_i}{\sqrt{2\pi}}$ where (ϕ_i) are any orthonormal coordinates on the Hilbert space of maps. It is sensible to maintain the above formula as a definition of the formal functional integral for the partition function even in the infinite-dimensional case. It is necessary then to give sense to the determinant of the positive operator G whose (discrete) eigenvalues λ_n, $n = 1, 2, \ldots$, behave as $\mathcal{O}(n^{-2/(d+1)})$. A convenient (but nonunique) way to do it is by the zeta-function regularization defining

$$\det\, G \;=\; \mathrm{e}^{-\zeta'_G(0)}$$

where $\zeta_G(s)$, given as $\sum_n \lambda_n^{-s}$ for $\mathrm{Re}\, s < -\frac{d+1}{2}$, extends to a meromorphic function analytic in the vicinity of zero. We shall stick to this definition of infinite determinants throughout the present lectures.

The next functional integrals we may like to compute are the ones for the correlations functions depending on a sequence $(x_i)_{i=1}^n$ of points in Σ

$$\langle \phi(x_1) \cdots \phi(x_n) \rangle \;\equiv\; \frac{\int\limits_{Map(\Sigma,\mathbf{R})} \phi(x_1) \cdots \phi(x_n)\; \mathrm{e}^{-S(\phi)}\; D\phi}{\int\limits_{Map(\Sigma,\mathbf{R})} \mathrm{e}^{-S(\phi)}\; D\phi}\,.$$

Again mimicking the case of finite-dimensional Gaussian integrals, we may define the formal functional integral on the right hand side by setting

$$\langle\phi(x_1)\cdots\phi(x_n)\rangle \;=\; \begin{cases} 0 & \text{for } n \text{ odd,} \\ G(x_1,x_2) & \text{for } n=2, \\ G(x_1,x_2)\,G(x_3,x_4)+G(x_1,x_3)\,G(x_2,x_4) & \\ +G(x_1,x_4)\,G(x_2,x_3) & \text{for } n=4, \\ \sum\limits_{\substack{\text{pairings}\\ \{(i_+,i_-)\}}}\prod\limits_{(i_+,i_-)} G(x_{i_+},x_{i_-}) & \text{for } n \text{ even} \end{cases}$$

where $G(x,y)$ denotes the kernel of operator G which is smooth for $x\neq y$ and exhibits a coinciding points singularity $\sim\ln\mathrm{dist}(x,y)$ for $d=1$ and $\sim\mathrm{dist}(x,y)^{-d+1}$ for $d>1$. Such a definition of the correlation functions is additionally substantiated by the fact that there exists a probability measure $d\mu_{_G}$ on the space of distributions $\mathcal{D}'(\Sigma)$ such that

$$\langle\phi(x_1)\cdots\phi(x_n)\rangle \;=\; \int\limits_{\mathcal{D}'(\Sigma)}\phi(x_1)\cdots\phi(x_n)\;d\mu_{_G}(\phi)$$

where the equality is understood in the sense of distributions. $\phi(x)$ may be then considered as a random distribution.

3. Feynman-Kac formula

Some people in the audience may wonder what it all has to do with Minkowski space field theory involving Hilbert space $\mathcal{H}$, quantum Hamiltonian H and quantum field operators acting in $\mathcal{H}$ since the (Euclidean) functional integral scheme has led us to entirely commutative structures as the Euclidean random distributions $\phi(x)$ which may, at most, be considered as a distribution with values in commuting multiplication operators acting in $L^2(\mathcal{D}'(\Sigma),d\mu_{_G})$. The relation of the two schemes is provided by the so called **Feynman-Kac formula**. Let us start from a simple quantum mechanical example.

Example 1. $d=0$, $\Sigma=[0,L]$ with the periodic identification of the ends and the standard metric. In this case, $d\mu_{_G}$ is supported by the space of continuous functions $\mathcal{C}_{per}([0,L])$ and is essentially a version of the Wiener measure. More exactly, it differs from the latter by the density $\sim\mathrm{e}^{-\frac{\beta m^2}{2\pi}\int_0^L\phi(x)^2}$. Suppose that $0\leq x_1\leq x_2\leq\ldots\leq x_n\leq L$. Then

$$\int\limits_{\mathcal{C}_{per}([0,L])}\phi(x_1)\cdots\phi(x_n)\;d\mu_{_G}(\phi) \;=\; \frac{\mathrm{tr}\;\mathrm{e}^{-x_1H}\varphi\,\mathrm{e}^{(x_1-x_2)H}\varphi\cdots\varphi\,\mathrm{e}^{(x_n-L)H}}{\mathrm{tr}\;\mathrm{e}^{-LH}} \tag{3}$$

where

$$H \;=\; -\tfrac{\pi}{\beta}\tfrac{d^2}{d\varphi^2}+\tfrac{\beta m^2}{4\pi}\varphi^2-\tfrac{m}{2} \;=\; m\left(-\sqrt{\tfrac{\pi}{\beta m}}\,\tfrac{d}{d\varphi}+\sqrt{\tfrac{\beta m}{4\pi}}\,\varphi\right)\left(\sqrt{\tfrac{\pi}{\beta m}}\,\tfrac{d}{d\varphi}+\sqrt{\tfrac{\beta m}{4\pi}}\,\varphi\right)\;\equiv\; m\,a^*a$$

is the Hamiltonian of a harmonic oscillator acting in $L^2(\mathbf{R},d\varphi)$. a and its adjoint a^* satisfy the canonical commutation relation

$$[a,a^*]=1\,.$$

The ground state Ω of H is proportional to $\mathrm{e}^{-\frac{\beta m}{4\pi}\varphi^2}$ and corresponds to the zero eigenvalue. Ω is annihilated by a and the higher H-eigenstates are obtained by aplying powers of a^* to Ω, each a^* raising energy (i.e. eigenvalue of H) by m (a is called the annihilation and a^* the creation operator). Hence the spectrum of H is $\{0, m, 2m, \ldots\} = m\mathbf{Z}_+$. With the use of the orthonormal basis $(\frac{1}{\sqrt{n!}}(a^*)^n\Omega)_{n=1}^\infty$ composed, up to normalizations, from the Hermite polynomials $H_n(\sqrt{\frac{\beta m}{\pi}}\varphi)$ times Ω, $L^2(\mathbf{R}, d\varphi)$ may be identified with (the Hilbert-space completion of) the symmetric algebra $S\mathbf{C}$ (the bosonic Fock space over $\mathbf{C}$). Note that

$$\varphi \;=\; \sqrt{\tfrac{\pi}{\beta m}}\,(a+a^*)\,.$$

Problem 1. Consider formula (3) for the 2-point function.

(a). Use the Fourier transform to write the left hand side. Show that its $L\to\infty$ limit $G_\infty(x_1,x_2)$ exists.

(b). Prove that for $x_1,\ldots,x_n>0$ and complex numbers $\lambda_1,\ldots,\lambda_n$,

$$\sum\limits_{k,l=1}^{n}\bar\lambda_k\lambda_l\,G_\infty(-x_k,x_l)\;\geq\;0\,. \tag{4}$$

(c). What is the $L\to\infty$ limit of the right hand side of Eq. (3) for $n=2$?

(d). Show that both sides of Eq. (3) with $n=2$ coincide at $L=\infty$. Prove (b) using this result.

(e). Prove relation (3) for $n=2$ and finite L.

It may be more natural to read the Feynman-Kac formula from the right to left. $\mathrm{e}^{-xH}(\phi,\phi')$ is nothing else but the transition probability to pass from ϕ to ϕ' in time x which may be used to define the Markov process $\phi(x)$ with the measure on the space of continuous realizations coinciding with $d\mu_{_G}$.

Example 2. $d>0$, $\Sigma=[0,L]^{d+1}$ with periodic identifications. Now $d\mu_{_G}$ is carried by genuinely distributional ϕ's. Let $(x_i=(x_i^0,\mathbf{x}_i))_{i=1}^n$ be s. t. $0<x_1^0<x_2^0<\ldots<x_n^0<L$. The Feynman-Kac formula now takes the form

$$\int\limits_{\mathcal{D}'([0,L]^{d+1})}\phi(x_1)\cdots\phi(x_n)\;d\mu_{_G}(\phi)\;=\;\frac{\mathrm{tr}\;\mathrm{e}^{-x_1^0H}\varphi(\mathbf{x}_1)\,\mathrm{e}^{(x_1^0-x_2^0)H}\varphi(\mathbf{x}_2)\cdots\varphi(\mathbf{x}_n)\,\mathrm{e}^{(x_n^0-L)H}}{\mathrm{tr}\;\mathrm{e}^{-LH}} \tag{5}$$

where the quantum Hamiltonian H is a positive self-adjoint operator in the Hilbert space $\mathcal{H}$, a tensor product of an infinite number of harmonic oscillators, one for each Fourier mode $\varphi_{\mathbf{k}}$ of the classical time zero field $\varphi_{\mathbf{k}}=\int_{[0,L]^d}\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}\phi(0,\mathbf{x})\,d\mathbf{x}$:

$$\mathcal{H}\;=\;\bigotimes\limits_{\pm\mathbf{k}\in\frac{2\pi}{L}\mathbf{Z}^d}L^2(\mathbf{C},d^2\varphi_{\mathbf{k}})\,.$$

The annihilation operators

$$a_{\mathbf{k}}\;=\;\sqrt{\tfrac{\pi L^d}{\beta k_0}}\,\frac{d}{d\varphi_{\mathbf{k}}}\;+\;\sqrt{\tfrac{\beta k_0}{4\pi L^d}}\,\varphi_{-\mathbf{k}}\,,$$

where $k_0 = \sqrt{\mathbf{k}^2 + m^2}$, and the creation operators $a^*_{\mathbf{k}}$ adjoint to them satisfy the canonical commutation relations

$$[a_{\mathbf{k}},\, a^*_{\mathbf{k}'}] \,=\, \delta_{\mathbf{k},\mathbf{k}'}$$

with all the other commutators vanishing. The quantum (time zero) field is

$$\varphi(\mathbf{x}) \,=\, \sum_{\mathbf{k}} \sqrt{\tfrac{\pi}{\beta k_0 L^d}}\, \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}}\, (a_{-\mathbf{k}} + a^*_{\mathbf{k}})\,.$$

The Hamiltonian

$$H \,=\, \sum_{\mathbf{k}} k_0\, a^*_{\mathbf{k}} a_{\mathbf{k}}$$

has $\Omega \sim \mathrm{e}^{\sum_{\mathbf{k}} \frac{\beta k_0}{4\pi L^d}\, |\varphi_{\mathbf{k}}|^2}$ as the ground state annihilated by all $a_{\mathbf{k}}$. The spectrum of H is $\sum_{\mathbf{k}} k_0 \mathbf{Z}_+$. There are three natural ways to look at the Hilbert space $\mathcal{H}$:

i/. $\mathcal{H}$ is an infinite tensor product of oscillator spaces $L^2(\mathbf{C},\, d^2\varphi_{\mathbf{k}})$ (how should it be defined?);

ii/. $\mathcal{H}$ is the Hilbert space completion of the symmetric algebra $S(l^2(\frac{2\pi}{L}\mathbf{Z}^d)) \cong S(L^2([0,L]^d))$; this is the Fock space picture;

iii/. $\mathcal{H}$ is a space of functionals of variables $\varphi_{\mathbf{k}}$ or of the time zero classical field $\phi(0,\mathbf{x})$ obtained by acting by creation operators $a^*_{\mathbf{k}}$ on the vacuum functional Ω.

One may introduce the (Minkowski) time dependence of the quantum field by defining

$$\varphi(t,\mathbf{x}) \,=\, \mathrm{e}^{itH}\, \varphi(\mathbf{x})\, \mathrm{e}^{-itH}\,.$$

Problem 2. Show that

$$\varphi(t,\mathbf{x}) \,=\, \sum_{\mathbf{k}} \sqrt{\tfrac{\pi}{\beta k_0 L^d}}\, (\mathrm{e}^{-itk_0 + i\mathbf{k}\cdot\mathbf{x}}\, a_{\mathbf{k}} \,+\, \mathrm{e}^{itk_0 - i\mathbf{k}\cdot\mathbf{x}}\, a^*_{\mathbf{k}})\,. \tag{6}$$

The infinite volume limit $L \to \infty$ of the formulae for H and $\varphi(t,\mathbf{x})$ may be easily taken if we introduce operators $a(\mathbf{k}) = L^{d/2} a_{\mathbf{k}}$. In the limit one obtains the operator-valued distributions $a(k)$ and their adjoints $a^*(k)$ acting in the Fock space $\overline{S(L^2(\mathbf{R}^d, đ\mathbf{k}))}$ ($đ\mathbf{k} \equiv d\mathbf{k}/(2\pi)^d$) and satisfying the commutation relations

$$[a(\mathbf{k}), a^*(\mathbf{k}')] \,=\, (2\pi)^d\, \delta(\mathbf{k} - \mathbf{k}')\,.$$

By identifying $L^2(\mathbf{R}^d, đ\mathbf{k})$ (by multiplication by $\sqrt{2k_0}$) with the space of functions on the upper mass hyperboloid $\{(k_0,\mathbf{k})\}$ square-integrable with the Lorentz-invariant measure $\frac{đ\mathbf{k}}{2k_0}$, one obtains the Minkowski scalar free field of mass m constructed in more abstract and explicitly Poincare-covariant way in David Kazhdan's lectures (check it!).

4. Massless free field with values in S^1

Let us pass to the next case where (Σ,γ) is again a general compact $(d+1)$-dimensional manifold, $M = \mathbf{R}/2\pi\mathbf{Z} \cong S^1$ and the action functional is that of the massless free field: $S(\phi) = \frac{\beta}{4\pi}\int_\Sigma |d\phi|^2\, dv$ (note that β has a natural interpretation of square of the radius ρ of the circle if we rewrite the classical action as $\frac{1}{4\pi}\int |d\phi|^2\, dv$ but use the metric $\rho^2 d\phi^2$ on the target). This is the case of a conformal (invariant) field theory with the conformal group acting (projectively) in the corresponding Hilbert space of states, transforming covariantly field operators. We shall get there slowly discussing in more detail the $d=1$ case where the conformal group is infinite-dimensional, essentially $= Diff(S^1)\times Diff(S^1)$. Let us start with an elementary treatment of the functional-integral.

How should we view the space of maps from Σ to S^1? A convenient way is to define

$$Map(\Sigma,\mathbf{R}/2\pi\mathbf{Z}) \;=\; \bigcup_{\chi\in Hom(\pi_1(\Sigma),2\pi\mathbf{Z})} Map(\tilde\Sigma,\mathbf{R})_\chi\,/\,2\pi\mathbf{Z}$$

where $\phi_\chi \in Map(\tilde\Sigma,\mathbf{R})_\chi$ is a a function on the universal cover $\tilde\Sigma$ of Σ equivariant with respect to the action of the fundamental group:

$$\phi_\chi(ax) \,=\, \phi_\chi(x) \,+\, \chi(a) \qquad \text{for } a\in\pi_1(\Sigma)\,.$$

Note that this definition makes sense for maps of arbitrary class (smooth continuous or distributional). $Hom(\pi_1(\Sigma),2\pi\mathbf{Z}) \cong H^1(\Sigma,2\pi\mathbf{Z})$ with χ given by the periods of $\alpha\in H^1(\Sigma,2\pi\mathbf{Z})$. Each $\phi_\chi\in Map_\chi$ may be uniquely decomposed according to

$$\phi_\chi \;=\; \int_{x_0}^{x}\alpha_h \,+\, \psi \;\equiv\; \phi_h \,+\, \psi \tag{7}$$

where α_h is the harmonic representative of $\alpha\in H^1$ corresponding to χ, x_0 is the base point of Σ and ψ is a univalued function on Σ. For the free field action we obtain

$$S(\phi_\chi) \;=\; \tfrac{\beta}{4\pi}\,\|\alpha_h\|^2_{L^2} \;+\; \tfrac{\beta}{4\pi}\,(\psi,-\Delta\psi)_{L^2}$$

(there is no mixed term, why?). This suggests the following definition of the functional integral for the partition function of the system:

$$\begin{aligned} Z \;=\; \int\limits_{Map(\Sigma,S^1)} \mathrm{e}^{-S(\phi)}\; D\phi \;&=\; \sum_{\alpha\in H^1(\Sigma,2\pi\mathbf{Z})} \mathrm{e}^{-\frac{\beta}{4\pi}\|\alpha_h\|^2_{L^2}} \int\limits_{Map(\Sigma,\mathbf{R})} \mathrm{e}^{-\frac{\beta}{4\pi}(\psi,-\Delta\psi)_{L^2}}\; D\psi \\ &=\; \sum_{\alpha\in H^1(\Sigma,2\pi\mathbf{Z})} \mathrm{e}^{-\frac{\beta}{4\pi}\|\alpha_h\|^2_{L^2}} \left(\frac{2\pi\,\mathrm{vol}_\Sigma}{\det{}'(-\frac{\beta}{2\pi}\Delta)}\right)^{1/2} \end{aligned} \tag{8}$$

where in $\det'$ the zero mode should be omitted (it contributes the factor $\sqrt{2\pi\,\mathrm{vol}_\Sigma}$, where $\mathrm{vol}_\Sigma = \int_\Sigma dv$, to the functional integral, why?).

Example 3. $d=0$, $\Sigma=[0,L]_{per}$. In this case, $\alpha_h = \frac{2\pi}{L}n\,dx$ and an easy calculation (see Problem 4 below) shows that

$$\det{}'(-\tfrac{\beta}{2\pi}\tfrac{d^2}{dx^2}) \;=\; 2\pi L^2/\beta\,.$$

Hence the $d=0$ partition function

$$Z \;=\; \sum_{n\in\mathbf{Z}} \mathrm{e}^{-\pi\beta L^{-1}n^2}\left(\frac{2\pi L}{\det'(-\frac{\beta}{2\pi}\frac{d^2}{dx^2})}\right)^{1/2} \underset{\substack{\text{Poisson}\\ \text{resummation}}}{=} \sum_{n\in\mathbf{Z}} \mathrm{e}^{-\pi\beta^{-1}Ln^2} \;=\; \mathrm{tr}\;\mathrm{e}^{-LH}$$

where now $H=-\frac{\pi}{\beta}\frac{d^2}{d\varphi^2}$ is the operator acting in $L^2(\mathbf{R}/2\pi\mathbf{Z},\,d\varphi)$ with the eigenvectors $\mathrm{e}^{in\varphi}$, $n\in\mathbf{Z}$, corresponding to eigenvalues $\pi n^2/\beta$.

Problem 3. Prove for $0\leq x_1\leq\ldots\leq x_n\leq L$ and $q_i\in\mathbf{Z}$ the Feynman-Kac formula

$$\int\prod_{i=i}^{n}\mathrm{e}^{iq_i\phi(x_i)}\;\mathrm{e}^{-\frac{\beta}{4\pi}\int_0^L(d\phi/dx)^2}\;D\phi \;=\; \mathrm{tr}\;\mathrm{e}^{x_1H}\,\mathrm{e}^{iq_1\varphi}\,\mathrm{e}^{(x_1-x_2)H}\,\mathrm{e}^{iq_2\varphi}\,\cdots\,\mathrm{e}^{iq_n\varphi}\,\mathrm{e}^{(x_n-L)H}$$

where the functional integral over $Map([0,L],S^1)$ on the left hand side is computed as the one for the partition function Z treated above. Infer that the left hand side may be also expressed as the expectation $\langle\mathrm{e}^{iq_1\phi(x_1)}\,\cdots\,\mathrm{e}^{iq_n\phi(x_n)}\rangle$ w. r. t. the Wiener measure on the periodic paths on S^1 constructed from the transition probabilities $\mathrm{e}^{-xH}(\varphi,\varphi')$.

Let us discuss in greater detail the case $d=1$ when (Σ,γ) is a Riemann surface of genus h_Σ with a fixed metric γ (inducing the complex structure of Σ). Let us chose a marking of Σ (a symplectic bases $(a_i,b_j)_{i,j=1}^{h_\Sigma}$ of $H_1(\Sigma,\mathbf{Z})$) with the corresponding basis $(\omega^i)_{i=1}^{h_\Sigma}$ of holomorphic 1,0-forms, $\int_{a_i}\omega^j=\delta^{ij}$, $\int_{b_i}\omega^j=\tau^{ij}$. The imaginary part τ_2 of the period matrix $\tau=(\tau^{ij})$ is positive. The equation

$$\alpha_h \;=\; \tfrac{\pi}{i}(\bar\tau\mathbf{m}+\mathbf{n})^t\tau_2^{-1}\omega\;+\;c.c.$$

for $\mathbf{m},\mathbf{n}\in\mathbf{Z}^g$ gives the harmonic forms in $H^1(\Sigma,2\pi\mathbf{Z})$ (with a_i-periods $-2\pi m_i$ and b_j-periods $2\pi n_j$).

$$\|\alpha_h\|^2_{L^2} \;=\; (2\pi)^2\,(\bar\tau\mathbf{m}+\mathbf{n})^t\tau_2^{-1}(\tau\mathbf{m}+\mathbf{n})$$

and the sum over α in eq. (8) may be done explicitly. After Poisson resummation over $\mathbf{n}$, one obtains the following result

$$\sum_{\alpha\in H^1(\Sigma,2\pi\mathbf{Z})}\mathrm{e}^{-\frac{\beta}{4\pi}\|\alpha_h\|^2_{L^2}} \;=\; \beta^{-h_\Sigma/2}(\det\,\tau_2)^{1/2}\,\vartheta_{Q_\beta}(\tau,\bar\tau) \tag{9}$$

where the "theta function" ϑ_{Q_β} is defined as follows. Let $\mathbf{E}_s$ be the s-dimensional Euclidean space. Let Q be a lattice in $\mathbf{E}_{s_+,s_-}=\mathbf{E}_{s_+}\oplus\mathbf{E}_{s_-}$ considered with the indefinite metric $|\cdot|^2_{\mathbf{E}_{s_+}}-|\cdot|^2_{\mathbf{E}_{s_-}}$. Then

$$\vartheta_Q(\tau,\bar\tau) \;=\; \sum_{(q_+,q_-)\,\in\,Q^{h_\Sigma}}\mathrm{e}^{\pi i\,(q_+,\,\tau\,q_+)\,-\,\pi i\,(q_-,\,\bar\tau\,q_-)}.$$

where the decomposition $q=(q_+,q_-)$ is according to that of $\mathbf{E}_{s_+,s_-}$. Above,

$$Q_\beta \;=\; \{\;(\tfrac{\beta^{1/2}m+\beta^{-1/2}n}{\sqrt2},\;\tfrac{\beta^{1/2}m-\beta^{-1/2}n}{\sqrt2})\;|\;m,n\in\mathbf{Z}\;\}\;\subset\;\mathbf{R}\oplus\mathbf{R}.$$

Inserting the relation (9) into eq. (8), we obtain

$$Z \;\equiv\; Z_\beta \;=\; \mathrm{e}^{(-6\ln 2\pi + 11\ln\beta/2)(h_\Sigma - 1)}\ \vartheta_{Q_\beta}(\tau,\bar\tau)\left(\frac{\mathrm{vol}_\Sigma\ \det\tau_2}{\det'(-\Delta)}\right)^{1/2}.$$

From eq. (8) it is obvious that the right hand side is marking-independent. Technically, this is due to the fact that, in the indefinite scalar product, the lattice Q_β is even (scalar-products are integers, scalar squares are even) and self-dual.

We may discard from Z any factor of the form $(\mathrm{const.})^{h_\Sigma - 1}$ by the addition to the action of a term proportional to the integral of the scalar curvature r of Σ since $\int_\Sigma r\, dv = 4\pi(1-h_\Sigma)$. Doing that we discover a somewhat miraculous equality

$$Z_\beta \;=\; Z_{1/\beta}\,, \tag{10}$$

a consequence of the obvious identity $Q_\beta = Q_{1/\beta}$. More directly, identity (10) follows from the Poisson resummation formula applied to the left hand side of eq. (9) and the fact that the lattice $H^1(\Sigma, 2\pi\mathbf{Z})$ with the L^2 scalar product is isomorphic to its dual (the isomorphism is induced by the intersection form). Eq. (10) is the simplest manifestation of the so called T-**duality** which states that the $1+1$-dimensional massless free fields with values in the circles of radius ρ and of radius ρ^{-1} are indistinguishable. This identification of inverse radia of free field compactification has a deep meaning in the string theory context and we shall return to it in a later discussion.

On the genus 1 curve $T_\tau = \mathbf{C}/(\mathbf{Z} + \tau\mathbf{Z})$ with τ in the upper half plane and the standard metric,

$$\det{}'(-\Delta) \;=\; \tau_2^2\, |\eta(\tau)|^4$$

where $\eta(\tau) = \mathrm{e}^{\pi i\tau/12}\prod_{n=1}^{\infty}(1-\mathrm{e}^{2\pi i n\tau})$ is the Dedekind eta function.

Problem 4 (a relatively complex calculation, going back to Kronecker 1890).

(a). Using the identity $\lambda^{-s} = ?(s)^{-1}\int_0^\infty t^{s-1}\mathrm{e}^{-\lambda t}\,dt$ show that $\zeta(0) = -\frac{1}{2}$ annd $\zeta(-1) = -\frac{1}{12}$ where ζ is the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ (for $\mathrm{Re}s > 1$, analytically continued elsewhere).

(b). For $\tau = \tau_1 + i\tau_2$ with τ_i real, $\tau_2 > 0$ show using the identity from (a) and the Poisson resummation that for $\mathrm{Re}s$ sufficiently large

$$\sum_{n=-\infty}^{\infty} |\tau + n|^{-2s} \;=\; \frac{\sqrt{\pi}}{\Gamma(s)}\left(\sum_{n\neq 0}\mathrm{e}^{2\pi i n\tau_1}\int_0^\infty t^{s-3/2}\,\mathrm{e}^{-\tau_2^2 t - \pi^2 n^2/t}\,dt \;+\; \tau_2^{-2s+1}\,?(s-1/2)\right).$$

Note that the right hand side is analytic in s around $s = 0$. Using (easy) relations $?(s)^{-1} = s + \mathcal{O}(s^2)$, $?(-\frac{1}{2}) = -2\sqrt{\pi}$ and $\int_0^\infty t^{-3/2}\mathrm{e}^{-x(t+t^{-1})}\,dt = \sqrt{\pi}\,x^{-1/2}\mathrm{e}^{-2x}$ obtain:

$$\frac{d}{ds}\Big|_{s=0}\sum_n |\tau + n|^{-2s} \;=\; -\ln|1-q|^2 - 2\pi\tau_2$$

with the standard notation $q \equiv \mathrm{e}^{2\pi i\tau}$.

(c). By taking $\tau \to 0$ in the last formula show that $\zeta'(0) = -\frac{1}{2}\ln(2\pi)$.

(d). Prove that for the periodic b.c. operator $\frac{d^2}{dx^2}$ on $[0, L]$ the zeta-regularized determinant

$$\det{}'(-\tfrac{\beta}{2\pi}\tfrac{d^2}{dx^2}) \;=\; 2\pi L^2/\beta\,.$$

(e). Show that the spectrum of the Laplacian Δ_τ on the torus $\mathbf{C}/(\mathbf{Z}+\tau\mathbf{Z})$ in the metric $|dz|^2$ is given by $\lambda_{m,n} = -(\frac{2\pi}{\tau_2})^2\,|\tau m+n|^2$ for $n, m \in \mathbf{Z}$.

(f). Proceeding as in (b) decompose

$$\sum_{(m,n)\neq(0,0)} |\tau m+n|^{-2s} \;=\; \sum_{m\neq 0,\ n} |\tau m+n|^{-2s} \,+\, 2\zeta(2s)$$

$$= \tfrac{\sqrt{\pi}}{\Gamma(s)}\left(\sum_{m,n\neq 0} \mathrm{e}^{2\pi i m n\tau_1}\int_0^\infty t^{s-3/2}\,\mathrm{e}^{-m^2\tau_2^2 t-\pi^2 n^2/t}\,dt \,+\, \sum_{m\neq 0} m^{-2s+1}\,\tau_2^{-2s+1}\,?\,(s-1/2)\right) + 2\zeta(2s)$$

and show that (after analytic continuation)

$$\zeta_{-\Delta'_\tau}(s) \;\equiv \sum_{(m,n)\neq(0,0)} (-\lambda_{m,n})^{-s} \;=\; -1 \,-\, 2s\,\ln|\prod_{m=1}^{\infty}(1-q^m)|^2 \,-\, 2s\,\ln\tau_2 \,+\, \tfrac{1}{3}\pi\,\tau_2\,s\,.$$

Infer that

$$\zeta_{-\Delta'_\tau}(0) \;=\; -1\,, \qquad \zeta'_{-\Delta'_\tau}(0) = -2\,\ln|\prod_{m=1}^{\infty}(1-q^m)|^2 \,-\, 2\,\ln\tau_2 \,+\, \tfrac{1}{3}\pi\,\tau_2\,,$$

and that

$$\det{}'(-\Delta_\tau) \;=\; \tau_2^2\,|\eta(\tau)|^4$$

where the Dedekind eta function $\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$.

Hence in the genus 1 case,

$$Z \;\equiv\; Z_\beta(\tau) \;=\; \vartheta_{Q_\beta}(\tau,\bar\tau)\;|\eta(\tau)|^{-2} \;=\; Z_{1/\beta}(\tau)\,. \tag{11}$$

The marking independence (together with the independence of Z_β on the normalization of the flat metric on T_τ, see below) implies that $Z(\tau)$ is a **modular invariant** function

$$Z_\beta(\tau) \;=\; Z_\beta(\tfrac{a\tau+b}{c\tau+d})$$

for $\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL_2(\mathbf{Z})$.

5. Toroidal compactifications: the partition functions

The above discussions may be easily generalized to the case of "toroidal compactifications" i.e. to the case of massless free field on (Σ,γ) with values in the N-dimensional torus $T^N=(\mathbf{R}/2\pi\mathbf{Z})^N$. Fix a constant metric $g=\sum_{ij}g_{ij}\,d\phi^i d\phi^j$ and a constant 2-form $\omega=\sum_{ij}b_{ij}\,d\phi^i\wedge d\phi^j$ on T^N and define the classical action of the field $\phi:\Sigma\to T^N$ as

$$S(\phi)\ =\ \tfrac{1}{4\pi}\,(\,\|d\phi\|^2_{_{L^2}}\ +\ i\int_\Sigma\phi^*\omega\,)\,. \tag{12}$$

Applying the same method as before (do it!) results in the formula

$$Z\ \equiv\ Z_d\ =\ \vartheta_{_{Q_d}}(\tau,\bar\tau)\,\left(\tfrac{\mathrm{vol}_\Sigma\ \det\tau_2}{\det'(-\Delta)}\right)^{N/2}\ =\ Z_{d^{-1}}$$

(T-duality again!) where $d=(d_{ij}=g_{ij}+b_{ij})$ and the lattice

$$Q_d\ =\ Q_{d^{-1}}\ =\ \{\ (\,\tfrac{dm+n}{\sqrt2}\,,\,\tfrac{d^tm-n}{\sqrt2})\ |\ m,n\in\mathbf{Z}^N\ \}\ \subset\ \mathbf{R}^N\oplus\mathbf{R}^N$$

is an even self-dual lattice in $\mathbf{R}^N\oplus\mathbf{R}^N$ with the indefinite scalar product $|(x,y)|^2=(x,g^{-1}x)-(y,g^{-1}y)$. At genus 1

$$Z\ \equiv\ Z_d(\tau)\ =\ \vartheta_{_{Q_d}}(\tau,\bar\tau)\,|\eta(\tau)|^{-2N}\ =\ Z_{d^{-1}}(\tau)\ =\ Z_d(\tfrac{a\tau+b}{c\tau+d})\,.$$

Example 4. Let T be the Cartan torus of a simply-laced, simple, simply-connected Lie group (the compact form of the A, D, E groups). By spanning the Lie algebra of T by the coroots $\alpha_i^\vee$, we may identify T with T^N where N is the rank of the group. Let $g_{ij}=\frac12\,\mathrm{tr}\,\alpha_i^\vee\alpha_j^\vee$ where tr is the Killing form normalized so that $g_{ii}=1$. We may write

$$2g_{ij}=d_{ij}+d_{ji}$$

for some integers d_{ij} and set

$$2b_{ij}=d_{ij}-d_{ji}$$

so that $d_{ij}=g_{ij}+b_{ij}$. The corresponding action of the toroidal compactification coincides (mod $2\pi i$) with the action of the WZW model with fields taking values in the corresponding simple group (the $\sim\int\phi^*\omega$ term is the remnant of the topological WZ term). Defining for $p^\vee\in(P^\vee)^{h_\Sigma}$, where $P^\vee$ is the coweight lattice dual to the root lattice,

$$\vartheta_{_{Q^\vee,p^\vee}}(\tau)\ =\ \sum_{q^\vee\,\in\,(Q^\vee)^{h_\Sigma}}\mathrm{e}^{\pi i\,\mathrm{tr}\,(p^\vee+q^\vee)^t,\,\tau\,(p^\vee+q^\vee)}$$

one obtains

$$\vartheta_{_{Q_b}}(\tau,\bar\tau)\ =\ \sum_{[p^\vee]\in(P^\vee/Q^\vee)^{h_\Sigma}}|\vartheta_{_{Q^\vee,p^\vee}}(\tau)|^2\,.$$

In particular at genus 1.

$$Z_d(\tau) \;=\; \sum_{[p^\vee]\in P^\vee/Q^\vee} \left|\frac{\vartheta_{_{Q^\vee,p^\vee}}(\tau)}{\eta(\tau)^N}\right|^2 \;=\; \sum_{[p^\vee]\in P^\vee/Q^\vee} |\mathrm{ch}^1_{[p^\vee]}(\tau)|^2 \tag{13}$$

where $\mathrm{ch}^1_{[p^\vee]}(\tau) = \frac{\vartheta_{_{Q^\vee,p^\vee}}(\tau)}{\eta(\tau)^N}$ runs through the characters of the level 1 representations of the corresponding Kac-Moody algebra. In particular, for the E_8 case $P^\vee = Q^\vee$ and the partition function is the absolute value squared of $\mathrm{ch}^1_0(\tau)$ which is a cubic root of the modular invariant function $j(\tau)$. In general, the right hand side of eq. (13) coincides with the genus 1 partition function of the level 1 WZW model. This remains true for higher genera and for the complete CFT's which is another miraculous coincidence of field theories with fields taking values in quite different target spaces (e.g. the $SU(2)$ WZW model at level 1 is equivalent to the free field with values in S^1 of radius 1).

The fact that the toroidal partition function (13) is a finite sesqui-linear combination of expressions holomorphic in τ is a characteristic feature of rational conformal theories.

Problem 5. Show that the free field compactified on a circle of rational radius squared ($=\beta$) is rational in the above sense.

For free fields with values in the Cartan tori of simply laced groups described above, the general partition functions are hermitian squares with respect to Quillen-like metric of holomorphic sections of projectively flat vector bundles over the moduli spaces of curves. We shall return to these issues during a more detailed discussion of the WZW models.

Example 5. Consider the toroidal compactification to T^2 equipped with the complex structure induced by the complex variable $\psi = \phi^1 + T\phi^2$ (T is in the upper half plane) and with a constant Kähler metric $g = (R_2/T_2)d\psi d\bar\psi$ with $R_2 > 0$ and a constant 2-form $\omega = i(R_1/T_2)\,d\psi\wedge d\bar\psi$. Set $R = R_1 + iR_2$. The partition function of the corresponding free field is

$$Z \;\equiv\; Z_{R,T} \;=\; \vartheta_{_{Q_{R,T}}}(\tau,\bar\tau)\,\left(\frac{\mathrm{vol}_\Sigma\ \det\,\tau_2}{\det'(-\Delta)}\right)$$

where

$$Q_{R,T} \;=\; \{\,(\,\frac{Rm^1+TRm^2+Tn^1-n^2}{\sqrt{2R_2T_2}}\,,\ \frac{\bar Rm^1+T\bar Rm^2+Tn^1-n^2}{\sqrt{2R_2T_2}}\,)\ |\ m^i,n^i\in\mathbf{Z}\ \}\ \subset\ \mathbf{C}\oplus\mathbf{C}$$

with the indefinite quadratic form $|(z_1,z_2)|^2 = |z_1|^2 - |z_2|^2$. Note that $Q_{T.R}$ may be obtained from $Q_{R,T}$ by complex conjugation on the second $\mathbf{C}$. We infer that

$$Z_{R,T} \;=\; Z_{T,R} \tag{14}$$

which is the simplest instance of **mirror symmetry** claiming identity of CFT's with fields in two different Calabi-Yau manifolds with the role of modular parameters of complex and (polarized) Kähler structures interchanged.

Problem 6. Show that, besides the relation (14), the partition function satisfies the identities

$$Z_{R,T}(\tau) \,=\, Z_{R+1,T}(\tau) \,=\, Z_{R,T+1}(\tau) \,=\, Z_{R,-T^{-1}}(\tau) \,=\, Z_{-R^{-1},-T^{-1}}(\tau)\,,$$

which imply the separate $SL_2(\mathbf{Z})$ invariance in R and T.

In general, the moduli space of N-dimensional toroidal compactifications is a double coset

$$O(N)\times O(N)\Big\backslash O(\mathbf{R}^N\oplus\mathbf{R}^N,\left(\begin{smallmatrix} I & 0\\ 0 & -I\end{smallmatrix}\right);\,\mathbf{R}^N\oplus\mathbf{R}^N,\left(\begin{smallmatrix} 0 & I\\ I & 0\end{smallmatrix}\right))\Big/O(\mathbf{R}^N\oplus\mathbf{R}^N,\left(\begin{smallmatrix} 0 & I\\ I & 0\end{smallmatrix}\right)|\,\mathbf{Z})$$

(in a, hopefully, self-explanatory notations) which coincides with the moduli of even self-dual lattices in $\mathbf{R}\oplus\mathbf{R}$ with the indefinite scalar product.

The action functional (12) of the (compactified) two-dimensional massless free field uses only the conformal class of the metric γ on Σ. The regularization of the free field determinants reintroduces however the dependence on the conformal factor of the metric, an effect called **conformal anomaly**. More exactly, one has

$$\frac{\delta}{\delta\sigma(x)}\bigg|_{\sigma=0}\ln\left(\frac{\det'(-\Delta)}{\mathrm{vol}_\Sigma}\right)\;=\;-\tfrac{1}{12\pi}\,r(x)\,,\tag{15}$$

where r is the scalar curvature of Σ, or, denoting the metric dependence of the partition function by the subscript,

$$\frac{\delta}{\delta\sigma(x)}\bigg|_{\sigma=0} Z_{\mathrm{e}^{\sigma}\gamma}\;=\;\tfrac{N}{24\pi}\,r(x)\;Z_\gamma\,.\tag{16}$$

Problem 7. Prove the relation (15) using the identity $\zeta_{-\Delta}(s)=\Gamma(s)^{-1}\int_0^\infty t^{s-1}\,\mathrm{tr}\,\mathrm{e}^{t\Delta}$ and the short time expansion of the heat kernel of $-\Delta$:

$$\mathrm{e}^{t\Delta}(x,x)\;=\;\tfrac{1}{4\pi t}\;+\;\tfrac{1}{12\pi}\,r(x)\;+\;\mathcal{O}(t)$$

Problem 8. Prove that the infinitesimal relation (16) is equivalent to the global one

$$Z_{\mathrm{e}^{\sigma}\gamma}\;=\;\mathrm{e}^{\,\frac{N}{96\pi}\,(\|d\sigma\|^2_{L^2}\,+\,4\int_\Sigma\sigma\,r\,dv)}\;Z_\gamma\tag{17}$$

6. Toroidal compactifications: the correlation functions

Besides the functional integrals for the partition functions, we would like to study the ones for the correlation functions of the massless field with values in S^1 of the type

$$\int\prod_{i=1}^n\mathrm{e}^{i\,q_i\,\phi(x_i)}\;\mathrm{e}^{-\frac{\beta}{4\pi}\int_\Sigma|d\phi|^2\,dv}\;D\phi\,,$$

for integer q_i, see Problem 3 for the $d=0$ example. This may be attempted by the same strategy as before by separating the field into the harmonic and univalued part, as in eq. (7), and then summing over the first and integrating over the second. This gives the expression

$$\sum_{\alpha\in H^1(\Sigma,2\pi\mathbf{Z})}\mathrm{e}^{-\frac{\beta}{4\pi}\|\alpha_h\|^2_{L^2}\,+\,i\sum_i q_i\,\phi_h(x_i)}\;\int\mathrm{e}^{-\frac{\beta}{4\pi}\,(\psi,-\Delta\psi)_{L^2}\,+\,i\sum_i q_i\,\psi(x_i)}\;D\psi\,.$$

The sum over H^1 may be expressed by partial Poisson resummation as an explicit theta-function. As for the functional integral, it may be formally performed by mimicking the finite-dimensional formulae:

$$\int \mathrm{e}^{-\frac{\beta}{4\pi}(\psi,-\Delta\psi)_{L^2}+i\sum_i q_i\psi(x_i)}\, D\psi \;=\; \delta_{\sum_i q_i,0}\;\mathrm{e}^{\frac{\pi}{\beta}\sum_{i,j=1}^{n} q_i q_j\, G(x_i,x_j)}\left(\frac{2\pi\,\mathrm{vol}_\Sigma}{\det'(-\frac{\beta}{2\pi}\Delta)}\right)^{1/2}$$

where the Kronecker delta is contributed by the integral over the constant mode of ψ and $G(x,y)=G(y,x)$ is a Green function of Δ satisfying $\Delta_x G(x,y)=\delta(x,y)-\frac{1}{\mathrm{vol}_\Sigma}$ (the constant ambiguity should drop out above due to the vanishing of $\sum q_i$). The obvious problem with the above formula is that

$$G(x,y)\;=\;\tfrac{1}{2\pi}\ln\mathrm{dist}(x,y)\;+\;\text{finite}$$

when $y\to x$ so that $G(x,x)$ is not defined. This is a standard problem with the short-distance singularities due to distributional character of typical configurations in the Gaussian free field measure. A possible treatment is to renormalize the above expression by replacing the divergent contributions by their finite parts

$$\tilde{G}(x,x)\;=\;\lim_{y\to x}\,G(x,y)\,-\,\tfrac{1}{2\pi}\,\ln\mathrm{dist}(x,y) \tag{18}$$

Upon division by the partition function, all this leads to a well defined renormalized expression for the correlation functions which we shall denote by

$$\langle :\mathrm{e}^{iq_1\phi(x_1)}:\,\cdots\,:\mathrm{e}^{iq_n\phi(x_n)}:\rangle_\gamma$$

where the colons remind the renormalization procedure (which is closely related to the Wick ordering discussed in Kazdan's lectures). The correlations depend on the "charges" q_i and positions x_i but also on the metric γ on Σ which is signaled by the subscript. In particular, on $\mathbf{CP}^1$ with $H^1=0$, one may take $G(x,y)=\frac{1}{2\pi}\ln|z(x)-z(y)|$ in the standard complex variable and for the metric $g=\mathrm{e}^\sigma\,dz d\bar{z}$ with a conformal factor e^σ, we obtain, setting $z_i\equiv z(x_i)$,

$$\langle :\mathrm{e}^{iq_1\phi(x_1)}:\,\cdots\,:\mathrm{e}^{iq_n\phi(x_n)}:\rangle_\gamma\;=\;\delta_{\sum_i q_i,0}\;\mathrm{e}^{-\sum_i\frac{q_i^2}{4\beta}\sigma(x_i)}\prod_{i<j}|z_i-z_j)|^{\frac{q_iq_j}{\beta}}. \tag{19}$$

The dependence on the conformal factor of the metric is solely due to the renormalization (18) and persists in general:

$$\langle :\mathrm{e}^{iq_1\phi(x_1)}:\,\cdots\,:\mathrm{e}^{iq_n\phi(x_n)}:\rangle_{\mathrm{e}^\sigma\gamma}\;=\;\mathrm{e}^{-\sum_i\Delta_i\sigma(x_i)}\langle :\mathrm{e}^{iq_1\phi(x_1)}:\,\cdots\,:\mathrm{e}^{iq_n\phi(x_n)}:\rangle_\gamma \tag{20}$$

where $\Delta_i=\frac{q_i^2}{4\beta}$ are the conformal dimensions of the (Euclidean) fields $:\mathrm{e}^{iq_i\phi}:$. The generalization to the toroidal compactifications is straightforward. The conformal dimensions of fields $:\mathrm{e}^{iq\phi}:$ where $q\in\mathbf{Z}^N$ is now $\frac{1}{4}(q,g^{-1}q)$.

The operator picture of the the free field compactified on S^1 is as follows. The quantum Hibert space is

$$\mathcal{H}\;=\;L^2(S^1,d\varphi_0)^{\mathbf{Z}}\otimes\mathcal{F}\otimes\tilde{\mathcal{F}}\,.$$

Above $L^2(S^1, d\varphi_0)^{\mathbf{Z}}$ is the infinite sum of copies of $L^2(S^1)$, each labeled by an integer ("winding number") w. The Fock space $\mathcal{F}$ is generated by applying operators α_n, $n = -1, -2, \ldots$, to a vector annihilated by α_n with $n = 1, 2, \ldots$,

$$[\alpha_n, \alpha_m] = n\,\delta_{n,-m}\,, \qquad \alpha_n^* = \alpha_{-n}\,,$$

and $\tilde{\mathcal{F}}$ is another copy of $\mathcal{F}$. Let $|p, w\rangle$ denote the function $\frac{1}{\sqrt{2\pi}}\mathrm{e}^{ipx}$ in the w^{th} copy of $L^2(S^1, d\varphi_0)^{\mathbf{Z}}$. Integer p is the eigenvalue of the operator $p_0 = \frac{1}{i}\frac{d}{d\varphi_0}$. $\mathcal{H}$ may be generated by applying operators α_n, $\tilde{\alpha}_n$ with negative n to vectors $|p, w\rangle$. The (multivalued) free field operator (with time dependence) is given by

$$\varphi(t, x) = \varphi_0 + \beta^{-1} p_0 t + w x + \frac{i}{\sqrt{2\beta}} \sum_{n\neq 0} \big(\tfrac{\alpha_n}{n}\,\mathrm{e}^{-i(t+x)n} - \tfrac{\tilde{\alpha}_n}{n}\,\mathrm{e}^{-i(t-x)n} \big).$$

The relation to the $m \to 0$ limit of the massive free field on periodic interval of length $L = 2\pi$ given by eq. (6) should be evident. Modulo the constant and winding modes, for $n = 1, 2, \ldots$,

$$\alpha_n = -i\sqrt{n}\,a_{-n}\,, \quad \alpha_{-n} = i\sqrt{n}\,a_{-n}^*\,, \quad \tilde{\alpha}_n = i\sqrt{n}\,a_n\,, \quad \tilde{\alpha}_{-n} = -i\sqrt{n}\,a_n^*\,.$$

The relabeling allows to separate the left-moving part (involving α_n) from the right-moving one (containing $\tilde{\alpha}$). The one-handed parts of $\varphi(t, x)$ are called chiral fields (the zero modes can be separated too). For the uncompactified massless free field, the $\mathbf{k} = 0$ mode contributes to the Hamiltonian the term $\sim -d^2/d\varphi_0^2$ acting on $L^2(\mathbf{R}, d\varphi_0)$ which has continuous spectrum. The compactification of the field (and consequently of its zero mode), restores the discreteness of the energy spectrum.

Hilbert space $\mathcal{H}$ carries a representation of two commuting copies of the Virasoro algebra with generators L_n and $\tilde{L}_n$. Explicitly,

$$L_n = \tfrac{1}{2} \sum_{m\in\mathbf{Z}} :\alpha_m \alpha_{n-m}:$$

(creator to the left of annihilators) and similarly for $\tilde{L}_n$ where we have set $\alpha_0 = \frac{1}{\sqrt{2}}(\beta^{1/2} w + \beta^{-1/2} p_0)$ and $\tilde{\alpha}_0 = \frac{1}{\sqrt{2}}(\beta^{1/2} w - \beta^{-1/2} p_0)$. The quantum Hamiltonian $H = L_0 + \tilde{L}_0$. $P = L_0 - \tilde{L}_0$ generates the space translations. $\Omega = |0, 0\rangle$ is the ground state of H (it is also annihilated by P).

Problem 9. Show that L_n's indeed satisfy the Virasoro algebra relations (with unit central charge)

$$[L_n, L_m] = (n - m)\,L_{n+m} + \tfrac{1}{12}(n^3 - n)\,\delta_{n,-m}\,.$$

Let us introduce the "vertex operators"

$$V_q(t, x) = :\mathrm{e}^{iq\varphi(t,x)}:$$

defined as the (formal) power series in α_n and $\tilde{\alpha}_n$ reordered by putting the creation operators with negative n indices to the left of the annihilation operators corresponding to positive n and also φ_0 operators to the left of p_0.

Problem 10. Using the operator relation $\mathrm{e}^A\mathrm{e}^B = \mathrm{e}^B\mathrm{e}^A\mathrm{e}^{[A,B]}$ holding if $[A,B]$ commutes with A and B, show that for $t_1 < t_2 < \cdots < t_n$,

$$(\,\Omega\,,\,V_{q_1}(it_1,x_1)\,\cdots\,V_{q_n}(it_n,x_n)\,\Omega\,)\;=\;\delta_{\sum_i q_i,0}\,\prod_{i<j}|z_i-z_j|^{\frac{q_iq_j}{\beta}}$$

where $z_j = \mathrm{e}^{-t_j+ix_j}$. This, together with eq. (19), provides a spherical version of the Feynman-Kac formula.

In variables $z = \mathrm{e}^{i(t+x)}$ and $\tilde{z} = \mathrm{e}^{i(t-x)}$, the commutation relations of L_n's and $\tilde{L}_n$'s with the vertex operators take the form

$$\begin{aligned}[L_n\,,\,V_q(z,\tilde{z})] \;&=\; (n+1)\,\Delta\,z^n\,V_q(z,\tilde{z})\,+\;z^{n+1}\,\partial_z\,V_q(z,\tilde{z})\,,\\ [\tilde{L}_n\,,\,V_q(z,\tilde{z})] \;&=\; (n+1)\,\Delta\,\tilde{z}^n\,V_q(z,\tilde{z})\,+\;\tilde{z}^{n+1}\,\partial_{\tilde{z}}\,V_q(z,\tilde{z})\end{aligned} \tag{21}$$

where above $\Delta = \frac{q^2}{4\beta}$ or $\Delta = \frac{1}{4}\,(q, g^{-1}q)$ stands for the conformal dimension of the operator. Later on, we shall see that these relations essentially follow from eq. (20) and the general covariance of the corresponding correlation functions. In the professional jargon, the fields satisfying such commutation relations are called primary Virasoro operators.

On the level of the Hilbert space $\mathcal{H}\equiv\mathcal{H}_\beta$, T-duality becomes the unitary transformation $U_T:\;\mathcal{H}_\beta\,\rightarrow\,\mathcal{H}_{1/\beta}$ such that

$$U_T|u,w\rangle\;=\;(-1)^{uw}\,|w,u\rangle\qquad\text{and}\qquad U_T\,\alpha_n\;=\;\alpha_n\,U_T\,,\quad U_T\,\tilde{\alpha}_n\;=\;-\,\tilde{\alpha}_n\,U_T\,,$$

where, for $u,w\in\mathbf{Z}$, $|u,w\rangle$ denotes the function $\frac{1}{\sqrt{2\pi}}\mathrm{e}^{iu\phi_0}$ in the w-component of $L^2(S^1,d\phi_0)^{\mathbf{Z}}$. U_T intertwines the action of the Virasoro algebras in $\mathcal{H}_\beta$ and $\mathcal{H}_{1/\beta}$ and maps the vertex operators in $\mathcal{H}_\beta$ to new operators which should be considered on the equal footing with the original ones.

Up to now, we have considered only S^1-valued free fields with periodic boundary conditions. In string theory aplications, one also considers fields on space with boundaries and with fixed boundary conditions like the Neumann ones (open strings). Quantizing such fields on space which is an interval $[0,\pi]$ $(d=1)$ one obtains quantum field

$$\varphi^N(t,x)\;=\;\varphi_0\;+\;\beta^{-1}pt\;+\;\frac{i}{\sqrt{2\beta}}\sum_{n\neq0}\Big(\,\frac{\alpha^N_n}{n}\,\mathrm{e}^{-i(t+x)n}\;+\;\frac{\alpha^N_n}{n}\,\mathrm{e}^{-i(t-x)n}\Big)$$

which may be realized in the subspace of the periodic b.c. Hilbert space $\mathcal{H}_\beta$ generated by applying operators $\frac{1}{\sqrt{2}}(\alpha_n-\tilde{\alpha}_n) = \alpha^N_n$ with negative n to vectors $|u,0\rangle$. The T-duality maps this field into the one corresponding to the Dirichlet boundary conditions

$$\varphi^D(t,x)\;=\;\varphi_0\;+\;wx\;+\;\frac{i}{\sqrt{2\beta}}\sum_{n\neq0}\Big(\,\frac{\alpha^D_n}{n}\,\mathrm{e}^{-i(t+x)n}\;-\;\frac{\alpha^D_n}{n}\,\mathrm{e}^{-i(t-x)n}\Big)$$

where φ_0 is fixed modulo $\pi\mathbf{Z}$. $\varphi^D(t,x)$ acts in the subspace of $\mathcal{H}_{1/\beta}$ generated by applying $\frac{1}{\sqrt{2}}(\alpha_n + \tilde\alpha_n) = \alpha^D_n$ to vectors $|0,w\rangle$. In toroidal compactifications with more dimensions of the target, one may have mixed "D-brane"-type boundary conditions with some coordinates of the field fixed to prescribed values at the ends of the space-interval $[0,\pi]$.

Problem 11 (Massless fermions on Riemann surface).

Let (Σ,γ) be a Riemann surface. Spin structure on Σ may be identified with the square root L of the canonical bundle $K = T^{*1,0}(\Sigma)$. A Dirac spinor $\Psi = (\psi,\tilde\psi)$ is an element of $?(L\oplus\bar L)$ where $\bar L$ is the bundle complex conjugate to L. The conjugate spinor is $\bar\Psi = (\tilde\chi,\chi) \in ?(\bar L \oplus L)$ and in the euclidean Dirac theory it should be treated as an independent field ($\chi = \psi,\ \tilde\chi = \tilde\psi$ for Majorana fermions). Denote by $\bar\partial_L$ the $\bar\partial$ operator of L and by $\partial_{\bar L}$ its complex conjugate which may be naturally identified with $\bar\partial_L^*$. The action is a function on the odd vector space $\Pi(?(L\oplus\bar L)\oplus ?(\bar L\oplus L))$:

$$S(\Psi,\bar\Psi) \,=\, -\tfrac{1}{\pi}\int_\Sigma (\chi\bar\partial_L\psi \,+\, \tilde\chi\partial_{\bar L}\tilde\psi)$$

(note that the integrand is naturally a 2-form). Partition functions of the Dirac fermions are given by the formal Berezin integral

$$Z_L \,=\, \int \mathrm{e}^{-S(\Psi,\bar\Psi)}\, D\bar\Psi\, D\Psi \,=\, \det(\partial_{\bar L})\,\det(\bar\partial_L) \,=\, \det(\bar\partial_L^*\bar\partial_L)\,.$$

The last determinant may be zeta-regularized giving a precise sense to the partition function Z_L of the Dirac field on Σ.

On the elliptic curve $\mathbf{C}/(\mathbf{Z}+\tau\mathbf{Z})$ with τ in the upper half-plane, the canonical bundle K may be trivialized by the section dz and spin structures correspond to the choice of periodic or anti-periodic boundary conditions under $z\to z+1$ and $z\to z+\tau$:

$$L \,=\, pp,\ pa,\ ap,\ aa\,.$$

(a). Show that the eigenvalues of $\bar\partial_L^*\bar\partial_L$ are

$$\lambda_{m,n} \,=\, (\tfrac{\pi}{\tau_2})^2\,|\tau m + n|^2$$

with

$$\begin{array}{llll} m\in\mathbf{Z}\,, & n\in\mathbf{Z} & \text{for} & L = pp\,,\\ m\in\mathbf{Z}\,, & n\in\mathbf{Z}+\frac12 & \text{for} & L = pa\,,\\ m\in\mathbf{Z}+\frac12\,, & n\in\mathbf{Z} & \text{for} & L = ap\,,\\ m\in\mathbf{Z}+\frac12\,, & n\in\mathbf{Z}+\frac12 & \text{for} & L = aa\,. \end{array}$$

(b). Infer that

$$Z_{pp}(\tau) \,=\, 0\,.$$

(c). Show that

$$\zeta_{\bar\partial^*_{pa}\bar\partial_{pa}}(s) \,=\, 2^{2s}\,(\zeta_{-\Delta'_{2\tau}}(s) \,-\, \zeta_{-\Delta'_{\tau}}(s))\,.$$

Infer from Eq. (11) that

$$Z_{pa}(\tau) \,=\, 4|q^{1/24}\prod_{n=1}^{\infty}(1+q^n)|^4.$$

In the Hilbert space picture

$$Z_{pa}(\tau) \,=\, \mathrm{tr}_{\mathcal{H}_R\otimes\tilde{\mathcal{H}}_R}\; q^{L_0-1/24}\bar{q}^{\tilde{L}_0-1/24}. \tag{22}$$

The "Ramond sector" Hilbert space is $\mathcal{H}_R\otimes\tilde{\mathcal{H}}_R$ with

$$\mathcal{H}_R \,=\, \mathbf{C}^2\otimes\Big(\wedge(\bigoplus_{n=1}^{\infty}\mathbf{C})\Big)^{\otimes 2} \tag{23}$$

and $\tilde{\mathcal{H}}_R$ is another copy of $\mathcal{H}_R$. L_0 acts in the first copy. It has eigenvalue $\frac{1}{8}$ on $\mathbf{C}^2$ (the "Ramond ground states") and the occupied n^{th} mode in the fermionic Fock space adds n to it. The periodic partition function is

$$Z_{pp}(\tau) \,=\, \mathrm{tr}_{\mathcal{H}_R\otimes\tilde{\mathcal{H}}_R}\,(-1)^{F+\tilde{F}}\; q^{L_0-1/24}\bar{q}^{\tilde{L}_0-1/24} \;\equiv\; \mathrm{str}_{\mathcal{H}_R\otimes\tilde{\mathcal{H}}_R}\;\, q^{L_0-1/24}\bar{q}^{\tilde{L}_0-1/24} \tag{24}$$

where $(1,0),(0,1)\,\in\,\mathbf{C}^2$ correspond to the eigenvalues $+1,-1$ of $(-1)^F$ and each occupied fermionic Fock space mode adds 1 to F. $Z_{pp}(\tau)$ vanishes since modes with odd and even Fermi numbers are paired.

(d). Show that

$$\zeta_{\bar{\partial}^*_{ap}\bar{\partial}_{ap}}(s) \,=\, 2^{4s}\zeta_{-\Delta'_{\tau/2}}(s)\,-\,2^{2s}\zeta_{-\Delta'_{\tau}}(s) \qquad\text{and}\qquad Z_{ap}(\tau) \,=\, |q^{-1/48}\prod_{n=0}^{\infty}(1-q^{n+1/2})|^4.$$

The Hilbert space interpretation is

$$Z_{ap}(\tau) \,=\mathrm{str}_{\mathcal{H}_{NS}\otimes\tilde{\mathcal{H}}_{NS}}\;\, q^{L_0-1/24}\bar{q}^{\tilde{L}_0-1/24} \tag{25}$$

where the "Neveu-Schwarz sector" Hilbert space is

$$\mathcal{H}_{NS} \,=\, \Big(\wedge(\bigoplus_{n=0}^{\infty}\mathbf{C})\Big)^{\otimes 2}\,. \tag{26}$$

The "Neveu-Schwarz ground state" has eigenvalue zero of L_0 and the n^{th} occupied zero mode contributes $(n+\frac{1}{2})$ to it. The fermion number of the NS-ground state vanishes and each occupied fermionic mode adds 1 to it.

(e). Show that

$$\zeta_{\bar{\partial}^*_{aa}\bar{\partial}_{aa}}(s) \,=\, 2^{2s}\zeta_{\bar{\partial}^*_{pa}\bar{\partial}_{pa}}(s)\Big|_{\tau/2}\,-\,\zeta_{\bar{\partial}^*_{pa}\bar{\partial}_{pa}}(s)\Big|_{\tau} \qquad\text{and}\qquad Z_{aa}(\tau) \,=\, |q^{-1/48}\prod_{n=0}^{\infty}(1+q^{n+1/2})|^4$$

and that

$$Z_{aa}(\tau) \,=\mathrm{tr}_{\mathcal{H}_{NS}\otimes\tilde{\mathcal{H}}_{NS}}\;\, q^{L_0-1/24}\bar{q}^{\tilde{L}_0-1/24}. \tag{27}$$

(f). Prove the modular properties:

$$\begin{aligned} &Z_{pa}(\tau+1) \,=\, Z_{pa}(\tau), &&Z_{ap}(\tau+1) \,=\, Z_{aa}(\tau), &&Z_{aa}(\tau+1) \,=\, Z_{ap}(\tau),\\ &Z_{pa}(-1/\tau) \,=\, Z_{ap}(\tau), &&Z_{ap}(-1/\tau) \,=\, Z_{pa}(\tau), &&Z_{aa}(-1/\tau) \,=\, Z_{aa}(\tau). \end{aligned}$$

Problem 12 Bosonization.

The spin structure is called even (odd) if the dimension of the kernel of $\bar{\partial}_L$ is even (odd). Denote by $\sigma(L)$ the parity of L. The bosonization formula asserts that

$$\tfrac{1}{2}\sum_L (-1)^{\sigma(L)}\, Z_L \;=\; C^{h_\Sigma - 1}\, Z_{1/2} \tag{28}$$

where on the right hand side we have the partition function of the bosonic free field with values in the circle of radius squared $\frac{1}{2}$, C is a constant and h_Σ the genus of the Riemann surface Σ. These equalities extend to correlations. For example, the fermionic fields $(\psi\tilde{\psi})(x)$ correspond to bosonic fields $:\mathrm{e}^{i\phi(x)}:$ and $(\tilde{\chi}\chi)(x)$ to $:\mathrm{e}^{-i\phi(x)}:$. What are their conformal weights? Prove identity (28) for $\Sigma = \mathbf{C}/(\mathbf{Z}+\tau\mathbf{Z})$ using the expression (11) for $Z_{1/2}(\tau)$ and the classical product expressions for the theta functions

$$\vartheta(z|\tau) \equiv \sum_{n\in\mathbf{Z}} \mathrm{e}^{\pi i\tau n^2 + 2\pi i n z} \;=\; \prod_{n=1}^{\infty}(1-q^n)\,(1+\mathrm{e}^{2\pi i z}\, q^{n-1/2})\,(1+\mathrm{e}^{-2\pi i z}\, q^{n-1/2})\,.$$

What is the Hilbert space interpretation of the left hand side of Eq. (28) on the elliptic curve?

In summary, by "calculating" the functional integrals for compactified massless free fields we have constructed models of two-dimensional CFT specified by giving the partition functions and correlation functions on general Riemann surfaces. In the next lecture(s), we shall examine the emerging CFT structure on a more abstract level.

References

A set of Gaussian integration formulae may be found e.g. in "Quantum Field Theory and Critical Phenomena" by J. Zinn-Justin (Sects. 1.1 and 1.2). See also Sects. 2.0-2.2 and 2.5 for the discussion of of the path integral and the Feynman-Kac formula. Of course, for the latter topic, "Quantum Mechanics and Path Integral" by Feynman-Hibbs is the physics classic. Rigorous theory of infinite-dimensional Gaussian integrals may be found e.g. in the 4^{th} volume of Gelfand-Vilenkin. See also Simon's "The Euclidean $P(\phi)_2$ Quantum Field Theory".

The free fields with values in S^1 are discussed briefly e.g. in the Ginspargs contributions to Les Houches 1988 School (Session XLIV "Fields, Strings and Critical Phenomena", eds. Brezin-Zinn-Justin) or in Drouffe-Itzykson: "Théorie Statistique des Champs", InterEditions 1989, (Sects. 3.2 and 3.6). For the case of fields with values in a torus see the paper by Narain-Sarmadi-Wittem in Nucl. Phys. B 279 (1987) p. 369 and for the case of complex torus read Vafa's contribution to "Essays on Mirror Symmetry", ed. S.-T. Yau, International Press, Hong Kong 1992.

Free fermions and bosonization on a Riemann surface are discussed in the paper by Alvarez-Gaumé-Bost-Moore-Nelson-Vafa in Commun. Math. Phys. 112 (1987), p. 503.

Lecture 2. Axiomatic approaches to conformal field theory

Contents:

1. Conformal field theory data

In the first lecture, we have discussed a functional-integral construction of the simplest models of CFT: the toroidal compactifications (of massless free fields). In this lecture we shall present a more general approach to CFT which, although not overly formalized, will be axiomatic in spirit using only the most general properties of the free field models. We shall assume that the basic data of a CFT model specify for each compact Riemann surface (Σ, γ) its partition function $Z_\gamma > 0$ and a set of its correlation functions $\langle\, \phi_{l_1}(x_1) \cdots \phi_{l_n}(x_n)\, \rangle_\gamma$ of the "primary fields" from a fixed set $\{\phi_l\}$. The correlation functions are symmetric in the pairs of arguments (x_i, l_i), are defined for non-coincident insertion points $x_i \in \Sigma$ and are assumed smooth. We shall also need later some knowledge of their short distance singularities. The dependence of both the partition and the correlation functions on the Riemannian metric γ will be assumed regular enough to assure existence of distributional functional derivatives of arbitrary order. The basic hypothesis are the following symmetry properties:

(i) diffeomorphism covariance

$$Z_\gamma \;=\; Z_{D^*\gamma}\ , \tag{2}$$

$$\langle\, \phi_{l_1}(D(x_1)) \cdots \phi_{l_n}(D(x_n))\, \rangle_\gamma \;=\; \langle\, \phi_{l_1}(x_1) \cdots \phi_{l_n}(x_n)\, \rangle_{D^*\gamma}\ , \tag{3}$$

(ii) local scale covariance

$$Z_{\mathrm{e}^\sigma\gamma} \;=\; \mathrm{e}^{\frac{c}{96\pi}\,(\|d\sigma\|^2_{L^2} + 4\int_\Sigma \sigma\, r\, dv)}\ Z_\gamma\ , \tag{4}$$

$$\langle\, \phi_{l_1}(x_1) \cdots \phi_{l_n}(x_n)\, \rangle_{\mathrm{e}^\sigma\gamma} \;=\; \prod_{i=1}^{n} \mathrm{e}^{-\Delta_{l_i}\sigma(x_i)}\ \langle\, \phi_{l_1}(x_1) \cdots \phi_{l_n}(x_n)\, \rangle_\gamma \tag{5}$$

where c is the central charge of the theory. In (**i**), we limit ourselves to orientation preserving diffeomorphism assuming that under the change of orientation of the surface,

$$Z_\gamma \mapsto Z_\gamma\ , \tag{6}$$

$$\langle\, \phi_{l_1}(x_1) \cdots \phi_{l_n}(x_n)\, \rangle_\gamma \mapsto \overline{\langle\, \phi_{\bar l_1}(x_1) \cdots \phi_{\bar l_n}(x_n)\, \rangle_\gamma}\ , \tag{7}$$

where $\phi_l \mapsto \phi_{\bar{l}}$ is an involution of the set of primary fields preserving their conformal weights ($: \mathrm{e}^{iq\phi} : \mapsto : \mathrm{e}^{-iq\phi} :$ for the toroidal compactifications). In what follows we shall first explore the implications of the above identities which we shall, jointly, call **conformal symmetries**. Other important properties of the correlation functions, for example those responsible for the Hilbert space interpretation of the theory, will be introduced and analyzed later.

Let us define new correlation functions with insertions of **energy-momentum tensor**[1] by setting

$$\begin{aligned}&\langle\, T_{\mu_1\nu_1}(y_1) \cdots T_{\mu_m\nu_m}(y_m)\, \phi_{l_1}(x_1) \cdots \phi_{l_n}(x_n)\,\rangle_\gamma \\ &= Z_\gamma^{\ -1}\, \frac{(4\pi)^m\, \delta^m}{\delta\gamma^{\mu_1\nu_1}(y_1) \cdots \delta\gamma^{\mu_m\nu_m}(y_m)}\, Z_\gamma\, \langle\, \phi_{l_1}(x_1)\ \cdots\ \phi_{l_n}(x_n)\rangle_\gamma\ ,\end{aligned} \tag{8}$$

where $\gamma^{\mu\nu}\partial_\mu\partial_\nu \equiv \gamma^{-1}$ is the inverse metric. In complex coordinates, energy-momentum tensor has the components

$$T_{zz} = \overline{T_{\bar{z}\bar{z}}} \qquad \text{and} \qquad T_{z\bar{z}} = T_{\bar{z}z} = \overline{T_{z\bar{z}}}\ .$$

By definition, the correlation functions $\langle\, T_{\mu_1\nu_1}(y_1) \cdots T_{\mu_m\nu_m}(y_m)\, \phi_{l_1}(x_1) \cdots \phi_{l_n}(x_n)\,\rangle_\gamma$ are distributions in their dependence on $y_1, \ldots, y_m$. As we shall see below, they are given by smooth functions for non-coincident arguments and away from x_i's, but we shall also have to study their distributional behavior at coinciding points.

2. Conformal Ward identities

Symmetries in QFT are expressed as **Ward identities** between correlation functions. Eqs. (3) and (5) are examples of such relations for group-like conformal symmetries. It is often useful to work out also Ward identities corresponding to infinitesimal, Lie algebra version of symmetries. We shall do this here for the infinitesimal conformal symmetries. The resulting formalism was the starting point of the 1984 Belavin-Polyakov-Zamolodchikov's paper. The approach presented here is close in spirit to the 1987 article by Eguchi-Ooguri (to some extend also to Friedan's 1982 Les Houches lecture notes). The general strategy is to expand the global symmetry identities to the second order in infinitesimal symmetries. This will be a little bit technical so you might wish to see first the results listed at the end of this section.

Let us start by exploring the infinitesimal version of the local scale covariance (4). Using the definition (8), we obtain the relation

$$4\pi\, Z_\gamma^{-1}\, \frac{\delta}{\delta\sigma}\bigg|_{\sigma=0} Z_{e^\sigma\gamma} = -\gamma^{zz}\,\langle T_{zz}\rangle_\gamma - 2\gamma^{z\bar{z}}\,\langle T_{z\bar{z}}\rangle_\gamma - \gamma^{\bar{z}\bar{z}}\,\langle T_{\bar{z}\bar{z}}\rangle_\gamma = \tfrac{c}{6}\, r\,. \tag{9}$$

Note that if $\gamma = |dz|^2$ then $\gamma_{zz} = \gamma_{\bar{z}\bar{z}} = \gamma^{zz} = \gamma^{\bar{z}\bar{z}} = 0$, $\gamma_{z\bar{z}} = \frac{1}{2}$ and $\gamma^{z\bar{z}} = 2$. Besides, the scalar curvature of γ vanishes. In such a metric, eq. (9) reduces to the equality

$$\langle T_{z\bar{z}}\rangle = 0 \tag{10}$$

which states that energy-momentum tensor in a CFT is traceless (in the flat metric, $\mathrm{tr}\, T_{\mu\nu} = 4T_{z\bar{z}}$). It is the first example of Ward identities expressing the infinitesimal conformal invariance on the quantum level. We shall see further identities of this type below.

[1] called also stress tensor in a static view of the Euclidean field theory

Notice that if $\gamma \mapsto e^\sigma\gamma$ with $\sigma = 1$ around the insertion points then the correlation functions do not change. Let us fix the complex structure of Σ and holomorphic complex coordinates around the insertion points of a correlation function. Call a metric γ locally flat if it is compatible with the complex structure of Σ and of the form $|dz|^2$ around the insertions. For such a choice of γ we shall drop the subscript "γ" in the notation for the correlation functions, like in eq. (10). We may restore the full dependence on the conformal factor by using the covariance relations (4) and (5). For example, for $\langle T_{zz}\rangle_\gamma$, we obtain

$$\langle T_{zz}\rangle_{e^\sigma\gamma} = \langle T_{zz}\rangle_\gamma + \tfrac{c}{24}\tfrac{\delta}{\delta\gamma^{zz}}\big(\|\partial\sigma\|^2_{L^2} + 4\int_\Sigma \sigma\, r\, dv\big)\,. \tag{11}$$

In order to compute the functional derivative on the right hand side, we shall need the following

Lemma. Let $\gamma^{z\bar z} = \gamma^{\bar z z} = 2$. To the first order in γ^{zz},

$$r = -\tfrac{1}{2}(\partial_z^2\gamma^{zz} + \partial_{\bar z}^2\gamma^{\bar z\bar z})\,. \tag{12}$$

Proof. Consider the inverse metric $\gamma^{-1} = \gamma^{zz}\,\partial_z^2 + 4\partial_z\partial_{\bar z} + \gamma^{\bar z\bar z}\partial_{\bar z}^2$. To compute the curvature to the first order in γ^{zz} we shall change the variables to $z' = z + \zeta(z,\bar z)$ so that in the new coordinate the metric is $(4+\rho)\partial_{z'}\partial_{\bar z'}$. Since

$$\partial_z = (1+\partial_z\zeta)\partial_{z'} + (\partial_z\bar\zeta)\partial_{\bar z'}\,, \qquad \partial_{\bar z} = (\partial_{\bar z}\zeta)\partial_{z'} + (1+\partial_{\bar z}\bar\zeta)\partial_{\bar z'}$$

then, retaining only the terms of the first order in γ^{zz}, ζ (and their complex conjugates), we obtain

$$\begin{aligned}
\gamma^{-1} \;=\;& (\gamma^{zz} - (\partial_x\gamma^{zz})\zeta - (\partial_{\bar z}\gamma^{zz})\bar\zeta + 2\gamma^{zz}\,\partial_z\zeta + 4\partial_{\bar z}\zeta)\,\partial_{z'}^2\\
+\;& (4 + 4\partial_z\zeta + 4\partial_{\bar z}\bar\zeta + 2\gamma^{zz}\,\partial_z\bar\zeta + 2\gamma^{\bar z\bar z}\,\partial_{\bar z}\zeta)\,\partial_{z'}\partial_{\bar z'}\\
+\;& (\gamma^{\bar z\bar z} - (\partial_{\bar z}\gamma^{\bar z\bar z})\bar\zeta - (\partial_z\gamma^{\bar z\bar z})\zeta + 2\gamma^{\bar z\bar z}\partial_{\bar z}\bar\zeta + 4\partial_z\bar\zeta)\,\partial_{\bar z'}^2
\end{aligned} \tag{13}$$

(we have kept more terms then needed for the Lemma for a future use). The requirement that $\gamma^{-1} = (4+\rho)\partial_{z'}\partial_{\bar z'}$ means in the leading order that $\partial_{\bar z}\zeta = -\frac{1}{4}\gamma^{zz}$. Hence to the first order in γ^{zz}

$$\begin{aligned}
r'v' = -i\,\bar\partial'\partial'\log\,(1+\partial_z\zeta+\partial_{\bar z}\bar\zeta) \;&=\; i\,\partial_{\bar z}\partial_z(\partial_z\zeta + \partial_{\bar z}\bar\zeta)\,dz\wedge d\bar z\\
&=\; -\tfrac{1}{2}\,(\partial_z^2\gamma^{zz} + \partial_{\bar z}^2\gamma^{\bar z\bar z})\,v'
\end{aligned}$$

where v' is the new volume form equal to $\frac{i}{2}dz\wedge d\bar z$ in the 0^{th} order.

Using the Lemma and the relation $\|d\sigma\|^2_{L^2} = \int_\Sigma(\partial_\mu\sigma)(\partial_\nu\sigma)\,\gamma^{\mu\nu}\,dv$, we obtain the relation

$$\tfrac{\delta}{\delta\gamma^{zz}}\big(\|d\sigma\|^2_{L^2} + 4\int_\Sigma \sigma\, r\, dv\big) = -2\,\partial_z^2\sigma + (\partial_z\sigma)^2 \tag{14}$$

which, substituted into eq. (11), gives the dependence on the conformal factors of the expectation value of T_{zz}:

$$\langle T_{zz}\rangle_{e^\sigma dz d\bar z} = \langle T_{zz}\rangle - \tfrac{c}{12}\big(\partial_z^2\sigma - \tfrac{1}{2}(\partial_z\sigma)^2\big)\,. \tag{15}$$

What are the transformation properties of $\langle T_{zz}\rangle$ under holomorphic changes $z \mapsto z' = f(z)$ of the local coordinate? Under such replacements the notion of a locally flat metric changes accordingly. By the diffeomorphism covariance and eq. (15), we have

$$\begin{aligned}
&(\tfrac{dz'}{dz})^2 \langle T_{z'z'}\rangle \,=\, \langle T_{zz}\rangle_{|dz'/dz|^2 dz d\bar z} \\
&= \langle T_{zz}\rangle - \tfrac{c}{12}\left(\partial_z^2 \log{(dz'/dz)} - \tfrac{1}{2}(\partial_z \log{(dz'/dz)})^2\right) \\
&= \langle T_{zz}\rangle - \tfrac{c}{12}\left(\tfrac{d^3z'/dz^3}{dz'/dz} - \tfrac{3}{2}(\tfrac{d^2z'/dz^2}{dz'/dz})^2\right) \equiv \langle T_{zz}\rangle - \tfrac{c}{12}\{z';z\}\ . \qquad (16)
\end{aligned}$$

The function $\{z';z\}$ is the **Schwarzian derivative** of the change of variables. As we see, in the correlation functions with locally flat metric, T_{zz} does not transform as a pure quadratic differential under general holomorphic changes of variables. The transformation law (16) defines what is called a **projective connection** on Σ.

Problem 1. Show that the Schwarzian derivative $\{z';z\}$ vanishes iff $z' = \frac{az+b}{cz+d}$ with $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in SL(2,\mathbf{C})$, i.e. for the Möbius transformations.

The further information about the correlation functions with energy-momentum tensor insertions will be obtained by studying deviations of the metric from the locally flat one. Applying to eq. (9) the operator $\frac{\pi}{Z_\gamma}\frac{\delta}{\delta\gamma^{ww}} Z_\gamma$ at γ locally flat[2] and using eq. (12), we obtain

$$\pi\,\delta^{(2)}(z-w)\langle T_{zz}\rangle + \langle T_{ww} T_{z\bar z}\rangle \,=\, \tfrac{\pi c}{12}\,\partial_z^2\delta^{(2)}(z-w)\ , \qquad (17)$$

where $\delta^{(2)}$ stands for the two-dimensional δ-function. Let us explore now the implications of the diffeomorphism covariance (2) and (3). Under an infinitesimal transformation

$$D(z) \,=\, z + \zeta(z,\bar z) \equiv z'\ , \qquad (18)$$

the change in the inverse metric $\delta\gamma^{-1} = {\gamma'}^{-1} - \gamma^{-1}$, where $D^*\gamma' = \gamma$, may be read from eq. (13):

$$\begin{aligned}
\delta\gamma^{zz} &= -(\partial_z\gamma^{zz})\zeta - (\partial_{\bar z}\gamma^{zz})\bar\zeta + 2\gamma^{zz}\partial_z\zeta + 4\partial_{\bar z}\zeta\ , \\
\delta\gamma^{z\bar z} &= 2\partial_z\zeta + 2\partial_{\bar z}\bar\zeta + \gamma^{zz}\partial_z\bar\zeta + \gamma^{\bar z\bar z}\partial_{\bar z}\zeta\ , \\
\delta\gamma^{\bar z\bar z} &= -(\partial_{\bar z}\gamma^{\bar z\bar z})\bar\zeta - (\partial_z\gamma^{\bar z\bar z})\zeta + 2\gamma^{\bar z\bar z}\partial_{\bar z}\bar\zeta + 4\partial_z\bar\zeta\ .
\end{aligned}$$

The diffeomorphism covariance implies that

$$\int_\Sigma (\langle T_{zz}\rangle_\gamma\,\delta\gamma^{zz} + 2\langle T_{z\bar z}\rangle_\gamma\,\delta\gamma^{z\bar z} + \langle T_{\bar z\bar z}\rangle_\gamma\,\delta\gamma^{\bar z\bar z})\, dv \,=\, 0\ .$$

Inserting the expressions for $\delta\gamma^{-1}$, stripping the resulting equation from the arbitrary function ζ and retaining only the first order terms in γ^{zz} around a locally flat metric, we obtain

$$(\partial_z\gamma^{zz})\langle T_{zz}\rangle_\gamma + 2\partial_z(\gamma^{zz}\langle T_{zz}\rangle_\gamma) + 4\partial_{\bar z}\langle T_{zz}\rangle_\gamma$$

[2] w and z refer to the complex coordinates of two nearby insertions taken in the same holomorphic chart

$$+4\partial_z\langle T_{z\bar z}\rangle_\gamma\,+\,2\partial_{\bar z}\,(\gamma^{\bar z\bar z}\langle T_{z\bar z}\rangle)\,+\,(\partial_z\gamma^{\bar z\bar z})\langle T_{\bar z\bar z}\rangle\,=\,0\;. \tag{19}$$

Specializing to $\gamma^{zz}=0$, we infer that

$$\partial_{\bar z}\langle T_{zz}\rangle\,=\,0\,=\,\partial_z\langle T_{\bar z\bar z}\rangle\;. \tag{20}$$

More generally, the component T_{zz} ($T_{\bar z\bar z}$) of energy-momentum tensor is analytic (anti-analytic) in correlation functions in a locally flat metric and away from other insertions. Eq. (20) is another conformal Ward identity.

At coinciding points, the correlation functions of energy-momentum tensor give rise to singularities which we shall study now. Application of $\frac{\pi}{Z_\gamma}\frac{\delta}{\delta\gamma^{ww}}Z_\gamma$ at γ locally flat to eq. (19) gives:

$$\pi\left(\partial_z\delta^{(2)}(z-w)\right)\langle T_{zz}\rangle\,+\,2\pi\,\partial_z\delta^{(2)}(z-w)\langle T_{ww}\rangle\,+\,\partial_{\bar z}\langle T_{zz}\,T_{ww}\rangle\,+\,\partial_z\langle T_{z\bar z}\,T_{ww}\rangle\,=\,0\;.$$

Using eq. (17) differentiated with respect z in order to replace $\partial_z\langle T_{z\bar z}\,T_{ww}\rangle$ in the last relation, we obtain

$$\begin{aligned}\partial_{\bar z}\langle T_{zz}\,T_{ww}\rangle\;&=\;-\pi\left(\partial_z\delta^{(2)}(z-w)\right)\langle T_{zz}\rangle\,-\,\pi\,\partial_z\delta^{(2)}(z-w)\langle T_{ww}\rangle\,-\,\tfrac{\pi c}{12}\,\partial_z^3\delta^{(2)}(z-w)\\ &=\;-\tfrac{\pi c}{12}\,\partial_z^3\delta^{(2)}(z-w)\,-\,2\pi\,\partial_z\delta^{(2)}(z-w)\langle T_{ww}\rangle\,+\,\pi\,\delta^{(2)}(z-w)\,\partial_w\langle T_{ww}\rangle\;.\end{aligned}$$

This is a distributional equation. Since $\delta^{(2)}(z-w)\,=\,\frac{1}{\pi}\partial_{\bar z}\frac{1}{z-w}$ in the sense of distributions, it follows that

$$\partial_{\bar z}\langle T_{zz}\,T_{ww}\rangle\,=\,\partial_{\bar z}\left(\frac{c/2}{(z-w)^4}\,+\,\frac{2}{(z-w)^2}\langle T_{ww}\rangle\,+\,\frac{1}{z-w}\,\partial_w\langle T_{ww}\rangle\right)\;, \tag{21}$$

which is still another conformal Ward identity. Since the only solutions of the distributional equation $\partial_{\bar z}f=0$ are analytic functions, one may rewrite the identity (21) as a short distance expansion encoding the ultraviolet properties of the CFT:

$$\langle T_{zz}\,T_{ww}\rangle\,=\,\frac{c/2}{(z-w)^4}\,+\,\frac{2}{(z-w)^2}\langle T_{ww}\rangle\,+\,\frac{1}{z-w}\,\partial_w\langle T_{ww}\rangle\,+\,\ldots\;, \tag{22}$$

where ". . ." stands for terms analytic in z around $z=w$. The complex conjugation of eq. (22) gives the singular terms of $\langle T_{\bar z\bar z}\,T_{\bar w\bar w}\rangle$, this time, up to anti-analytic terms. Expansions of the type (22) are usually called the **operator product expansion** (OPE) in accordance with the operator interpretation of correlation functions to be discussed in the next section. We shall follow this terminology.

What about the mixed insertions? Differentiating ($Z_\gamma\,\times$) eq. (19) with respect to $\gamma^{\bar w\bar w}$ at γ locally flat, we obtain

$$\partial_{\bar z}\langle T_{zz}\,T_{\bar w\bar w}\rangle\,+\,\partial_z\langle T_{z\bar z}\,T_{\bar w\bar w}\rangle\,+\,\pi\,(\partial_z\delta^{(2)}(z-w))\,\langle T_{\bar z\bar z}\rangle\,=\,0\;.$$

With the use of the complex conjugate version of eq. (17) to eliminate $\langle T_{z\bar z}\,T_{\bar w\bar w}\rangle$, this reduces to

$$\partial_{\bar z}\langle T_{zz}\,T_{\bar w\bar w}\rangle\,=\,-\tfrac{\pi c}{12}\,\partial_z\,\partial_{\bar z}^2\,\delta^{(2)}(z-w)$$

which, stripped of $\partial_{\bar z}$, gives

$$\langle\, T_{zz}\, T_{\bar w\bar w}\,\rangle \;=\; -\tfrac{\pi c}{12}\,\partial_z\,\partial_{\bar z}\,\delta^{(2)}(z-w)\;+\;\;.\;.\;.\;\;, \tag{23}$$

i.e. a contact term with support at $z=w$ plus a function analytic in z and anti-analytic in w.

The other source of singular contributions to the correlation functions of T_{zz} or $T_{\bar z\bar z}$ are insertions of the (primary) fields $\phi_l(x)$. Let us compute these singularities. Proceeding similarly as before, we apply $\frac{\pi}{Z_\gamma}\frac{\delta}{\delta\sigma}$ to $Z_{e^\sigma\gamma}\,\langle\,\phi_l(x)\,\rangle_{e^\sigma\gamma}$ at $\sigma=0$ and γ locally flat obtaining with the help of eqs. (4) and (5) the relation

$$\langle\, T_{z\bar z}\;\phi_l(w,\bar w)\,\rangle \;=\; \pi\,\Delta_l\,\delta^{(2)}(z-w)\,\langle\,\phi_l(w,\bar w)\,\rangle \tag{24}$$

(we have replaced the point x in the argument of ϕ_l by its local coordinate w and its complex conjugate to stress the non-holomorphic dependence on x of the $\phi_l(x)$ insertion). Next we exploit the diffeomorphism covariance. For $D(z)=z+\zeta(z,\bar z)\equiv z'$ and $\gamma=D^*\gamma'$,

$$\langle\,\phi_l(w',\bar w')\,\rangle_{\gamma'}\,Z_{\gamma'} \;=\; \langle\,\phi_l(w,\bar w)\,\rangle_\gamma\,Z_\gamma\;.$$

Since for $\gamma=|dz|^2$

$$\delta\gamma^{-1}\;\equiv\;(\gamma')^{-1}-\gamma^{-1}\;=\;4(\partial_{\bar z}\zeta)\partial_z^2\;+\;4(\partial_z\zeta\;+\;\partial_{\bar z}\bar\zeta)\partial_z\partial_{\bar z}\;+\;4(\partial_z\bar\zeta)\partial_{\bar z}^2,$$

to the first order in ζ, see eq. (13), we infer that

$$\pi\,\delta^{(2)}(z-w)\;\partial_w\,\langle\,\phi_l(w,\bar w)\,\rangle\;-\;\partial_{\bar z}\,\langle\,T_{zz}\,\phi_l(w,\bar w)\,\rangle\;-\;\partial_z\,\langle\,T_{z\bar z}\;\phi_l(w,\bar w)\,\rangle\;=\;0\;.$$

Using the last equation to eliminate $\partial_z\,\langle\,T_{z\bar z}\;\phi_l(w,\bar w)\,\rangle$ from eq. (24) acted upon by ∂_z, we obtain the relation

$$\partial_{\bar z}\,\langle\,T_{zz}\,\phi_l(w,\bar w)\,\rangle\;=\;-\pi\,\Delta_l\,\delta^{(2)}(z-w)\,\langle\,\phi_l(w,\bar w)\,\rangle\;+\;\pi\,\delta^{(2)}(z-w)\,\partial_w\langle\,\phi_l(w,\bar w)\,\rangle$$

which may be conveniently rewritten as an OPE of the product of the T_{zz} component of energy-momentum tensor with a primary field:

$$\langle\,T_{zz}\,\phi_l(w,\bar w)\,\rangle\;=\;\left(\frac{\Delta_l}{(z-w)^2}\;+\;\frac{1}{z-w}\,\partial_w\right)\langle\,\phi_l(w,\bar w)\,\rangle\;+\;\;.\;.\;.\;.\; \tag{25}$$

Finally note, that under the holomorphic change of the local coordinate $z\mapsto z'=f(z)$

$$\langle\,\phi_l(z',\bar z')\,\rangle\;=\;\langle\,\phi_l(z,\bar z)\,\rangle_{|dz'/dz|^2\,dz d\bar z}\;=\;|\tfrac{dz'}{dz}|^{-2\,\Delta_l}\,\langle\,\phi_l(z,\bar z)\,\rangle$$

or

$$\langle\,\phi_l(z',\bar z')\,\rangle\,(dz')^{\Delta_l}(d\bar z')^{\Delta_l}\;=\;\langle\,\phi_l(z,\bar z)\,\rangle\,(dz)^{\Delta_l}(d\bar z)^{\Delta_l}$$

so that ϕ_l behaves like a (Δ_l,Δ_l)-form in the correlation functions with locally flat metric. One often needs to consider also primary fields with weights $(\Delta_l,\tilde\Delta_l)$ and $\Delta_l-\tilde\Delta_l$ integer (or half-integer). $d_l=\Delta_l+\tilde\Delta_l$ is the scaling dimension of such a field and $s_l=\Delta_l-\tilde\Delta_l$ its spin.

Geometrically, the correlation functions of such fields are sections of the $s^{\rm th}$ power of the sphere subbundle in the cotangent bundle $T^*\Sigma$.

Let us collect the relations obtained in this section for low point insertions in the correlation functions. Since all the considerations were local, the same equalities hold in correlation functions with other insertions as long as their points stay away from the insertions taken together. Adding also the relations involving the complex conjugate components of energy-momentum tensor and introducing simplified notation $T \equiv T_{zz}$, $\bar{T} = T_{\bar{z}\bar{z}}$, we obtain:

i/. **identities**

$$T_{z\bar{z}} = 0 = T_{\bar{z}z} \,, \tag{26}$$

$$\partial_{\bar{z}} T = 0 = \partial_z \bar{T} \,, \tag{27}$$

ii/. **operator product expansions**

$$\begin{aligned}
T(z)\,T(w) &= \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}\,T(w) + \frac{1}{z-w}\,\partial_w T(w) + \ \ldots\,, \\
\bar{T}(\bar{z})\,\bar{T}(\bar{w}) &= \frac{c/2}{(\bar{z}-\bar{w})^4} + \frac{2}{(\bar{z}-\bar{w})^2}\,\bar{T}(\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\,\partial_{\bar{w}} \bar{T}(\bar{w}) + \ \ldots\,, \\
T(z)\,\bar{T}(\bar{w}) &= -\frac{\pi c}{12}\,\partial_z\,\partial_{\bar{z}}\,\delta^{(2)}(z-w) + \ \ldots\,,
\end{aligned} \tag{28}$$

$$T(z)\,\phi_l(w,\bar{w}) = \left(\frac{\Delta_l}{(z-w)^2} + \frac{1}{z-w}\,\partial_w\right)\phi_l(w,\bar{w}) + \ \ldots\,, \tag{29}$$

$$\bar{T}(\bar{z})\,\phi_l(w,\bar{w}) = \left(\frac{\Delta_l}{(\bar{z}-\bar{w})^2} + \frac{1}{\bar{z}-\bar{w}}\,\partial_{\bar{w}}\right)\phi_l(w,\bar{w}) + \ \ldots\,, \tag{30}$$

iii/. **transformation laws**

$$T(z')\,(dz')^2 = T(z)\,(dz)^2 - \frac{c}{12}\,\{z';z\}\,(dz)^2 \,, \tag{31}$$

$$\bar{T}(\bar{z}')\,(d\bar{z}')^2 = \bar{T}(\bar{z})\,(d\bar{z})^2 - \frac{c}{12}\,\overline{\{z';z\}}\,(d\bar{z})^2 \,, \tag{32}$$

$$\phi_l(z',\bar{z}')\,(dz')^{\Delta_l}(d\bar{z}')^{\tilde{\Delta}_l} = \phi_l(z,\bar{z})\,(dz)^{\Delta_l}(d\bar{z})^{\tilde{\Delta}_l}\,. \tag{33}$$

3. Physical positivity and Hilbert space picture

Up to now we have analyzed abstract conformal fields in the Euclidean formalism, probabilistic in its nature and distinct from the traditional operator approach. The operator formalism of QFT fits into the general quantum mechanical scheme with the

(i) Hilbert space of states,

(ii) representation of the symmetry group or algebra,

(iii) distinguished family of operators

as its basic triad. This is a fundamental fact of QFT that the passage between the Euclidean and the operator formalisms, which we have discussed already for free fields, may be done in quite general circumstances. This fact is responsible for the deep relation between critical phenomena and quantum fields and it has strongly marked the developments of QFT. CFT, which is not an exception in this respect, has largely profited from the unity of two approaches. In the present section we shall discuss how the operator picture may be recovered from the Euclidean formulation of CFT presented above assuming the **physical** (or Osterwalder-Schrader) **positivity** formulated as a condition on correlation functions on the Riemann sphere $\mathbf{C}P^1$. Analysis of the genus zero situation will allow to recover the Hilbert space of states and to translate the operator product expansions of the last section into an action of the Lie algebra of conformal symmetries and of the primary field operators in the space of states. Later we shall describe the operator formalism on higher genus Riemann surfaces which permits to relate naturally CFT in different space-time topologies.

Let us consider the map $\vartheta : \mathbf{C}P^1 \to \mathbf{C}P^1$, $\vartheta(z) = \bar{z}^{-1}$. ϑ interchanges the disc $D = \{\,|z| \leq 1\,\}$ with $D' \equiv \{\,|z| \geq 1\,\}$ and leaves invariant their common boundary $\{\,|z| = 1\,\}$. Suppose that we are given a Riemannian metric γ on D, compatible with the complex structure, which is of the form $|z|^{-2}|dz|^2$ around ∂D (we shall call such a metric flat at boundary). $\vartheta^*\gamma$ is a metric on D' and it glues smoothly with γ on D to the metric $\vartheta^*\gamma \vee \gamma$ on $\mathbf{C}P^1$. Consider formal expressions

$$X \,=\, \prod_i \phi_{l_i}(z_i, \bar{z}_i) \tag{34}$$

for distinct points z_i in the interior of D (with the empty product case included). Denote

$$\Theta\phi_l(z, \bar{z}) \,\equiv\, (-\bar{z}^{-2})^{\Delta_l}(-z^{-2})^{\tilde{\Delta}_l}\, \phi_{\bar{l}}(\tfrac{1}{\bar{z}}, \tfrac{1}{z}) \tag{35}$$

where $l \mapsto \bar{l}$ is the same involution that appeared in eq. (7). For X as above, we set

$$\Theta X \,=\, \prod_i \Theta\phi_{l_i}(z_i, \bar{z}_i)\ . \tag{36}$$

The physical positivity requires that for each family (λ_α) of complex numbers, each family (X_α) of expressions (34) and each family (γ_α) of metrics on D flat at boundary

$$\sum_{\alpha_1,\,\alpha_2} \bar{\lambda}_{\alpha_2}\, \lambda_{\alpha_1}\, Z_{\vartheta^*\gamma_{\alpha_2}\vee\gamma_{\alpha_1}}\, \langle\, (\Theta X_{\alpha_2})\, X_{\alpha_1}\,\rangle_{\vartheta^*\gamma_{\alpha_2}\vee\gamma_{\alpha_1}} \;\geq\; 0\ . \tag{37}$$

These properties hold for the free field compactifications. The condition (37) may be rewritten using correlation functions with energy-momentum insertions and a fixed metric. Set

$$\begin{aligned} \Theta T(z) \,&\equiv\, \bar{z}^{-4}\, T(\tfrac{1}{\bar{z}})\ , \\ \Theta\bar{T}(\bar{z}) \,&\equiv\, z^{-4}\, \bar{T}(\tfrac{1}{z}) \end{aligned} \tag{38}$$

and extend the definition (34) to expressions

$$Y \,=\, \prod_m T(z_m) \prod_n \bar{T}(\bar{z}_n) \prod_i \phi_{l_i}(z_i, \bar{z}_i) \tag{39}$$

(with all points in the interior of D and distinct) for which

$$\Theta Y \;=\; \prod_m \Theta T(z_m) \prod_n \Theta \bar{T}(\bar{z}_n) \prod_i \Theta\phi_{l_i}(z_i,\,\bar{z}_i)\;. \tag{40}$$

One may infer from the property (37) that

$$\sum_{\alpha_1,\,\alpha_2} \bar{\lambda}_{\alpha_2}\,\lambda_{\alpha_1}\,\langle\,(\Theta Y_{\alpha_2})\,Y_{\alpha_1}\,\rangle \;\geq\; 0 \tag{41}$$

where $\langle\,\cdots\,\rangle$ denotes the correlation functions in a locally flat metric.

Problem 2. Show that (37) implies (41).

The construction of the Hilbert space $\mathcal{H}$ of states is now simple. The expression

$$\sum_{\alpha,\,\beta} \bar{\lambda}'_\beta\,\lambda_\alpha\,\langle\,(\Theta Y'_\beta)\,Y_\alpha\,\rangle$$

defines a hermitian form on the space V_D of formal linear combinations of products (39). Due to (41), this form is positive and becomes positive definite on the quotient by its null subspace $V_D^{\rm null}$. One sets

$$\mathcal{H} \;=\; \overline{V_D/V_D^{\rm null}}. \tag{42}$$

We shall denote by ι the canonical map from V_D to $\mathcal{H}$, by $\mathcal{H}_0$ its image $\subset \mathcal{H}$ and by $\mathcal{Y}$ the image $\iota(Y)$ of Y. The empty product in (39) gives rise to the "**vacuum vector**" Ω. The scalar product is given by

$$(\,\mathcal{Y}',\,\mathcal{Y}\,) \;=\; \langle\,(\Theta Y')\,Y\,\rangle.$$

$\mathcal{H}$ carries an anti-unitary involution $\mathcal{I}$ mapping vector $\mathcal{Y}$ to $\bar{\mathcal{Y}}$ where $\mathcal{Y}$ corresponds to

$$\bar{Y} \;=\; \prod_m T(\bar{z}_m) \prod_n \bar{T}(z_n) \prod_l \phi_{\bar{l}_i}(\bar{z}_i,\,z_i)\;. \tag{43}$$

4. Virasoro algebra and its primary fields

Define the action of dilations by $q \in \mathbf{C},\ 0 < |q| \leq 1$, on the fields by setting

$$S_q T(z) = q^2 T(qz),\quad S_q \bar{T}(\bar{z}) = \bar{q}^2 T(\bar{q}\bar{z}),\quad S_q \phi_l(z,\bar{z}) = q^{\Delta_l}\bar{q}^{\bar{\Delta}_l}\phi_l(qz,\bar{q}\bar{z}).$$

For Y given by (39), we put

$$S_q Y \;=\; \prod_m S_q T(z_m) \prod_n S_q \bar{T}(\bar{z}_n) \prod_l S_q \phi_{l_i}(z_l,\,\bar{z}_l)\;.$$

Problem 3. Using the conformal symmetries of the correlation functions, verify that

$$\langle (\Theta Y') S_q Y \rangle = \langle (\Theta S_{\bar{q}} Y') Y \rangle . \tag{44}$$

In the Hilbert space, we may define the dilation operator $\mathcal{S}_q$ by the equality

$$\mathcal{S}_q \mathcal{Y} = \iota(S_q Y).$$

Note that eq. (44) implies that $\mathcal{S}_q$ is well defined on the dense invariant domain $\mathcal{H}_0$. In fact, the family of operators $(\mathcal{S}_q)$ forms a semigroup: $\mathcal{S}_{q_1}\mathcal{S}_{q_2} = \mathcal{S}_{q_1 q_2}$. Applying many times the Schwartz inequality, identity (44) and the semigroup property of $\mathcal{S}_q$, one obtains following Osterwalder-Schrader:

$$\begin{aligned} |(\mathcal{Y}', \mathcal{S}_q \mathcal{Y})| &\leq \|\mathcal{Y}'\| \, \|\mathcal{S}_q \mathcal{Y}\| = \|\mathcal{Y}'\| \, (\mathcal{Y}, \mathcal{S}_{\bar{q}q} \mathcal{Y})^{1/2} \\ &\leq \cdots\cdots \leq \|\mathcal{Y}'\| \, \|\mathcal{Y}\|^{\frac{1}{2}+\dots+\frac{1}{2^{n-1}}} \, (\mathcal{Y}, \mathcal{S}_{(\bar{q}q)^{2^{n-1}}} \mathcal{Y})^{1/2^n} . \end{aligned} \tag{45}$$

Assume now that for each $\epsilon > 0$, there exists a constant C_ϵ s.t.

$$|\langle (\Theta Y') S_t Y \rangle| \leq C_\epsilon \, t^{-\epsilon} \tag{46}$$

when $t \to 0$. What it means is that when the distances of a group of insertions are uniformly shrunk to zero the singularity of the correlation functions is not stronger then the power law given by the overall scaling dimension of the group. Using bound (46) on the right hand side of (45) and taking n to infinity, we infer that

$$|(\mathcal{Y}', \mathcal{S}_q \mathcal{Y})| \leq \|\mathcal{Y}'\| \, \|\mathcal{Y}\| ,$$

i.e. that the dilation semigroup $\mathcal{S}_q$ is composed of contractions of $\mathcal{H}$. Eq. (44) implies now that $\mathcal{S}_q^* = \mathcal{S}_{\bar{q}}$. The weak continuity of the semigroup $(\mathcal{S}_q)$ on $\mathcal{H}$ follows from that on $\mathcal{H}_0$ which is evident. By the abstract semigroup theory

$$\mathcal{S}_q = q^{L_0} \, \bar{q}^{\tilde{L}_0} \tag{47}$$

for strongly commuting self-adjoint operators L_0 and $\tilde{L}_0$ s.t. $L_0 + \tilde{L}_0 \geq 0$. Clearly, $\mathcal{H}_0$ is inside the domain of L_0 and of $\tilde{L}_0$ and

$$L_0 \mathcal{Y} = \partial_q|_{q=1} \mathcal{S}_q \mathcal{Y} , \quad \tilde{L}_0 \mathcal{Y} = \partial_{\bar{q}}|_{q=1} \mathcal{S}_q \mathcal{Y} . \tag{48}$$

It also follows that $\mathcal{S}_q \mathcal{H}_0$ is dense in $\mathcal{H}$ for all q.

L_0, $\tilde{L}_0$ are only the tip of an operator iceberg. To see more of it, define operators $\mathcal{T}(z)$, $\bar{\mathcal{T}}(\bar{z})$ and $\varphi_l(z, \bar{z})$, with $\mathcal{S}_z \mathcal{H}_0$ as the (dense) domain ($|z| < 1$), by setting

$$\mathcal{T}(z) \mathcal{Y} = \iota(T(z) Y), \qquad \bar{\mathcal{T}}(\bar{z}) \mathcal{Y} = \iota(\bar{T}(\bar{z}) Y), \qquad \varphi_l(z, \bar{z}) \mathcal{Y} = \iota(\phi_l(z, \bar{z}) Y).$$

It is easy to see that the operators $\mathcal{T}(z)$, $\bar{\mathcal{T}}(\bar{z})$ and $\varphi_l(z, \bar{z})$ are well defined. Note that for Y given by eq. (39) and with the absolute values of all insertion points different,

$$\mathcal{Y} = R\left(\prod_m \mathcal{T}(z_m) \prod_n \bar{\mathcal{T}}(\bar{z}_n) \prod_i \varphi_{l_i}(z_i, \bar{z}_i) \right) \Omega \tag{49}$$

where $R(\cdots)$ reorders the operators so that they act in the order of increasing $|z|$. This is the reason why the operator scheme described here is often called **radial quantization**. Under the conjugation by the anti-unitary involution $\mathcal{I}$ of $\mathcal{H}$,

$$\mathcal{I}\varphi_l(z,\bar{z})\mathcal{I} = \varphi_{\bar{l}}(\bar{z},z), \quad \mathcal{I}\mathcal{T}(z)\mathcal{I} = \mathcal{T}(\bar{z}), \quad \mathcal{I}\bar{\mathcal{T}}(\bar{z})\mathcal{I} = \bar{\mathcal{T}}(z)\ . \tag{50}$$

It will be useful to introduce Fourier components of the operators $\mathcal{T}(z)$ and $\bar{\mathcal{T}}(\bar{z})$:

$$L_n = \frac{1}{2\pi i}\oint_{|z|=r<1} z^{n+1}\,\mathcal{T}(z)\,dz\ , \tag{51}$$

$$\tilde{L}_n = -\frac{1}{2\pi i}\oint_{|z|=r<1} \bar{z}^{n+1}\,\bar{\mathcal{T}}(\bar{z})\,d\bar{z}\ . \tag{52}$$

Since the insertion of $T(z)$ in the correlation functions $\langle\cdots\rangle$ is analytic in z as long as the other insertions are not met, the matrix elements $(\mathcal{Y}',\,L_n\mathcal{Y})$ (and hence the vector $L_n\mathcal{Y}$ itself) does not depend on r as long as $r<1$ and the contour $|z|=r$ surrounds the insertions of Y (similarly for $\tilde{L}_n$). Notice that

$$\begin{aligned}(\mathcal{Y}',\,L_n\mathcal{Y}) &= \frac{1}{2\pi i}\oint_{|z|=1-\epsilon} z^{n+1}\,\langle(\Theta Y')\,T(z)\,Y\rangle\,dz \\ &= \frac{1}{2\pi i}\oint_{|z|=1+\epsilon} z^{n-3}\,\langle\,(\Theta(Y'T(\tfrac{1}{\bar{z}}))\,Y\,\rangle\,dz\end{aligned} \tag{53}$$

where we have moved the integration contour slightly, representing $T(z)$ with $|z|=1+\epsilon$ as $z^{-4}\,\Theta T(\frac{1}{\bar{z}})$. The right hand side is equal to

$$\begin{aligned}&(\,-\frac{1}{2\pi i}\oint_{|z|=1+\epsilon}\bar{z}^{n-3}\,\hat{T}_{zz}(\tfrac{1}{\bar{z}})\,d\bar{z}\,\mathcal{Y}',\ \mathcal{Y}\,) \\ &= (\,\frac{1}{2\pi i}\oint_{|w|=(1+\epsilon)^{-1}} w^{-n+1}\,T(w)\,dw\,\mathcal{Y}',\ \mathcal{Y}\,) = (\,L_{-n}\mathcal{Y}',\,\mathcal{Y}\,)\ .\end{aligned}$$

It follows, that operators L_n (and $\tilde{L}_n$) are closable[3] and their adjoints satisfy

$$L_n^* = L_{-n}\ , \qquad \tilde{L}_n^* = \tilde{L}_{-n}\ . \tag{54}$$

L_n's and $\tilde{L}_n$'s commute with the anti-involution $\mathcal{I}$ of $\mathcal{H}$. It will be convenient to somewhat extend the domain of definition of the operators introduced above. Let us admit in expressions Y of (39) integrated insertions $\oint_{|z|=r} z^{n+1}\,T(z)\,dz$ and similarly for $\bar{T}(\bar{z})$. Denote by $\mathcal{H}_1$ the resulting subspace of $\mathcal{H}$. Of course $\mathcal{H}_1$ contains $\mathcal{H}_0$ and is invariant under L_n's and $\tilde{L}_n$'s. Operators $\mathcal{T}(z)$, $\bar{\mathcal{T}}(\bar{z})$ and $\varphi_l(z,\bar{z})$ may be clearly extended to $\mathcal{S}_z\mathcal{H}_1$ and we shall assume below that this has been done[4].

The calculation which we shall do now is an example of an argument which translates (certain) OPE's into commutation relations and is used in CFT again and again. A devoted student should

[3] we keep the same symbols for their closures

[4] let us remark that these operators are not closable so their domains should be handled with special care

memorize its idea once for all. We start with the OPE (28) for $T(z)$ which will give commutation relations between L_n's. Let us consider the matrix element

$$
\begin{aligned}
& (\,\mathcal{Y}',\,[L_n,\,\mathcal{T}(w)]\,\mathcal{Y}\,) && (55)\\
&= \tfrac{1}{2\pi i}\left(\oint_{|z|=|w|+\epsilon} dz \,-\, \oint_{|z|=|w|-\epsilon} dz\right) z^{n+1}\,\langle\,(\Theta Y')\,T(z)\,T(w)\,Y\,\rangle && (56)\\
&= \tfrac{1}{2\pi i}\oint_{|z-w|=\epsilon} z^{n+1}\,dz\,\langle\,(\Theta Y')\,\left(\tfrac{c/2}{(z-w)^4} + \tfrac{2}{(z-w)^2}\,T(w) + \tfrac{1}{z-w}\,\partial_w T(w)\right) Y\,\rangle\,, && (57)
\end{aligned}
$$

where we have used the fact (49) that the order of operators is determined by the radial order of insertions in the correlation function. In the last line we have collapsed the contour of integration to a small circle around w and inserted the OPE (28). Expanding z^{n+1} around $z=w$

$$
\begin{aligned}
z^{n+1} \,=\, ((z-w)+w)^{n+1} \,=\, \tfrac{n^3-n}{6}\,(z-w)^3\,w^{n-2} \,+\, \tfrac{n^2+n}{2}\,(z-w)^2\,w^{n-1}\\
+\,(n+1)\,(z-w)\,w^n \,+\, w^{n+1} \,+\, \ldots
\end{aligned}
$$

and retaining only the terms which contribute to the residue at $z=w$ in the last integral of eq. (57), we obtain

$$
\begin{aligned}
(\,\mathcal{Y}',\,[L_n,\,\mathcal{T}(w)]\,\mathcal{Y}\,) \,=\, \langle\,(\Theta\,Y')\,\{\,\tfrac{c}{12}\,(n^3-n)\,w^{n-2} \,+\, 2(n+1)\,w^n\,T(w)\\
+\,w^{n+1}\,\partial_w T(w)\,\}\,Y\,\rangle
\end{aligned}
$$

which is the weak form of relations

$$
[L_n,\,\mathcal{T}(w)] \,=\, \tfrac{c}{12}\,(n^3-n)\,w^{n-2} \,+\, 2(n+1)\,w^n\,\mathcal{T}(w) \,+\, w^{n+1}\,\partial_w\mathcal{T}(w)\;. \tag{58}
$$

Similarly, the OPE (28) implies that

$$
[\tilde{L}_n,\,\bar{\mathcal{T}}(\bar{w})] \,=\, \tfrac{c}{12}\,(n^3-n)\,\bar{w}^{n-2} \,+\, 2(n+1)\,\bar{w}^n\,\bar{\mathcal{T}}(\bar{w}) \,+\, \bar{w}^{n+1}\,\partial_{\bar{w}}\bar{\mathcal{T}}(\bar{w})\;. \tag{59}
$$

By virtue of eq. (28), the mixed commutators $[L_n,\,\bar{\mathcal{T}}(\bar{w})]$ and $[\tilde{L}_n,\,\mathcal{T}(w)]$ vanish.

Performing a contour integral over w on both sides of eq. (58) multiplied by z^{m+1}, we obtain the commutation relations

$$
[L_n,\,L_m] \,=\, (n-m)\,L_{n+m} \,+\, \tfrac{c}{12}\,(n^3-n)\,\delta_{n+m,0}\;. \tag{60}
$$

The (infinite-dimensional) Lie algebra with generators L_n and a central element $\mathcal{C}$ (called the **central charge**) and with relations (60), where c is replaced replaced by $\mathcal{C}$, is known as the **Virasoro algebra**. We shall denote it by Vir. It is closely related to the (Witt) Lie algebra of polynomial vector fields $Vect(S^1)$ on the circle $\{\,|z|=1\,\}$ with generators $l_n = -z^{n+1}\partial_z$ and relations

$$
[l_n,\,l_m] \,=\, (n-m)\,l_{n+m}\;.
$$

More exactly, Vir is a central extension of $Vect(S^1)$, i.e. we have an exact sequence of Lie algebras

$$
0 \,\longrightarrow\, \mathbf{C} \,\longrightarrow\, Vir \,\longrightarrow\, Vect(S^1) \,\longrightarrow\, 0\,,
$$

where the second arrow sends 1 to $\mathcal{C}$ and the third one maps L_n to l_n.

Eq. (59) gives rise to another set of Virasoro commutation relations

$$[\tilde{L}_n,\, \tilde{L}_m] \,=\, (n-m)\,\tilde{L}_{n+m} \,+\, \tfrac{c}{12}(n^3-n)\,\delta_{n+m,0}\ . \tag{61}$$

L_n's and $\tilde{L}_m$'s commute. Both (60) and (61) hold on the invariant dense domain $\mathcal{H}_1 \in \mathcal{H}$, As we see, the Hilbert space of states $\mathcal{H}$ of a CFT carries a densely defined unitary (i.e. with property (54)) representation of the algebra $Vir \oplus Vir$ with central charges acting as the multiplication by c.

The representation theory of the Virasoro algebra has played an important role in the construction of models of CFT. We shall include for completeness a brief sketch of its elements in the next section. But why did the Virasoro algebra appear in CFT in the first place? As we have mentioned, Vir is the central extension of an algebra of vector fields on the circle. But $Vect(S^1)\oplus Vect(S^1)$ may be identified with the Lie algebra of (polynomial) conformal vector fields on the two-dimensional cylinder $\{\,(t,x)\mid x \bmod 2\pi\,\}$ with the Minkowski metric $\gamma_M \equiv dt^2 - dx^2$. By definition, the conformal vector fields X satisfy $\mathcal{L}_X\,\gamma_M = f_X\,\gamma_M$ for some function f_X, where $\mathcal{L}_X$ denotes the Lie derivative w.r.t. X. The identification assigns to generators l_n and $\bar{l}_n$ the conformal vector fields $-z^{n+1}\partial_z$ and $-\bar{z}^{n+1}\partial_{\bar{z}}$, respectively, with $z \equiv \mathrm{e}^{i(t+x)}$ and $\bar{z} \equiv \mathrm{e}^{i(t-x)}$. In particular, $i(l_0+\bar{l}_0)$ is the infinitesimal shift of the Minkowski time t and $i(l_0-\bar{l}_0)$ the infinitesimal shift of x. Hence $Vect(S^1)\oplus Vect(S^1)$ is the Lie algebra of Minkowskian conformal symmetries and representations of $Vir\oplus Vir$ describe its projective actions realizing such conformal symmetries on the quantum-mechanical level (projective representations correspond to genuine actions of symmetries on the rays in the Hilberts space representing (pure) quantum states). $H \equiv L_0 + \tilde{L}_0$ is the quantum Hamiltonian[5] and $P = L_0 - \tilde{L}_0$ is the quantum momentum operator. The unitarity conditions (54) correspond to the natural real form of the algebra composed of real vector fields: such vector fields are represented by skew-adjoint operators so that the corresponding global conformal transformations act by unitary operators. $Vect(S^1)$ may be viewed as the Lie algebra of the group $Diff_+(S^1)$ of orientation preserving diffeomorphisms of the circle. Let $\widetilde{Diff}_+(S^1)$ denote the group of diffeomorphisms of the line commuting with the shifts by 2π.

$$0 \longrightarrow \mathbf{Z} \longrightarrow \widetilde{Diff}_+(S^1) \longrightarrow Diff_+(S^1) \longrightarrow 0\ .$$

The group $\mathcal{D} \equiv (\widetilde{Diff}_+(S^1)\times\widetilde{Diff}_+(S^1))\,/\,\mathbf{Z}_{\rm diag}$ (which acts on the light-cone variables $x^\pm \equiv t\pm x$ is the group of conformal, orientation and time-arrow preserving diffeomorphism of the Minkowski cylinder. $Vir\oplus Vir$ action in $\mathcal{H}$ integrates to the projective unitary representation of $\mathcal{D}$.

We shall need more information about the representations of $Vir\times Vir$ which appear in CFT. This may be obtained by studying the behavior of the primary field operators with respect to the Virasoro algebra action.

Problem 4. (a). Show by employing the contour integral technique that

$$[L_n,\, \varphi_l(w,\bar{w})] \,=\, \Delta_l\,(n+1)\,w^n\,\varphi_l(w,\bar{w}) \,+\, w^{n+1}\,\partial_w\varphi_l(w,\bar{w})\ , \tag{62}$$

$$[\tilde{L}_n,\, \varphi_l(w,\bar{w})] \,=\, \bar{\Delta}_l\,(n+1)\,\bar{w}^n\,\varphi_l(w,\bar{w}) \,+\, \bar{w}^{n+1}\,\partial_{\bar{w}}\varphi_l(w,\bar{w})\ . \tag{63}$$

[5]later, we shall see that it is more natural to shift H by a constant

(b). Using the above relations and eqs. (58) and (59) prove that the operators L_0, $\tilde{L}_0$ given by (51) and (52) coincide with the generators of the semigroup $(\mathcal{S}_q)$ introduced earlier.

Eqs. (62) and (63) express on the operator level the properties of the (Virasoro) primary fields of conformal weights $(\Delta_l, \tilde{\Delta}_l)$. Comparing them to the last two equations of Lecture 1, we infer that operators $\varphi_l(w,\bar{w})$ for $(w,\bar{w}) = (\mathrm{e}^{-t+ix}, \mathrm{e}^{-t-ix})$ should be interpreted as the imaginary time versions of Minkowski fields. Note that the components $\mathcal{T}(z)$ and $\bar{\mathcal{T}}(\bar{z})$ fail to be Virasoro primary fields of weights $(2,0)$ and $(0,2)$, respectively, due to the anomalous term proportional to c in the relations (58) and (59).

Recall, that (as a generator of a self-adjoint semigroup of contractions) the self-adjoint operator $H = L_0 + \tilde{L}_0$ has to be positive. In Minkowskian QFT with Poincare invariance the positivity of the Hamiltonian implies the spectral condition $H \pm P \geq 0$ where P is the momentum operator. The same is true in CFT with its Hilbert space corresponding to cylindrical Minkowski space. The Virasoro commutation relations imply,

$$L_0 \geq 0 \,, \qquad \tilde{L}_0 \geq 0 \,. \tag{64}$$

Indeed. Let E_B be a non-vanishing joint spectral projector of L_0 and $\tilde{L}_0$ corresponding to eigenvalues in a small ball B, with the L_0 eigenvalues negative and such that $E_{B-(1,0)} = 0$. Then, for any normalized vector ψ with $E_B \psi = \psi$, we have

$$L_1 \psi = L_1 E_B \psi = E_{B-(1,0)} L_1 \psi = 0 \,.$$

On the other hand,

$$0 \leq (L_{-1}\psi, L_{-1}\psi) = (\psi, L_1 L_{-1}\psi) = (\psi, [L_1, L_{-1}]\psi) = 2(\psi, L_0 \psi) < 0$$

which shows that L_0 cannot have negative spectrum. Similarly for $\tilde{L}_0$. Hence only **positive energy representations** of the Virasoro algebra with $L_0 \geq 0$ ($\tilde{L}_0 \geq 0$) appear in CFT with the Hilbert space interpretation. The techniques of CFT apply, however, also to certain scaling limits of statistical mechanical models without physical positivity where a wider class of Virasoro representations intervenes.

Relations (62) and (63) provide further spectral information about L_0 and $\tilde{L}_0$. They imply the equalities

$$\begin{aligned}
&L_n \Omega = 0 \,, \quad n \geq -1 && \tilde{L}_n \Omega = 0 \,, \quad n \geq -1 && (65)\\
&L_n \varphi_l(0)\Omega = 0 \,, \quad n > 0 \,, && \tilde{L}_n \varphi_l(0)\Omega = 0 \,, \quad n > 0 \,, && (66)\\
&L_0 \varphi_l(0)\Omega = \Delta_l \varphi_l(0)\Omega \,, && \tilde{L}_0 \varphi_l(0)\Omega = \tilde{\Delta}_l \varphi_l(0)\Omega &&
\end{aligned}$$

where, by definition, $\varphi_l(0)\Omega = \lim_{z\to 0} \varphi_l(z,\bar{z})\Omega$. In particular, it follows that the vacuum vector Ω is an eigenvector of L_0 and $\tilde{L}_0$ with the lowest possible eigenvalues 0. We shall assume that it is a unique vector, up to normalization, with this property (although there are CFTs without this property). In fact, Ω is annihilated by the $sl_2 \times sl_2$ Lie subalgebra generated by L_0, $L_{\pm 1}$, $\tilde{L}_0$ and $\tilde{L}_{\pm 1}$ but not by the entire symmetry algebra $Vir \times Vir$ of the theory: the conformal symmetry is spontaneously broken.

$\varphi_l(0)\,\Omega$ are also eigenvectors of L_0 and $\tilde{L}_0$, with eigenvalues $(\Delta_l, \tilde{\Delta}_l)$ and it follows that $\Delta_l,\ \tilde{\Delta}_l > 0$. Also $\mathcal{I}\,\Omega = \Omega$ and $\mathcal{I}\,\varphi_l(0)\,\Omega = \varphi_{\bar{l}}(0)\,\Omega$. In fact, vectors $\varphi_l(0)\,\Omega$ are annihilated by all Virasoro generators with positive indices. The eigenvectors of $L_0,\ \tilde{L}_0$ (and $\mathcal{C}$) with such property are called Virasoro highest weight (HW) vectors.

Summarizing, we have shown that the Hilbert space of states in a CFT carries a (densely defined) positive energy representation of two commuting copies of the Virasoro algebra with the same central charge c. The primary field operators $\varphi_l(z,\bar{z})$ applied to the vacuum become in the limit $z \to 0$ HW vectors of the Virasoro representations.

5. Highest weight representations of Vir

For completeness, we include a brief sketch of representation theory of the Virasoro algebra.

An important class of representations of the Virasoro algebra is constituted by the so called **highest weight** (HW) representations. Let $\theta = \mathbf{C}L_0 \oplus \mathbf{C}\mathcal{C}$, $\mathcal{N}_+ = \bigoplus_{n=1}^{\infty} \mathbf{C}L_n$, $\mathcal{N}_- = \bigoplus_{n=1}^{\infty} \mathbf{C}L_{-n}$. $Vir = \mathcal{N}_- \oplus \theta \oplus \mathcal{N}_+$ is the triangular decomposition of the Virasoro algebra. Let $\lambda \in \theta^*$, the dual space to θ, $\lambda(\mathcal{C}) = c$, $\lambda(L_0) = \Delta$. A Vir-module (representation) $M_{c,\Delta}$ is called a HW module of HW λ if there exists a vector $v_0 \in V$ such that

$$\begin{array}{lcl} \mathcal{N}_+ v_0 & = & 0\,, \\ \mathcal{U}(\mathcal{N}_-) v_0 & = & M_{c,\Delta}\,, \\ x\,v_0 & = & \lambda(x)\,v_0 \quad \text{for} \quad x \in \theta \end{array}$$

where $\mathcal{U}(\,\cdot\,)$ denotes the enveloping algebra. v_0 is called the HW vector, c the central charge and Δ the conformal weight of the HW representation. It follows that $M_{c,\Delta}$ is the linear span of the vectors $L_{-n_r}L_{-n_{r-1}}\cdots L_{-n_1}v_0$ with $0 < n_1 \le n_2 \le \ldots \le n_r$, but these vectors are not necessarily linearly independent. $N = \sum_{i=1}^{r} n_i$ is called the level of the vector $L_{-n_r}L_{-n_{r-1}}\cdots L_{-n_1}v_0$. A level N vector is an eigenvector of L_0 with eigenvalue $N + \Delta$. We shall denote the subspace of the level N vectors by $M_{c,\Delta}^{(N)}$. Clearly vectors of different levels are linearly independent, thus we have

$$M_{c,\Delta} = \bigoplus_{N=1}^{\infty} M_{c,\Delta}^{(N)}$$

with the $\dim M_{c,\Delta}^{(N)} \le p(N)$, the partition number of N. A HW module with $\dim(M_{c,\Delta}^{(N)}) = p(N)$ for all N, i.e. where all the vectors of the form $L_{-n_r}L_{-n_{r-1}}\cdots L_{-n_1}v_0$ (with ordered n_i's) are linearly independent is called the Verma module $V_{c,\Delta}$. It exists for all complex c, Δ and is unique up to isomorphisms. In the analysis of the HW modules of the Virasoro algebra an important role is played by **singular vectors**. A non-zero vector $v_s^{(N)}$ of level N is called singular, if $L_n v_s^{(N)} = 0$ for all $n > 0$. Any vector v_s with $L_n v_s = 0$ for all $n > 0$ is a a sum of singular vectors (its non-zero homogeneous components). Any singular vector $v_s^{(N)}$ generates a submodule in $V_{c,\Delta}$ isomorphic to $V_{c,\Delta+N}$. We have:

(i) Any submodule of $V_{c,\Delta}$ is generated by its singular vectors.

(ii) Any HW module $M_{c,\Delta}$ is isomorphic to a quotient of the Verma module $V_{c,\Delta}$.

(iii) The factor module of the Verma module by the maximal proper submodule is the unique (up to isomorphisms) irreducible HW module $H_{c,\Delta}$.

A Vir-module M is called unitary, if there exists a (positive, hermitian) scalar product $(\cdot,\cdot)$ on M s.t.

$$(v, L_n w) = (L_{-n} v, w) \text{ for all } n \in \mathbf{Z} \text{ and for all } v, w \in V . \tag{67}$$

It follows then that $(v, \mathcal{C}w) = (\mathcal{C}v, w)$ for all $v, w \in V$ and that the eigenvalues of $\mathcal{C}$ and L_0 are real. On each Verma module $V_{c,\Delta}$ with c and Δ real there exists a unique hermitian (Shapovalov) form $\langle\cdot,\cdot\rangle$ s.t. $\langle v, L_n w\rangle = \langle L_{-n}v, w\rangle$ for all $v, w \in V_{c,\Delta}$ and that $\langle v_0, v_0\rangle = 1$. Define $Null_{c,\Delta} = \{v \in V_{c,\Delta} \mid \langle v, w\rangle = 0 \text{ for all } w \in V_{c,\Delta}\}$.

(i) $\langle v^{(N)}, v^{(N')}\rangle = 0$ if $v^{(N)}$ ($v^{(N')}$) is a level N (N') vector and $N \neq N'$,

(ii) any singular vector of positive level belongs to $Null_{c,\Delta}$.

(iii) $Null_{c,\Delta}$ is the maximal proper submodule of $V_{c,\Delta}$, i.e. $H_{c,\Delta} = V_{c,\Delta}/Null_{c,\Delta}$

Let us investigate the conditions under which $H_{c,\Delta}$ is a unitary Vir-module, or, equivalently, under which the hermitian form on $H_{c,\Delta}$ induced by the Shapovalov form is positive. Since $\langle L_{-n}v_0, L_{-n}v_0\rangle = 2n\Delta + \frac{c}{12}(n^3 - n)$, necessarily $c \geq 0$ and $\Delta \geq 0$. Now consider the two $2n$ level vectors $v = L_n^{\ 2} v_0$ and $w = L_{-2n}v_0$ and suppose that $c = 0$. The Gram determinant

$$\det \begin{pmatrix} \langle v, v\rangle & \langle v, w\rangle \\ \langle w, v\rangle & \langle w, w\rangle \end{pmatrix} = -20\, n^4 \Delta^2 + 32\, n^3 \Delta^3 .$$

Thus for $c = 0$ a necessary (and sufficient) condition is $\Delta = 0$. $c, \Delta = 0$ correspond to the trivial one-dimensional representations. So there exists no non-trivial unitary HW representation of the Lie algebra $Vect(S^1)$ of the (polynomial) vector fields on the circle. It is enough to study the positivity of the Shapovalov form restricted the the subspaces $H^{(N)}_{c,\Delta}$ of the given level. Let $m^{(N)}_{c,\Delta}$ denote the $p(N) \times p(N)$ matrix with the entries $\langle L_{-n_r} L_{-n_{r-1}} \cdots L_{-n_1} v_0, L_{-n'_r} L_{-n'_{r-1}} \cdots L_{-n'_1} v_0\rangle$ where (n_i) and (n'_i) are ordered and $\sum n_i = \sum n'_i = N$. Unitarity of $H_{c,\Delta}$ is equivalent to the conditions $m^{(N)}_{c,\Delta} \geq 0$ for all N. In particular, $\det(m^{(N)}_{c,\Delta})$ has to be non-negative for all N when $H_{c,\Delta}$ is unitary. Direct calculation for level 1 and 2 gives

$$\begin{aligned} \det(m^{(1)}_{c,\Delta}) &= 2\Delta , \\ \det(m^{(2)}_{c,\Delta}) &= 4\Delta \left(\tfrac{1}{2}c + (c-5)\Delta + 8\Delta^2\right) . \end{aligned}$$

A general formula for the determinant of $m^{(N)}_{c,\Delta}$ was given by Kac and was proven by Feigin and Fuchs:

$$\det(m^{(N)}_{c,\Delta}) = \kappa_N \prod_{\substack{1 \leq rs \leq N \\ r,s \in \mathbf{N}}} (\Delta - \Delta_{r,s}(m))^{p(N-rs)}$$

where $\Delta_{r,s}(m) = \frac{((m+1)r-ms)^2-1}{4m(m+1)}$, m is a root of the equation $c = 1 - \frac{6}{m(m+1)}$, and $\kappa_N = \prod_{\substack{1\leq rs\leq N \\ r,s\in\mathbf{N}}} ((2r)^s s!)^{n(r,s)}$ with $n(r,s)$ denoting the number of partitions of N in which r appears s times.

For $\Delta \to \infty$, $m^{(N)}_{c,\Delta}$ goes to a diagonal matrix with positive entries. From the Kac formula it follows that $\det(m^{(N)}_{c,\Delta}) > 0$ for $c > 1$, $\Delta > 0$. Therefore $m^{(N)}_{c,\Delta}$ is non degenerate and positive for $c > 1$, $\Delta > 0$ and non-negative for $c \geq 1$, $\Delta \geq 0$. Thus $V_{c,\Delta}$ is irreducible for $c > 1$, $\Delta > 0$ and $H_{c,\Delta}$ is unitary for $c \geq 1$, $\Delta \geq 0$. Since, for $c = 1$

$$\det(m^{(N)}_{1,\Delta}) = \kappa_N \prod_{1\leq rs\leq N} \left(\Delta - \frac{(r-s)^2}{4}\right)^{p(N-rs)} ,$$

it follows that $V_{1,\Delta}$ is irreducible if and only if $\Delta \neq \frac{n^2}{4}$, $n \in \mathbf{Z}$.

For $0 \leq c < 1$, the situation is more interesting providing the first example of the selective power of conformal invariance.

Theorem. For (c, Δ) with $0 \leq c < 1$ and $\Delta \geq 0$, unitarity of the irreducible HW representation $H_{c,\Delta}$ requires that (c, Δ) belong to the following discrete series of points:

$$\left.\begin{array}{rcl} c & = & 1 - \frac{6}{m(m+1)} \\ \Delta_{r,s}(m) & = & \frac{((m+1)r-ms)^2-1}{4m(m+1)} \end{array}\right\} \begin{array}{l} m = 2,3,\ldots\ , \\ 1 \leq r \leq m-1\ , \\ 1 \leq s \leq r\ . \end{array}$$

The theorem was proven by Friedan-Qiu-Shenker by a careful analysis of the geometry of lines $(c(m),\ \Delta_{r,s}(m))$ in the (c, Δ) plane. This, in conjunction with the Kac determinant formula, allowed subsequent elimination of portions of the (c, Δ) plane were negative norm vectors appear, by an induction on N. At the end, only the points listed above were left. Goddard-Kent-Olive constructed explicitly unitary irreducible HW representations of the Virasoro algebra for the above series of (c, Δ) employing the so called "coset construction".

All unitary HW representations integrate to projective unitary representations of $Diff_+(S^1)$ in the Hilbert space completion of $H_{c,\Delta}$.

All unitary representations M of the Virasoro algebra s.t. (the closure of) L_0 is a positive self-adjoint operator with a discrete spectrum of finite multiplicity in the Hilbert space completion $\overline{M}$ of M is essentially isomorphic to a direct sum of unitary representations (the isomorphism may require a different choice of the common invariant dense domain for L_n's in $\overline{M}$). We shall see in the next section that in CFT the operators $\mathcal{S}_q = q^{L_0}\,\bar{q}^{\tilde{L}_0}$ should be trace class for $|q| < 1$ from which it follows that L_0 and $\tilde{L}_0$ have a discrete spectrum of finite multiplicity. Hence the Virasoro algebra representations which appear in CFT are direct sums of the unitary HW representations.

The algebra $Vect(S^1) \oplus Vect(S^1)$, and hence also $Vir \oplus Vir$ acts naturally on the space $\Lambda^{\Delta\tilde{\Delta}}$

of $(\Delta,\tilde{\Delta})$-forms $f(dz)^\Delta(d\bar{z})^{\tilde{\Delta}}$ on $D\setminus\{0\}$ by Lie derivatives:

$$l_n f \,=\, -\Delta(n+1)\,z^n f \,-\, z^{n+1}\partial_z f\,,\qquad \bar{l}_n f \,=\, -\tilde{\Delta}(n+1)\,\bar{z}^n f \,-\, \bar{z}^{n+1}\partial_{\bar{z}} f\,.$$

The commutation relations (62) and (63) express the fact that the operators φ_l intertwine the action of $Vir\oplus Vir$ in $\mathcal{H}_1\subset\mathcal{H}$ and $\mathcal{H}_1\otimes\Lambda^{\Delta_l\tilde{\Delta}_l}$. This gives a representation theory interpretation of the primary fields of CFT.

6. Segal's axioms and vertex operator algebras

Up to now we have studied QFT on closed Riemann surfaces. Let us consider now a compact Riemann surface Σ (connected or disconnected) with the boundary composed of the connected components $\mathcal{C}_i$, $i\in I$. We shall parametrize each $\mathcal{C}_i$ in a real analytic way by the standard unit circle $S^1\subset\mathbf{C}$. Components $\mathcal{C}_i$ may be divided into "in" and "out" ones, depending on whether the parametrization disagrees or agrees with the orientation of $\mathcal{C}_i$ inherited from Σ. This induces the splitting $I=I_{\rm in}\cup I_{\rm out}$ of the set of indices i. We may invert the orientation of $\mathcal{C}_i$ by composing its parametrization with the map $z\mapsto z^{-1}$ of S^1. To Σ with parametrized boundary, we may uniquely assign a compact surface $\hat{\Sigma}$ without boundary by gluing a copy of disc D to each boundary component $\mathcal{C}_i$ with $i\in I_{\rm in}$ and a copy of disc D' to each $\mathcal{C}_i$ with $i\in I_{\rm out}$. Conversely, given a closed Riemann surface $\hat{\Sigma}$ with holomorphically embedded disjoint discs D and D' ("local parameters"), by removing their interiors we obtain the surface Σ with boundary parametrized by the standard circles $\{\,|z|=1\}$. We shall call a metric h on Σ (compatible with its complex structure) flat at boundary if, for each i, it is of the form $|z|^{-2}|dz|^2$ in the local holomorphic coordinate around $\mathcal{C}_i$ extending its parametrization. We shall consider only such metrics on surfaces with boundary.

First, let us note that a metric γ on Σ and metrics γ_i on D, all flat at boundary, give rise by gluing to the metric

$$\tilde{\gamma}\,=\,(\vee_{i\in I_{\rm in}}\gamma_i)\vee\gamma\vee(\vee_{i\in I_{\rm out}}\vartheta^*\gamma_i)$$

on $\hat{\Sigma}$. It follows from the local scale covariance formula (4) that the combination of partition functions

$$Z_{\tilde{\gamma}}\,\prod\limits_{i\in I}(Z_{\vartheta^*\gamma_i\vee\gamma_i})^{-1/2}\;\equiv\;Z_\gamma \tag{68}$$

is independent of (the conformal factors) of γ_i. Besides, the transformation of Z_γ under $\gamma\mapsto{\rm e}^\sigma\gamma$ with σ vanishing around the boundary is still given by eq. (4). It is sensible to call Z_γ the partition function of the Riemann surface Σ with boundary. Let $\mathcal{Y}_i\in\mathcal{H}_0$. Consider the matrix elements defined by

$$(\,\otimes_{i\in I_{\rm out}}\mathcal{Y}_i\,,\;A_{\Sigma,\gamma}\;\otimes_{i\in I_{\rm in}}\mathcal{Y}_i\,)\;=\;Z_\gamma\,\langle\,\prod\limits_{i\in I_{\rm out}}(\Theta Y_i)\;\prod\limits_{i\in I_{\rm in}}Y_i\,\rangle\,. \tag{69}$$

We shall postulate that the above formula defines operator "amplitudes" $A_{\Sigma,\gamma}$ mapping the tensor products of the CFT Hilbert spaces associated to the boundary components of Σ. In particular, for Σ without boundary, eq. (69) should be read as the identity $A_{\Sigma,\gamma}=Z_\gamma$. Amplitudes $A_{\Sigma,\gamma}$ (or rather their holomorphic counterparts) were considered as the defining data of CFT in the work of Segal. Let us state the Segal axioms (adapted to the real setup).

(i) We are given the Hilbert space of states $\mathcal{H}$ with an anti-unitary involution $\mathcal{I}$ and for each compact Riemann surface (with parametrized boundary or without boundary, connected or disconnected) the operator[6]

$$A_{\Sigma,\gamma}\,:\,\bigotimes_{i\in I_{\rm in}}\mathcal{H}\;\longrightarrow\;\bigotimes_{i\in I_{\rm out}}\mathcal{H}\,,\tag{70}$$

which we assume trace class.

(ii) If Σ is a disjoint union of Σ_1 and Σ_2, then

$$A_{\Sigma,\gamma}\;=\;A_{\Sigma_1,\gamma}\otimes A_{\Sigma_2,\gamma}\;.$$

(iii) If $D:\Sigma_1\to\Sigma_2$ is a diffeomorphism reducing to identity around the (parametrized) boundary then

$$A_{\Sigma_2,\gamma}\;=\;A_{\Sigma_1,D^*\gamma}\;.$$

(iv) If $\bar\Sigma$ denotes the Riemann surface with conjugate complex structure (and opposite orientation) then

$$A_{\bar\Sigma,\gamma}\;=\;A^\dagger_{\Sigma,\gamma}$$

(v) The inversion of boundary parametrization acts on the amplitude $A_{\Sigma,\gamma}$ by the isomorphism $\mathcal{H}\cong\mathcal{H}^*$ induced by $\mathcal{I}$ in the corresponding Hilbert space factor.

(vi) If Σ' is obtained from Σ by gluing of the $\mathcal{C}_{i_0}$ and $\mathcal{C}_{i_1}$ boundary components then

$$A_{\Sigma',\gamma}\;=\;{\rm tr}_{i_0,i_1}\,A_{\Sigma,\gamma}\;,$$

where ${\rm tr}_{i_0,i_1}$ denotes the partial trace in the $\mathcal{H}$ factors corresponding to $\mathcal{C}_{i_0}$ and $\mathcal{C}_{i_1}$.

(vii) For any function σ on Σ vanishing around the boundary

$$A_{\Sigma,{\rm e}^\sigma\gamma}\;=\;{\rm e}^{\frac{c}{96\pi}\,(\|d\sigma\|^2_{L^2}+4\int_\Sigma\sigma\,r\,dv)}\;A_{\Sigma,\gamma}$$

In the approach in which we start from the data specifying partition functions and correlation functions on surfaces without boundaries, property **(i)**, in conjunction with eq. (69), imposes certain new regularity conditions on the correlation functions. Property **(ii)** may be viewed as a definition of the amplitudes for disconnected surfaces. All the remaining properties except for **(vi)** follow easily. The gluing axiom brings an essential novel element allowing to compare the correlation functions of different surfaces (with different complex structures and different topologies). In the heuristic approach employing functional integrals, it expresses locality of the latter. Let us explain this point in more detail.

[6] the empty tensor product should be interpreted as $\mathbf{C}$

We shall think about the partition function of a QFT on a closed Riemann surface as being given by a functional integral of the type

$$Z_\gamma = \int \mathrm{e}^{-S_\Sigma(\phi)}\, D\phi \tag{71}$$

where $S_\Sigma(\phi)$ is the action functional and $D\phi = \prod_{x\in\Sigma} d\nu(\phi(x))$ is the formal volume element on the space of fields. On a Riemann surface with boundary Σ, we could consider an analogous functional integral with fixed boundary values $\phi|_{\mathcal{C}_i} \equiv \varphi_i$ of the fields:

$$A_{\Sigma,\gamma}((\varphi_i)_{i\in I}) = \int\limits_{\phi|_{\mathcal{C}_i}=\varphi_i} \mathrm{e}^{-S_\Sigma(\phi)}\, D\phi\,. \tag{72}$$

It is a function of the field boundary values. The space of functionals $\mathcal{F}(\varphi)$ of the fields on the circle with the formal L^2 scalar product may be thought of as the Hilbert space $\mathcal{H}$ of states of the system (we have seen a concrete realization of this idea for the free compactified field). We may then interpret the functional $A_{\Sigma,\gamma}((\varphi_i)_{i\in I})$ as the kernel of an operator mapping the tensor product of state spaces, one for each "in" component of the boundary, to the similar product for the "out" components.

The space $\mathcal{H}$ may be equipped with a formally anti-unitary involution $\mathcal{I}$, $(\mathcal{I}\mathcal{F})(\varphi) \equiv \overline{\mathcal{F}(\varphi^\vee)}$ where $\varphi^\vee(z) \equiv \varphi(z^{-1})$ which allows to turn the operators in $\mathcal{H}$ into bilinear forms and vice versa and to identify the operator amplitudes when we invert the orientation of some boundary components.

The basic formal property of the functional integral (72) is that one should be able to compute it iteratively. Suppose that the surface Σ' is obtained by identifying a (parametrized) "out" component $\mathcal{C}_{i_0}$ with an "in" component $\mathcal{C}_{i_1}$ of a (connected or disconnected) surface Σ. The functional integral on Σ' may be done by keeping first the value of the field on the gluing circle fixed and integrating over it only subsequently. In other words, the equality

$$\int\limits_{\substack{\phi|_{\mathcal{C}_i}=\varphi_i,\\ i\neq i_0,i_1}} \mathrm{e}^{-S_{\Sigma'}(\phi)}\, D\phi = \int \Big(\int\limits_{\substack{\phi|_{\mathcal{C}_i}=\varphi_i,\ i\neq i_0,i_1\\ \phi|_{\mathcal{C}_{i_0}}=\varphi_0=\phi|_{\mathcal{C}_{i_1}}}} \mathrm{e}^{-S_\Sigma(\phi)}\, D\phi \Big)\, D\varphi_0$$

should hold and this is exactly a formal version of the gluing property (**vi**).

What is the functional integral interpretation of eq. (69)? One should interpret vectors $\mathcal{Y}\in\mathcal{H}$ as corresponding to functions

$$\mathcal{F}_Y(\varphi) = (Z_{\vartheta*\gamma\vee\gamma})^{-1/2} \int\limits_{\phi|_{\partial D}=\varphi} Y\, \mathrm{e}^{-S_D(\phi)}\, D\phi$$

of fields on the circle. The complex conjugate functions are then given by the D'-functional integrals

$$(Z_{\vartheta*\gamma\vee\gamma})^{-1/2} \int\limits_{\phi|_{\partial D}=\varphi} \Theta Y\, \mathrm{e}^{-S_{D'}(\phi)}\, D\phi$$

so that the formal L^2 scalar product for the functions of fields φ on the boundary circle gives

$$\int \overline{\mathcal{F}_Y(\varphi)}\, \mathcal{F}_Y(\varphi)\, D\varphi \;=\; (Z_{\vartheta^*\gamma\vee\gamma})^{-1} \int (\Theta Y)\, Y\, \mathrm{e}^{-S_{D\cup D'}(\phi)}\, D\phi$$
$$= \langle\, (\Theta Y)\, Y\,\rangle \;=\; (\mathcal{Y}, \mathcal{Y})$$

(γ has been assumed locally flat above). Formula (69) follows now by iterative calculation of the functional integral

$$Z_{\tilde{\gamma}}\, \langle \prod_{i\in I_{\mathrm{out}}} (\Theta Y_i) \prod_{i\in I_{\mathrm{in}}} Y_i \,\rangle \;=\; \int \prod_{i\in I_{\mathrm{out}}} (\Theta Y_i) \prod_{i\in I_{\mathrm{in}}} Y_i \;\mathrm{e}^{-S_{\hat{\Sigma}}(\phi)}\, D\phi$$

by first fixing the values of ϕ on $\partial\Sigma$ and then integrating over them.

Any two-dimensional QFT should give rise to operator amplitudes with properties (**i**) to (**vi**). In particular, the so called $P(\phi)_2$ theories corresponding to the actions $S(\phi) \sim \|\phi\|^2_{L^2} + \int P(\varphi)\, dv$, where P is a bounded below polynomial, give undoubtly rise to such structures. The special property, which distinguishes the CFT models from other two-dimensional QFT's is, of course, the local scale covariance (**vii**) (conjecturally, special limits of the $P(\phi)_2$ theories should possess this property).

As already stressed, if we consider eq. (69) as the definition of the amplitudes $A_{\Sigma,\gamma}$ then the properties (**i**) to (**vii**) above become further requirements on the correlation functions. The point of Segal was, however, that the amplitudes $A_{\Sigma,\gamma}$ satisfying (**i**) to (**vii**) may be taken as the defining objects of CFT from which the rest of the structure, like the Virasoro algebra action, the primary fields and their correlation functions, etc. may be extracted. In this sense, the requirements (**i**) to (**vii**) may be viewed as the basic axioms of CFT, encapsulating its mathematical structure. Let us briefly sketch Segal's arguments. They require looking at the amplitudes $A_{\Sigma,\gamma}$ for the simplest geometries.

Discs

For $\Sigma = D$, $A_{D,\gamma} \in \mathcal{H}$ and $(Z_{\vartheta^*\gamma\vee\gamma})^{-1/2} A_{D,\gamma}$ is a metric independent vector which, according to eq. (69) we should interpret as the vacuum vector:

$$(Z_{\vartheta^*\gamma\vee\gamma})^{-1/2} A_{D,\gamma} \;\equiv\; \Omega\,. \tag{73}$$

Similarly, $A_{D',\vartheta^*\gamma}$ is the linear functional on $\mathcal{H}$,

$$(Z_{\vartheta^*\gamma\vee\gamma})^{-1/2} A_{D',\vartheta^*\gamma} \;=\; (\,\Omega,\;\cdot\;)\,.$$

It follows from the properties (**iv**) and (**v**) that $\Omega = \mathcal{I}\,\Omega$.

Annuli

Let us pass to surfaces Σ with annular topology. Under gluing of an "out" and an "in" boundary components of two such surfaces their amplitudes $A_{\Sigma,\gamma}$ form a semigroup due to the property (**vi**). This is the semigroup which encodes the Virasoro action on $\mathcal{H}$. In particular, for the annuli $\Sigma = \{\, |q| \leq |z| \leq 1\,\} = C_q$ with the "out" boundary parametrized by $z \mapsto z$ and the "in" one by $z \mapsto qz$, using the metric $\gamma_0 = |z|^{-2}|dz|^2$ on C_q, we obtain a semigroup almost identical to the semigroup $(\mathcal{S}_q)$ considered before

$$(\mathcal{Y}', A_{C_q,\gamma_0}\mathcal{Y}) = Z_{\gamma_0}\langle(\Theta Y)\, S_q Y\rangle = Z_{\gamma_0}(\mathcal{Y}', \mathcal{S}_q\mathcal{Y}) \tag{74}$$

where Z_{γ_0} is given by eq. (68).

Problem 5. Show that $Z_{\gamma_0} = (q\bar{q})^{-c/24}$.

The immediate consequence of this relation and of eqs. (74) and (47) is the identity

$$A_{C_q,\gamma_0} = q^{L_0-c/24}\,\bar{q}^{\tilde{L}_0-c/24}\,. \tag{75}$$

Note that the mapping $z \mapsto \mathrm{e}^{iz}$ establishes a holomorphic diffeomorphism between the finite cylinder $C_\tau = \{\, z \mid 0 \le \mathrm{Im}\, z \le 2\pi\tau_2 \,\}/2\pi\mathbf{Z}$ and C_q where $q = \mathrm{e}^{2\pi i\tau}$. The pullback of the metric γ_0 by $z \mapsto \mathrm{e}^{iz}$ is $|dz|^2$. The amplitude $A_{C_\tau,|dz|^2}$ is equal to $\mathrm{e}^{-2\pi\tau_2 H}\,\mathrm{e}^{2\pi i\tau_1 P}$, where $\tau = \tau_1 + i\tau_2$ and H is the Hamiltonian of the theory quantized in periodic volume $\mathbf{R}/2\pi\mathbf{Z}$ and P is the momentum operator. Comparison with eq. (75) yields

$$H = L_0 + \tilde{L}_0 - \tfrac{c}{12}\,, \qquad P = L_0 - \tilde{L}_0\,.$$

The requirement that the amplitudes be trace class operators implies that L_0 and $\tilde{L}_0$ have discrete spectrum with finite multiplicity, so that their eigenvalue zero corresponding to eigenvector Ω is isolated (with possible multiplicity). Note, that energy of the vacuum state becomes equal to $\frac{c}{12}$ now. If we work on the space $\mathbf{R}/L\mathbf{Z}$ instead, the energy spectrum is multiplied by $\frac{2\pi}{L}$ so that the lowest energy becomes $\frac{\pi c}{6L}$. This permits to calculate (following Cardy) the central charge of the conformal models from the finite size dependence of the ground state energy in the periodic interval.

Gluing together the boundaries of C_τ, one obtains the elliptic curve $T_\tau = \mathbf{C}/(\mathbf{Z} + \tau\mathbf{Z})$ (with metric $4\pi^2|dz|^2$). The properties (**iii**) and (**vi**) of the operator amplitudes imply then the following expression for the **toroidal partition function**

$$Z(\tau) \equiv A_{T_\tau,|dz|^2} = \mathrm{tr}\; q^{L_0-c/24}\,\bar{q}^{\tilde{L}_0-c/24}\,. \tag{76}$$

which is necessarily a modular invariant function (recall why?). The modular invariance of the partition functions of CFT has played an important role in the search and the classification of models.

The amplitudes for other annular surfaces allow to obtain the action of other generators of the double Virasoro algebra in the Hilbert space of states $\mathcal{H}$. If $D \ni z \mapsto f(z) \in D$ is a holomorphic embedding of D into its interior preserving the origin then $\Sigma_f := D \setminus f(\mathrm{int}(D))$ is in a natural way an annulus with parametrized boundary (one "in" $f(\partial D)$ and one "out" ∂D component). Such annuli form a semigroup. In particular, for $f_{q,\alpha,n}(z) =\mapsto q^{z\partial_z}\,\mathrm{e}^{\alpha z^{n+1}\partial_z}z$ for $n > 0$, $|q| < 1$ and $|\alpha|$ sufficiently small,

$$A_{\Sigma_{f_{q,\alpha,n}},\gamma} = Z_\gamma\; \mathrm{e}^{\alpha L_n}\, q^{L_0}\, \mathrm{e}^{\bar{\alpha}\tilde{L}_n}\, \bar{q}^{\tilde{L}_0}$$

encoding the action of the operators L_n, $\tilde{L}_n$ for $n>0$ (the complex-conjugate annuli $\bar{\Sigma}_{f_{q,n,\alpha}}$ give amplitudes encoding L_n, $\tilde{L}_n$ for $n<0$). The Virasoro HW vectors $\mathcal{X}_l\in\mathcal{H}$ of conformal weights $(\Delta_l,\tilde{\Delta}_l)$ may be characterized by the equalities

$$Z_\gamma^{-1}\,A_{\Sigma_f,\gamma}\,\mathcal{X}_l\;=\;(\tfrac{df(0)}{dz})^{\Delta_l}\,(\tfrac{d\bar{f}(0)}{d\bar{z}})^{\tilde{\Delta}_l}\,\mathcal{X}_l\,. \tag{77}$$

In particular, $L_0\mathcal{X}_l=\Delta_l\mathcal{X}_l$, $L_n\mathcal{X}_l=0$ for $n>0$ and similarly for $\tilde{L}_n$'s.

Each Virasoro HW vector gives rise to a primary field and the correlation functions of the primary fields may be recovered from the operator amplitudes by the following construction. Let $(x_1,\dots,x_n)$ be a sequence of non-coincident points in the surface $\hat{\Sigma}$ without boundary. Specify local parameters at points x_j by embedding discs D into $\hat{\Sigma}$ so that their images centered at points x_j do not intersect. $\hat{\Sigma}$ may be then viewed as a surface Σ with boundary capped with discs D. The correlation functions of the primary fields φ_{l_i} corresponding to the HW vectors $\mathcal{X}_{l_i}$ may be defined in a locally flat metric by the formula

$$\langle\,\phi_{l_1}(x_1)\cdots\phi_{l_n}(x_n)\,\rangle\;=\;\tfrac{1}{Z_\gamma}\,A_{\Sigma,\gamma}\,\otimes_i\mathcal{X}_{l_i} \tag{78}$$

(in accordance with relation (69)). The characteristic property (77) of the HW vectors assures that the left hand side is unambiguously defined as a $\Delta_{l_j},\tilde{\Delta}_{l_j}$-form in variable x_j:

Problem 6. Using relation (77) and the gluing axiom show that the right hand side of eq. (78) picks only a factor $(\frac{df(0)}{dz})^{\Delta_{l_j}}\,(\frac{d\bar{f}(0)}{d\bar{z}})^{\tilde{\Delta}_{l_j}}$ under the change $z\mapsto f(z)$ of the $j^{\rm th}$ local parameter of surface $\hat{\Sigma}$.

As we see, the relation between the primary fields and the HW vectors $\phi_l\mapsto\varphi_l(0)\,\Omega$, discovered before is one to one (if we include among the primary fields the trivial "identity" field whose insertions have no effect in correlation functions and which corresponds to the HW vector Ω).

Pants

More generally, it is possible to associate to each vector $\mathcal{X}$ in the Hilbert space of states $\mathcal{H}$, in the domain of ${q_1}^{-L_0}\,{\bar{q}_1}^{-\tilde{L}_0}$ for some q_1 with $|q_1|<1$, an operator-valued field $\varphi(\mathcal{X};w,\bar{w})$ defined for $0<|w|<1$ in the following way. For $0<|q|<|w|-|q_1|<1-2|q_1|$, consider the Riemann surface

$$P_{q,q_1,w}\;=\;\{\,|q|\leq|z|\leq1,\;\;|z-w|\geq|q_1|\,\}$$

with the "out" boundary component $\{\,|z|=1\,\}$ parametrized by $z\mapsto z$ and the "in" ones by $z\mapsto qz$ and by $z\mapsto w+q_1z$. Such a surface is often called "pants" for obvious reasons. The operator $\varphi(\mathcal{X};w,\bar{w})$ will be essentially defined as the amplitude of the pants. More exactly, we shall put

$$\varphi(\mathcal{X};w,\bar{w})\,\mathcal{Y}\;=\;\tfrac{1}{Z_\gamma}\,A_{P_{q,q_1,w},\gamma}\,(\,q^{-L_0}\,\bar{q}^{-\tilde{L}_0}\,\mathcal{Y}\otimes{q_1}^{-L_0}\,{\bar{q}_1}^{-\tilde{L}_0}\,\mathcal{X}\,)\;.$$

Note that $\varphi(\mathcal{X};w,\bar{w})\,\mathcal{Y}$ is independent of q_1 and of q as long as $\mathcal{X}$ is in the domain of ${q_1}^{-L_0}\,{\bar{q}_1}^{-\tilde{L}_0}$ and $\mathcal{Y}$ in the domain of $q^{-L_0}\,\bar{q}^{-\tilde{L}_0}$.

Problem 7. (a). Show that $\varphi(\mathcal{X};0)\,\Omega \equiv \lim\limits_{w\to 0}\ \varphi(\mathcal{X};w,\bar{w})\,\Omega = \mathcal{X}$.
(b). Show that for a HW vector $\mathcal{X}_l$, $\varphi(\mathcal{X}_l;w,\bar{w})$ coincides with the operator $\varphi_l(w,\bar{w})$ assigned to the primary field ϕ_l corresponding to $\mathcal{X}_l$.
(c). Show that $\varphi(L_{-2}\Omega;w,\bar{w}) = \mathcal{T}(w)$ and that $\varphi(\tilde{L}_{-2}\Omega;w,\bar{w}) = \bar{\mathcal{T}}(\bar{w})$.

The operators $\varphi(\mathcal{X};w,\bar{w})$ satisfy an important identity

$$\varphi(\mathcal{Y};z,\bar{z})\,\varphi(\mathcal{X};w,\bar{w}) \;=\; \varphi(\varphi(\mathcal{Y};z-w,\bar{z}-\bar{w})\,\mathcal{X}\,;\,w,\bar{w}) \tag{79}$$

holding for $0<|w|<|z|$, $0<|z-w|<1$. Eq. (79) follows from the two ways that one may obtain the disc with three holes by gluing two discs with two holes.

Problem 8. Prove eq. (79) using the gluing property (**vi**) of the operator amplitudes and treating with care the normalizing factors Z_γ^{-1}.

The relation (79) may be viewed as a global form of the OPE's, the local forms following from it by expanding the vector $\varphi(\mathcal{Y};z-w,\bar{z}-\bar{w})\,\mathcal{X}$ into terms homogeneous in $z-w$ and $\bar{z}-\bar{w}$.

Since any Riemann surface can be built from discs, annuli and pants the general amplitudes $A_{\Sigma,\gamma}$ may be expressed by the Virasoro generators and operators $\varphi(\mathcal{X};w,\bar{w})$. This permits to formulate the basic mathematical structure of CFT in an even more economic (and more algebraic) way than through the amplitudes $A_{\Sigma,h}$ with the properties (**i**) to (**vii**), getting rid of the Riemann surface burden. Instead, one obtains a set of axioms for the action of the $Vir\times Vir$ algebra and of the **vertex operators** $\varphi(\mathcal{X};w,\bar{w})$ in the Hilbert space $\mathcal{H}$, with the relation (79), the main consistency condition, playing a prominent role. That was, essentially, the idea which, in the holomorphic version of CFT, has led to the concept of a **vertex operator algebra** developed by Frenkel-Lepowsky-Meurman and by Borcherds. The latter, by studying the algebras arising in the context of toroidal compactifications (on Minkowki targets) was led to the concept of generalized Kac-Moody or (Borcherds) algebras which promise to play an important role in physics and mathematics.

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Lecture 3. Sigma models

Contents:

1. 1PI effective action and large deviations

It will be useful to describe another relation between a field theoretic and a probabilistic concept. Consider a positive measure

$$d\mu(\xi) = \mathrm{e}^{-S(\xi)}\, D\xi\,.$$

on a (finite dimensional) euclidean space $\mathbf{E}$. We shall assume that $S(\xi)$ grows faster than linearly at infinity and, for convenience, that the measure is normalized. The characteristic functional of $d\mu$

$$\mathrm{e}^{W(J)} = \int \mathrm{e}^{\langle\xi,J\rangle}\, d\mu(\xi)$$

is then an analytic function of J in the complexified dual of $\mathbf{E}$. Let $N\zeta = \sum_1^N \xi_j$ be the sum of N independent random variables equally distributed with measure $d\mu$. The probability distribution of $N\zeta$ is

$$\begin{aligned} P_N(\zeta) &= \int \delta(N\zeta - \sum_{j=1}^N \xi_j) \prod_{j=1}^N d\mu(\xi_j) \\ &= \int DJ \int \mathrm{e}^{-\langle N\zeta-\sum\xi_j,J\rangle} \prod_{j=1}^N d\mu(\xi_j) = \int \mathrm{e}^{-N\langle\zeta,J\rangle+NW(J)}\, DJ\,. \end{aligned} \tag{2}$$

where the J-integration is over an imaginary section of $\mathbf{E}^*_{\mathbf{C}}$ with DJ denoting the properly normalized Lebesque measure on it[7]. We shall be interested in the behavior of P_N for large N. It is not difficult to see that

$$P_N(\zeta) = \mathrm{e}^{N \inf\limits_{J\in\mathbf{E}^*}[-\langle\zeta,J\rangle+W(J)] \;+\; o(N)} \equiv \mathrm{e}^{-N\Gamma(\zeta) \;+\; o(N)}\,. \tag{3}$$

[7] another reading of eq. (2) says that the N-fold convolution becomes the N^{th} power in the Fourier language

In probability theory,

$$?(\zeta) \;=\; \sup_{J\in\mathbf{E}^*}\,[\langle\zeta,J\rangle - W(J)] \;=\; [\langle\zeta,J\rangle - W(J)]\Big|_{\zeta=W'(J)} \tag{4}$$

(W' denotes the derivative of W) is called the "large deviation (rate) function". It is the Legendre transform of $W(J)$ which is a strictly convex function on $\mathbf{E}^*$. It controls the regime where $\sum\xi_j = \mathcal{O}(N)$ as opposed to the central limit theorem which probes $\sum(\xi_j - \langle\xi\rangle) = \mathcal{O}(N^{1/2})$ where $\langle\xi\rangle$ is the mean value of ξ_j. Since $?'(\zeta) = J$, the minimum of $?$ occurs at $\zeta = W'(0) = \langle\xi\rangle$. The central limit theorem sees only the second derivative of $?$ at $\langle\xi\rangle$:

$$\lim_{N\to\infty} P_N(\langle\xi\rangle + N^{-1/2}\psi) \;\sim\; \mathrm{e}^{-\frac{1}{2}\Gamma''(\langle\xi\rangle)(\psi^2)}.$$

W may be recovered from $?$ by the inverse Legendre transform:

$$W(J) \;=\; \sup_{\zeta\in\mathbf{E}}\,[\langle\zeta,J\rangle - ?(\zeta)] \;=\; [\langle\zeta,J\rangle - ?(\zeta)]\Big|_{J=\Gamma'(\zeta)}. \tag{5}$$

We may compute $W(J)$ and $?(\zeta)$ as formal power series introducing a coefficient $\hbar$ (the Planck constant), Taylor expanding $S(\xi)$ and separating the quadratic contribution to it:

$$\begin{aligned}\mathrm{e}^{W(J)} \;&=\; \int \mathrm{e}^{\frac{1}{\hbar}[\langle\xi,J\rangle - S(\xi)]}\,D\xi\,\Big|_{\hbar=1}\\ &=\; \Big[\mathrm{e}^{-\frac{1}{\hbar}S(0)}\int \mathrm{e}^{\frac{1}{\hbar}\langle\xi,J-S'(0)\rangle - \frac{1}{3!\hbar}S'''(0)(\xi^3) - \dots\dots}\,d\mu_{\hbar S''(0)^{-1}}(\xi)\;\int \mathrm{e}^{-\frac{1}{2\hbar}S''(0)(\xi^2)}\,D\xi\Big]\Big|_{\hbar=1}\end{aligned} \tag{6}$$

where the Gaussian measure

$$d\mu_{\hbar S''(0)^{-1}}(\xi) \;=\; \frac{\mathrm{e}^{-\frac{1}{2\hbar}S''(0)(\xi^2)}\,D\xi}{\int \mathrm{e}^{-\frac{1}{2\hbar}S''(0)(\xi^2)}\,D\xi}. \tag{7}$$

Expanding the exponential under the $d\mu$ integral into the power series and performing the Gaussian integration, we obtain an expansion in powers of $\hbar$ which, as discussed in Kazhdan's lectures, gives upon exponentiation the relation

$$\mathrm{e}^{W(J)} \;=\; \Big[\mathrm{e}^{-\frac{1}{\hbar}S(0) + \sum\limits_{\substack{\text{vacuum}\\ \text{graphs } G}} \hbar^{\#\{\text{loops of } G\}-1}\, I_G(J,S)}\int \mathrm{e}^{-\frac{1}{2\hbar}S''(0)(\xi^2)}\,D\xi\Big]\Big|_{\hbar=1} \tag{8}$$

where, by definition, graphs G are connected graphs[8] made of 1-leg vertices J or $-S'(0)$, 3-leg vertices $-S'''(0)$, 4-leg vertices $-S^{(4)}(0)$ etc., with propagators $S''(0)^{-1}$ on the internal lines[9] and no propagators on the external lines. The vacuum graphs are the ones without external lines. The amplitudes $I_G(J,S)$ are associated to the graphs in a natural way, with the symmetry factors of the graphs included. If S is a polynomial and $S'(0) = 0$, then there is only a finite number of graphs with a given number of J-vertices and a given number of loops and we may

[8]recall that the exponential of the sum over connected graphs is the sum over connected and disconnected graphs

[9]we count lines ending at the 1-leg vertices as internal

view $W(J)$ as a formal series in J and in the number of loops. More exactly, comparing the left and the right hand sides of eq. (8), we infer that

$$W(J) \;=\; -S(0) \;+\; \sum_{\substack{\text{vacuum}\\ \text{graphs } G}} I_G(J,S) \;+\; \ln\big(\textstyle\int \mathrm{e}^{-\frac{1}{2}S''(0)(\xi^2)}\, D\xi\big)\,.$$

As discussed by Kazdan, by cutting all the lines of the graphs whose removal makes the graph disconnected, we obtain the second representation for $W(J)$:

$$W(J) \;=\; -?(0) \;+\; \sum_{\substack{\text{vacuum}\\ \text{trees } T}} I_T(J,?)$$

where the "1PI effective action" $?(\zeta)$ is defined by its formal Taylor series

$$\begin{aligned}
?(0) \;&=\; S(0) - \sum_{\substack{\text{1PI vacuum}\\ \text{graphs } G}} I_G(S) - \ln\big(\textstyle\int \mathrm{e}^{-\frac{1}{2}S''(0)(\xi^2)}\, D\xi\big), \qquad & ?'(0) \;&=\; -\sum_{\substack{\text{1PI graphs } G\\ \text{with 1 external line}}} I_G(S)\,,\\
\tfrac{1}{2}\,?''(0) \;&=\; \tfrac{1}{2}\,S''(0) - \sum_{\substack{\text{1PI graphs } G\\ \text{with 2 external lines}}} I_G(S)\,, & \tfrac{1}{3!}\,?'''(0) \;&=\; -\sum_{\substack{\text{1PI graphs } G\\ \text{with 3 external lines}}} I_G(S)\,,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned} \tag{9}$$

where "1PI" stands for (amputated) 1-particle irreducible graphs without J-vertices. Rewriting eq. (8) with S replaced by $?$, i.e. as an expansion for $\int \mathrm{e}^{\frac{1}{\hbar}[\langle\zeta,J\rangle - \Gamma(\zeta)]}\, D\zeta$, but keeping only the leading terms at $\hbar$ small, we obtain finally the equality

$$\sup_{\zeta\in\mathbf{E}}\,[\,\langle\zeta,J\rangle - ?(\zeta)\,] \;=\; -?(0) \;+\; \sum_{\substack{\text{vacuum}\\ \text{trees } T}} I_T(J,?) \;=\; W(J)\,.$$

Comparing this to eq. (5) we see that eqs. (9) provide a perturbative interpretation of the large deviation function $?$.

2. Geometric sigma models

We have already discussed a simple way to write down a conformal invariant action for maps $\phi:\Sigma\to M$ where (Σ,γ) is a Riemann surface and (M,g) a Riemannian manifold. The functional

$$S_g(\phi) \;=\; \tfrac{1}{4\pi}\|d\phi\|^2_{L^2} \;=\; \tfrac{i}{2\pi}\int_\Sigma g_{ij}(\phi)\,\partial\phi^i\wedge\bar\partial\phi^j \tag{10}$$

(summation convention!) depends only on the conformal class of γ. We could add to $S_g(\phi)$ also a "topological" term

$$S_{top}(\phi) \;=\; \tfrac{i}{4\pi}\int_\Sigma \phi^*\omega \;=\; \tfrac{i}{2\pi}\int_\Sigma b_{ij}(\phi)\,\partial\phi^i\wedge\bar\partial\phi^j \tag{11}$$

involving a 2-form $\omega = b_{ij}(\phi)\,d\phi^i\wedge d\phi^j$ on M which does not depend on γ but only on the orientation of Σ. Renormalizability forces addition of two more terms to the action which break classical conformal invariance:

$$S_{tach}(\phi) = \tfrac{1}{4\pi}\int_\Sigma u\circ\phi\; dv \qquad\text{and}\qquad S_{dil} \;=\; \tfrac{1}{4\pi}\int_\Sigma w\circ\phi\; r\, dv \tag{12}$$

where u, w are functions on M called, respectively, the tachyon and the dilaton potentials[10], dv stands for the volume measure and r for the scalar curvature of Σ.

One may also consider a supersymmetric version of the model (see Problem set 3) with the action

$$S_g^{\rm SUSY}(\Phi) \;=\; \tfrac{1}{2\pi i}\int g_{ij}(\Phi)\, D\Phi\; \tilde{D}\Phi\;\, dz\wedge d\bar{z}\wedge d\theta\wedge d\tilde{\theta} \tag{13}$$

where the superfield

$$\Phi \;=\; \phi \,+\, \theta\psi \,+\, \tilde{\theta}\tilde{\psi} \,+\, \theta\tilde{\theta}\, F\,.$$

and $D=\partial_\theta+\theta\partial_z\,,\ \tilde{D}=\partial_{\tilde{\theta}}+\tilde{\theta}\partial_{\bar{z}}\,.$ In components, after elimination of the auxiliary field F through its equation of motion, one obtains

$$\begin{aligned} S_g^{\rm SUSY}(\phi,\psi,\tilde{\psi}) \;=\; \tfrac{i}{2\pi}\int \Big(g_{ij}(\phi)\,\partial_z\phi^i\,\partial_{\bar{z}}\phi^j \;-\; g_{ij}(\phi)\,\big(\psi^i\nabla_z\psi^j \,+\, \tilde{\psi}^i\nabla_{\bar{z}}\tilde{\psi}^j\big)\\ -\,\tfrac{1}{2}\,R_{ijkl}(\phi)\,\psi^i\psi^j\tilde{\psi}^k\tilde{\psi}^l\Big)\,dz\wedge d\bar{z} \end{aligned}\tag{14}$$

where $\nabla_z\psi^j \;=\; \partial_z\psi^i \,+\, ?^{\,j}_{kl}\,\partial_z\phi^k\,\psi^l$ with $?^{\,j}_{kl} \;=\; \{{}^{\,j}_{kl}\}$ standing for the Levi-Civita connection symbols and similarly for $\nabla_{\bar{z}}\tilde{\psi}^i$ and where R_{ijkl} denotes the curvature tensor of M. Addition of the 2-form $\omega=b_{ij}(\phi)\,d\phi^i\wedge d\phi^j$ term corresponds to the change $g_{ij}\mapsto g_{ij}+b_{ij}$ in eq. (13). In the component formula (14) it results in the same replacement of g_{ij} and, additionally, in the replacement of the Levi-Civita symbols in $\nabla_z\psi^j$ ($\nabla_{\bar{z}}\tilde{\psi}^j$) by symbols of a metric connection with torsion $\{{}^{\,j}_{kl}\}\pm\frac{3}{2}g^{jm}\,H_{klm}$, respectively, where H_{klm} is the antisymmetric tensor representing $d\omega$. The curvature R_{ijkl} in eq. (14) becomes that of the connection with the plus sign[11].

The two-dimensional field theory with action (10) is usually called a **sigma model**. The stationary points of $S(\phi)$ are harmonic maps from Σ to M and correspond to the classical solutions. Can one quantize sigma models by giving sense to functional integrals

$$\int\limits_{Map(\Sigma,M)} F(\phi)\;{\rm e}^{-\,S(\phi)}\;D\phi \tag{15}$$

with $S\,=\,S_g+S_{top}+S_{tach}+S_{dil}$ where for $F(\phi)$ one may for example take $\prod_j u_j(\phi(x_j))$ for some functions u_j on M? We have already seen that this was easily doable for M a torus with a constant metric and a constant 2-form ω, with vanishing (or constant) tachyon and dilaton potentials. The corresponding functional integral was essentially Gaussian and the resulting theory was a little decoration of the free massless field. Here we would like to examine the case with an arbitrary topology and geometry of the target by treating the functional integrals of the type (15) in perturbation theory and also, possibly, going beyond the purely perturbative considerations employing powerful methods of the renormalization group approach to quantum field theories.

[10] the names come from the string theory context
[11] it is equal to R_{klij} for the minus sign connection

3. Regularization and renormalization

We may anticipate problems with the definition of functional integrals (15) even in a perturbative approach. We shall attempt to remove these problems by using freedom to change the parameters of the theory, namely the metric on M and the tachyon and dilaton potentials (for the sake of simplicity, we shall discard the topological term in the action). The strategy to make sense of functional integrals of type (15) will then be as follows:

1. (**regularization**) we modify the theory introducing a (short distance) cutoff Λ into it to make functional integral exist;

2. (**renormalization**) we try to choose the metric g and the tachyon and dilaton potentials entering the action in a Λ-dependent way so that the cutoff versions of integrals (possibly after further multiplication by a Λ-dependent factor) converge to a non-trivial limit.

There are many ways to introduce a short distance (ultra-violet) cutoff into the theory. To simplify the problem further let us assume that Σ is the periodic box $[-0,L]^2$ (that will do away with the contribution of the dilaton potential). One possibility to introduce the UV cutoff is to consider the lattice version of the sigma model. Let $\Sigma_\Lambda\subset[0,L]^2$ be composed of points with coordinates in $\frac{1}{\Lambda}\mathbf{Z}$ where ΛL is a power of 2. The lattice version of $\phi:[0,L]^2\rightarrow M$ is the map $\phi:\Sigma_\Lambda\rightarrow M$ and for the cutoff action we may put

$$S^\Lambda_{g,u}(\phi)\ =\ \frac{1}{8\pi}\sum\limits_{\substack{x,y\in\Sigma_\Lambda\\ |x-y|=\Lambda^{-1}}}d_g^2(\phi(x),\phi(y))+\frac{1}{4\pi}\sum\limits_{x\in\Sigma_\Lambda}\Lambda^{-2}u(\phi(x))$$

where d_g stands for the metric distance on M. If ϕ is the restriction of a fixed smooth (periodic) M-valued map on $[0,L]^2$ and u is continuous then in the limit $\Lambda\rightarrow\infty$ we recover the value of the original action $S_g+S_{tach}\equiv S_{g,u}$. The cutoff version of the normalized integral (15) with $F(\phi)=\prod u_j(\phi(x_j))$ becomes now

$$\frac{\int\prod\limits_{j=1}^n u_j(\phi(x_j))\ \mathrm{e}^{-S^\Lambda_{g,u}(\phi)}\ D_g\phi}{\int\mathrm{e}^{-S^\Lambda_{g,u}(\phi)}\ D_g\phi}\tag{16}$$

where $D_g\phi=\prod\limits_{x\in\Sigma_\Lambda}dv_g(\phi(x))$ with dv_g denoting the metric volume measure on M. The integral is finite e.g. for compact M and u_j, say, continuous. The lattice sigma models for $M=S^{N-1}$ with a metric proportional to that of the unit sphere in $\mathbf{R}^N$ are essentially well known "spin" models in classical (as opposed to quantum) statistical mechanics ($N=1$ corresponds to the Ising model, $N=2$ to a version of the XY model, $N=3$ to a slightly modified classical Heisenberg model; u proportional to a coordinate in $\mathbf{R}^N$ describes the coupling to the magnetic field).

We would like to study if, after renormalization, the cutoff may be removed in the correlation functions[12] (16). More precisely we would like to show that the limits

$$\lim\limits_{\Lambda\rightarrow\infty}\ \frac{\int\prod\limits_{j=1}^n Z(\Lambda)u_j(\phi(x_j))\ \mathrm{e}^{-S^\Lambda_{g(\Lambda),u(\Lambda)}(\phi)}\ D_{g(\Lambda)}\phi}{\int\ \mathrm{e}^{-S^\Lambda_{g(\Lambda),u(\Lambda)}(\phi)}\ D_{g(\Lambda)}\phi}\tag{17}$$

[12] from the point of view of statistical mechanics which reformulates the problem in terms of the system with a fixed lattice spacing, this is a question about the large distance behavior of correlation functions

exist for a cutoff-dependent linear map[13] $Z(\Lambda)$ on the space of functions on M and for cutoff-dependent choices of the metric $g(\Lambda)$ and of the tachyon potential $u(\Lambda)$ on M. We would also like to parametrize possible limits (17) defining the correlation functions of the quantum two-dimensional sigma models.

4. Renormalization group effective actions

One could study the questions raised above first by perturbative methods applied directly to the lattice correlation functions (16). It is important, however to set the perturbative scheme in the way that does not destroy the geometric features of the model (i.e. in a way covariant under diffeomorphisms of M). One way to assure this is to study, instead of correlation functions, objects known under the name of renormalization group effective actions.

Fix μ such that Λ/μ is a power of 2 and for $y\in\Sigma_\mu$ denote by $B(y)$ the set of $x\in\Sigma_\Lambda$ in the square $y+[0,\frac{\mu}{\Lambda}L[^2$, Call a point $\phi\in M$ a barycenter of a set of points $\phi_j\in M$, $j=1,\dots,N$, if ϕ extremizes $\sum_j d_g^2(\phi,\phi_j)$. Clearly, if M is a euclidean space then $\phi=\frac{1}{N}\sum_j\phi_j$. Suppose that we fix a map $\phi:\Sigma_\mu\to M$ and compute the integral

$$\mathrm{e}^{-S^{\mu}_{eff}(\phi)}\,D\phi\;=\;\int\prod_{y\in\Sigma_\mu}\delta\big(\tfrac{1}{2}\nabla_{\phi(y)}\sum_{x\in B(y)}d_g^2(\phi(y),\varphi(x))\big)\;\mathrm{e}^{-S^{\Lambda}_{g,u}(\varphi)}\;D_g\varphi\,. \tag{18}$$

The right hand side is naturally a measure on M^{Σ_μ}. It essentially computes the probability distribution of the barycenters $\phi(y)$ of "spins" $\varphi(x)$ with x in blocks $B(y)$[14]. $S^{\mu}_{eff}(\phi)$, the logarithm of the the density of the right hand side w.r.t. to some reference measure $D\phi$ on M^{Σ_μ}, is called the ("block spin") **renormalization group** (RG) **effective action** on scale μ.

The renormalizability problem may now be reformulated as the question about existence of the $\Lambda\to\infty$ limit of S^{μ}_{eff}, more exactly, of the normalized measure

$$d\nu^{\mu}_{eff}(\phi)=\frac{\mathrm{e}^{-S^{\mu}_{eff}(\phi)}\,D\phi}{\int\mathrm{e}^{-S^{\mu}_{eff}(\phi)}\,D\phi} \tag{19}$$

on M^{Σ_μ}, if we choose "bare" g and u in the Λ-dependent way and keep μ fixed. With the modification of the definition of S^{μ}_{eff} described in the footnote, one may show that the two formulations of the renormalizability problem are essentially equivalent.

The limiting measures $d\nu^{\mu}_{eff}$ may be viewed as describing the $\Lambda=\infty$ theory averaged over variations of the fields on distance scales $\leq\mu^{-1}$. Pictorially, they describe the system viewed from far away when we do not distinguish details of length $\lesssim\frac{1}{\mu}$. An important observation at the core of the RG analysis is that this averaging may be done inductively by first eliminating the variations on the smallest scales, then on the larger ones, and so on until scale μ is reached[15]. In

[13] recall that even the free field case required a multiplicative renormalization of the correlation functions of exponents of field ϕ

[14] this would hold if the barycenters were unique and with the normalizing factor $\frac{1}{\int\delta(\frac{1}{2}\nabla_{\phi(y)}\sum_{x\in B(y)}d_g^2(\phi(y),\varphi(x)))\,D_g\phi}$ under the integral on the right hand side of eq. (18)

[15] this is not exactly the case for our definition of S^{μ}_{eff} but ignore this for a moment

the infinite volume ($L=\infty$), the process may be viewed as a repeated application of a map on a space of unit lattice actions. If under the iterations the effective actions are driven to a simple attractor (like an unstable manifold of a fixed point) then the renormalization consists of choosing the initial "bare" actions so that the $\Lambda=\infty$ effective actions S^{μ}_{eff} end up on the attractor. In the vicinity of a fixed point this would be possible if the family of the bare actions (parametrized by bare couplings) crosses transversally the stable manifold. The renormalized couplings parametrize then the unstable manifold (a drawing would be helpful here). This dynamical system view of renormalization developed by K. G. Wilson is extremely important and will hopefully be explained in much more details in future lectures.

We have suppressed in the notation the dependence on the size L of the box. If the choice of $g(\Lambda)$ and $u(\Lambda)$ involved in the $\Lambda\to\infty$ limit can be done in an L independent way, we automatically obtain a family of measures parametrized by μ and L. The measures with the product $L\mu$ fixed (to a power of 2) are related by the rescaling of space-time distances[16]. If the infinite volume limit $L\to\infty$ of the theory exists, μ becomes a continuous parameter. Suppose that the effective actions S^{μ}_{eff} of possible continuum limits (i.e. the attractor of the RG map in the dynamical system view) may be parametrized by (dimensionless) "renormalized" metrics g and tachyon potentials u. You should think that S^{μ}_{eff} is equal to $S^{\mu}_{g,\mu^2 u}$ plus less important (higher dimension) terms separated by a precise rule. We would say then that the theory is renormalizable by a metric and a tachyon potential renormalization. This is the scenario realized in perturbation theory, see below. In such a situations the $\Lambda, L=\infty$ theories are characterized by the "running" metric $\mu\mapsto g(\mu)$ and tachyon potential $\mu\mapsto u(\mu)$ describing S^{μ}_{eff} on different scales μ in the passive view of the scale-dependence of the renormalized theory. In the active view, the μ-dependence of g and u is generated by the action of rescalings of distances on the limiting theory. The infinitesimal scale transformations generate a vector field $\beta\partial_g+\gamma\partial_u$ in the space of (g,u) defined by

$$\beta(g,u)=\mu\tfrac{\partial}{\partial\mu}g\,,\qquad \gamma(g,u)=\mu\tfrac{\partial}{\partial\mu}u\,. \tag{20}$$

$\beta(g,u)$ and $\gamma(g,u)$ are called in the physicists jargon the RG "**beta**" and "**gamma functions**". In the dynamical system language, $\beta\partial_g+\gamma\partial_u$ is a vector field on the attractor of the RG map and it extends the map to a flow. The importance of the RG functions lies in the fact that, computed in perturbation expansion, they allow to go beyond it, providing for example a consistency check on the latter: by solving the RG eqs. (20) with β and γ given by few perturbative terms we may check whether the trajectories $\mu\mapsto(g(\mu),u(\mu))$ stay or are driven out for large μ (that is at short distances) from the region of the (g,u)-space where the perturbative calculation may be trusted. It is clear that the zeros of the (β,γ) vector field should play an important role. They correspond to scale invariant (and hence conformal invariant) field theories and, in the dynamical system picture, to fixed points of the RG map (since they lie already on the attractor).

How to generate the perturbation expansion for the RG effective actions $\mathrm{e}^{-S^{\mu}_{eff}}$? A helpful observation is that the delta-function in the definition (18) can be rewritten in simple terms if we use the exponential map $\mathrm{e}: T_\phi M\to M$.

[16] i. e. of the metric on Σ

Problem 1 (geometric). Show that for vectors ξ_j in a small ball in $T_\phi M$,

$$\tfrac{1}{2}\nabla_\phi(\sum_{j=1}^{N} d_g^2(\phi, \mathrm{e}^{\xi_j}\phi) \;=\; \sum_{j=1}^{N}\xi_j\,.$$

It follows that ϕ is a barycenter of the set of points $\{\mathrm{e}^{\xi_j}\phi\}$ iff $\sum\xi_j = 0$.

Substituting in eq. (18) $\varphi(x) = \mathrm{e}^{\xi(x)}\phi(y)$ for $\xi(x) \in T_{\phi(y)}M$ if $x \in B(y)$, or, in a shorthand notation, $\varphi = \mathrm{e}^{\xi}\tilde{\phi}$ where $\tilde{\phi}(x) = \phi(y)$ for $x \in B(y)$, we obtain[17]

$$\mathrm{e}^{-S^{\,\mu}_{eff}(\phi)} \;=\; \Big[\int \prod_{y\in\Sigma_\mu} \delta(\sum_{x\in B(y)}\xi(x))\; \mathrm{e}^{-\frac{1}{\hbar}S^{\Lambda}_{g,u}(\mathrm{e}^{\xi}\tilde{\phi})}\; D_g\mathrm{e}^{\xi}\tilde{\phi}\Big]\Big|_{\hbar=1}. \tag{21}$$

Note that the lattice field ξ takes values in a vector space $\{\,\xi \mid \xi(x) \in T_{\phi(y)}M\,,\ \sum\limits_{x\in B(y)}\xi(x) = 0\,\}$. The loop expansion for $S^{\,\mu}_{eff}$ may just be generated in the standard way by expanding in powers of ξ on the right hand side of eq. (21) all terms except for the quadratic contribution to $S_{g,u}$ which is used to produce a Gaussian measure. At each loop order the result will be invariant under the simultaneous action of diffeomorphisms of M on ϕ, g and u. When $\Lambda\to\infty$, divergences will appear in the perturbative expressions. The perturbative renormalizability of the theory may be studied by replacing the "bare" g and u on the right hand side of eq. (21) by

$$g(\Lambda) \;=\; g + \sum_{n=1}^{\infty}\hbar^n\,\delta g_n(g,u,\Lambda/\mu)\,,\qquad u(\Lambda) \;=\; \mu^2\left(u + \sum_{n=1}^{\infty}\hbar^n\,\delta_n u(g,u,\Lambda/\mu)\right). \tag{22}$$

We may attempt to fix the above series by choosing some way to extract the renormalized metric g and the renormalized potential u from effective actions $S^{\,\mu}_{eff}$. We would then like to show that the above substitution cancels the $\Lambda\to\infty$ divergences in each loop order of $S^{\,\mu}_{eff}$ resulting in a family of perturbative RG effective actions parametrized by "running" metric $g(\mu)$ and tachyon potential $u(\mu)$. Differentiation of the series (22) over $\ln\mu$ with $g(\Lambda), u(\Lambda)$ fixed would then produce in the $\Lambda\to\infty$ limit the loop expansion for the beta and gamma RG functions.

5. Background field effective action

In practice, the lattice perturbative calculations are prohibitively complicated. It would be easier to work with continuum regularization and renormalization which allow to calculate the diagram amplitudes by momentum space integrals and to make use of rotational invariance. We have seen in Witten's lectures on perturbative renormalization of the scalar field theories with the ϕ^3 or ϕ^4 interactions that it was convenient to analyze directly the "1PI effective action" ? given by the Legendre transform of the free energy functional W. The latter was defined as the logarithm of the integral of type of (15), with $F(\phi) = \mathrm{e}^{\langle\phi,J\rangle}$. The definition of both W and ? , however, as well as their perturbative analysis, used heavily the linear structure in the space of maps from the space-time to the target M, inherited from the linear structure of M. Such structure is missing if M is a general manifold. It is possible, nevertheless, to introduce

[17] the fact that the exponential parametrization may work only locally does not impede the perturbative analysis

for sigma models an effective action mimicking the construction of the large deviations function (see eq. (3)) and somewhat similar in spirit to the RG effective actions for the lattice version of sigma models discussed in the previous section. Instead of fixing the block barycenters in a single lattice theory, we shall take N independent copies of continuum theories with fields ϕ_j and shall fix for each x the barycenters $\phi(x)$ of $\phi_j(x)$ defining the functional

$$P_N(\phi) \;=\; \int \prod_x \delta(\tfrac{1}{2}\nabla_{\phi(x)} \sum_{j=1}^{N} d_g^2(\phi(x), \phi_j(x))) \; \prod_j \mathrm{e}^{-S_{g,u}(\phi_j)} \; D_g\phi_j \tag{23}$$

by a formal functional integral. Note that the right hand side reduces to a well defined integral for a lattice version of the theory. For a map $\phi : \Sigma \to M$ and for a section ξ of the pullback ϕ^*TM of the bundle tangent to M, denote by $\mathrm{e}^{\xi}\phi$ the map from Σ to M whose value at point x is obtained by applying the exponential map to $\xi(x) \in T_{\phi(x)}M$. Reparametrizing in eq. (23) $\phi_j = \mathrm{e}^{\xi_j}\phi$, we obtain

$$P_N(\phi) \;=\; \int \delta(\sum_{j=1}^{N} \xi_j) \; \prod_j \mathrm{e}^{-S(\mathrm{e}^{\xi_j}\phi)} \; D(\mathrm{e}^{\xi_j}\phi). \tag{24}$$

We may try to extract the "**background field effective action**" $?_b(\phi)$ from the leading contribution to P_N at large N:

$$P_N(\phi) \;=\; \mathrm{e}^{N\,\Gamma_b(\phi) \;+\; o(N)}. \tag{25}$$

It should be clear that $?_b(\phi)$ coincides then with the effective action $?_\phi(\zeta = 0)$ of the ξ-field theory (depending on ϕ as a parameter) corresponding to the functional integral

$$\int \; - \; \mathrm{e}^{-S(\mathrm{e}^{\xi}\phi)} \; D(\mathrm{e}^{\xi}\phi). \tag{26}$$

Fields ξ take values in a vector space of sections of ϕ^*TM so that the perturbative treatment of the ξ-theory is more standard. Note that only the geometric structure on M was used in the formal definition of $?_b(\phi)$.

We shall reformulate the renormalizability problem (17) for the second time as the question about existence of the $\Lambda \to \infty$ limit of the regularized version of the background-field effective action $?_b^{\Lambda}(\phi)$ for a cutoff-dependent theory with the action $S_{g(\Lambda),u(\Lambda)}(\phi)$. We could regularize the functional integral (24) by putting fields ξ_j on a lattice with spacing $\frac{1}{\Lambda}$ while keeping ϕ as a continuum field. This would not produce a big computational gain in comparison to the perturbative calculation of the RG effective actions. It is possible, however, to regularize the loop expansion of the background field effective action just by introducing an ultraviolet cutoff in the momentum space integrals for the 1PI vacuum amplitudes in the ξ-field theory whose fields form a vector space of sections of ϕ^*TM. $?_{b,\Lambda}(\phi)$ obtained this way will be covariant under the diffeomorphisms of M in each order of the loop expansion. The perturbative renormalization will consist of choosing the "bare" parameters of the theory in a cuttoff-dependent way as in eqs. (22) and such that the $\Lambda \to \infty$ limit of $?_b^{\Lambda}(\phi)$ exists order by order in the loop expansion. The perturbative limits will be parametrized by the "running" metric $g(\mu)$ and potential $u(\mu)$ on M, with the change of μ induced by rescaling of distances on Σ.

6. Dimensional regularization

We shall prove the perturbative renormalizability of the background field effective action in the $2D$ sigma model only in the leading order of the loop expansion, concentrating instead on the discussion of the renormalization group aspects of the 1 loop result. To avoid calculational (but not conceptual) difficulties, we shall work in the flat euclidean space-time $\Sigma = \mathbf{E}^2$. We shall also use a specific scheme for regularization of divergent diagrams: the **dimensional regularization** and a particular way to renormalize the theory (i. e. to chose $g(\Lambda)$ and $u(\Lambda)$): the **minimal subtraction**. Briefly, the idea is to

1. regularize the momentum space integrals by rewriting them as integrals into which the space-time dimension D enters as an analytic (complex) parameter, then

2. to calculate the integral for the values of D where it converges and, finally,

3. to analytically continue to the physical values of D extracting the pole parts of the result at the physical dimension as the divergence to be removed by the renormalization.

In order to gain some practice let us compare how the simplest divergent diagram of the 4-dimensional ϕ^3 theory –O– is regularized and renormalized first in the more conventional momentum space regularization used in Witten's lecture and then in the dimensional regularization - minimal subtraction scheme. The momentum space amplitude $\hat{I}(k)$ corresponding to the (amputated) graph was given by the integral

$$\hat{I}_D(k) \;=\; \tfrac{g^2}{4}\int \frac{đ\, q}{(q^2+m^2)((q-k)^2+m^2)} \;=\; \tfrac{g^2}{4}\int_0^1 d\alpha \int \frac{đ\, q}{(q^2+\alpha(1-\alpha)k^2+m^2)^2} \tag{27}$$

where k is the external momentum, q is that of the loop (both euclidean) and $đ\, q \equiv \frac{d^D q}{(2\pi)^D}$. In space-time dimension $D=4$ the q-integral diverges logarithmically. It may be regularized by restricting the integration to $|q| \leq \Lambda$.

$$\hat{I}_4^{\Lambda}(k) \;=\; \tfrac{g^2}{4}\int_0^1 d\alpha \int\limits_{|q|\leq\Lambda} \frac{đ\, q}{(q^2+\alpha(1-\alpha)k^2+m^2)^2} \;=\; \tfrac{g^2}{32\pi^2}\ln\tfrac{\Lambda}{\mu} + \hat{K}_4^{\Lambda}(\tfrac{k^2}{\mu^2},\tfrac{m}{\mu})$$

where

$$\lim_{\Lambda\to\infty}\hat{K}_4^{\Lambda}(\tfrac{k^2}{\mu^2},\tfrac{m}{\mu}) \;=\; -\tfrac{g^2}{64\pi^2}\left(\int_0^1 d\alpha\, \ln[\alpha(1-\alpha)\tfrac{k^2}{\mu^2}+\tfrac{m^2}{\mu^2}] + 1\right) \tag{28}$$

The renormalization idea is then to substitute

$$g(\Lambda) \;=\; \mu\lambda, \qquad m^2(\Lambda) \;=\; \mu^2\left(r + \sum_{n=1}^{\infty}\hbar^n\,\delta r_n(\lambda, r, \tfrac{\Lambda}{\mu})\right) \tag{29}$$

for the coupling constant and the mass squared in the initial action (recall that the theory does not need renormalization of g and only 1 loop counterterm would do). The powers of μ make λ and r dimensionless. With a choice

$$\delta r_1 \;=\; \tfrac{\lambda^2}{16\pi^2}\ln\tfrac{\Lambda}{\mu}$$

the contribution to the 1 loop amplitude $I_4^\Lambda(k)$ diverging when $\Lambda\to\infty$ is canceled resulting in the renormalized value of the amplitude

$$\hat I^\Lambda_{4,ren}(k) \;=\; -\tfrac{\mu^2\lambda^2}{64\pi^2}\left(\int_0^1 d\alpha\;\ln\left[\alpha(1-\alpha)\tfrac{k^2}{\mu^2}+r\right]\,+\,1\right)$$

The RG functions $\beta(\lambda,r)\;=\;\mu\frac{d}{d\mu}\lambda$, $\gamma_2(\lambda,r)\;=\;\mu\frac{d}{d\mu}r$ describing the scale-dependence of the renormalized couplings are obtained by differentiating eq. (29) over $\ln\mu$ with $\Lambda, g(\Lambda)$ and $m^2(\Lambda)$ held fixed:

$$\begin{aligned} 0 &= \mu\lambda\,+\,\mu^2\tfrac{d\lambda}{d\mu}\,,\\ 0 &= \mu\tfrac{d}{d\mu}[\mu^2 r\,+\,\hbar\,\tfrac{g^2}{16\pi^2}\,\ln\tfrac{\Lambda}{\mu}\,+\,\mathcal{O}(\hbar^2)]\;=\;2\mu^2 r\,+\,\mu^3\tfrac{dr}{d\mu}\,-\,\hbar\,\mu^2\tfrac{\lambda^2}{16\pi^2}\,+\,\mathcal{O}(\hbar^2)\end{aligned}$$

so that

$$\beta(\lambda,r)\;=\;-\lambda\,,\qquad \gamma_2(\lambda,r)\;=\;-2r\,+\,\hbar\,\tfrac{\lambda^2}{16\pi^2}\,+\,\mathcal{O}(\hbar^2)\,. \tag{30}$$

Let us see how the same problem is treated in the dimensional regularization - minimal subtraction scheme. Using the relation $\int_0^\infty \mathrm{e}^{-a\sigma}\,\sigma\,d\sigma\;=\;a^{-2}$, we may rewrite the integral for $\hat I_D(k)$ in the form[18]

$$\hat I_D(k)\;=\;\tfrac{g^2}{4}\int_0^1 d\alpha\int_0^\infty d\sigma\int\sigma\,\mathrm{e}^{-[q^2+\alpha(1-\alpha)k^2+m^2]\sigma}\,đ\,q\,. \tag{31}$$

Performing the q-integral first, we obtain

$$\hat I_D(k)\;=\;\tfrac{g^2}{2^{D+2}\pi^{D/2}}\int_0^1 d\alpha\int_0^\infty d\sigma\;\sigma^{1-D/2}\;\mathrm{e}^{-[\alpha(1-\alpha)k^2+m^2]\sigma}\,.$$

The latter integral converges for any complex D with $\mathrm{Re}D<4$. It gives explicitly

$$\hat I_D(k)\;=\;\tfrac{g^2}{2^{D+2}\pi^{D/2}}\;?\,(2-\tfrac{D}{2})\int_0^1 d\alpha\;[\alpha(1-\alpha)k^2+m^2]^{\frac{D}{2}-2}\,.$$

The divergence in four dimensions manifests itself as a pole in the expression at $D=4$;

$$\hat I_D(k)\;=\;\tfrac{g^2}{32\pi^2}\,\tfrac{1}{4-D}\,-\,\tfrac{g^2}{64\pi^2}\left(\int_0^1 d\alpha\;\ln\left[\alpha(1-\alpha)k^2+m^2\right]+\ln 4\pi\,+\,C\right)\,+\,\mathcal{O}(4-D)$$

where $C\;=\;-?'(1)$ is the Euler constant. In the minimal subtraction renormalization scheme, the amplitude is renormalized by substituting for the original coupling and mass squared

$$g\;=\;\mu^{3-D/2}\lambda\,,\qquad m^2\;=\;\mu^2\left(r\,+\,\sum_{n=1}^{\infty}\hbar^n\,\delta r_n(\lambda,r)\right) \tag{32}$$

[18]for those who do not remember Feynman's famous formula (I don't), we could have used twice the identity $\int_0^\infty \mathrm{e}^{-a_i\sigma_i}\,d\sigma_i\;=\;a_i^{-1}$ in the original expression for $\hat I_D(k)$ changing then the variables to (α,σ) where $\sigma_1\;=\;\alpha\sigma\,,\;\sigma_2=(1-\alpha)\sigma$

with the pure pole dependence of δr_n on the dimension:

$$\delta r_n = \sum_{j=1}^{k_n} \delta r_{n,j}(\lambda, r)\, \tfrac{1}{(4-D)^j}$$

chosen to cancel exactly the pole part of the dimensionally regularized amplitudes. In reality, only the 1 loop amplitude has a pole which is simple. With

$$\delta r_1 \;=\; \tfrac{\lambda^2}{16\pi^2 (4-D)}\,,$$

we obtain the renormalized amplitude

$$\hat{I}^{\Lambda}_{4,ren}(k) \;=\; -\tfrac{\mu^2\lambda^2}{64\pi^2}\left(\int_0^1 d\alpha\;\ln\left[\alpha(1-\alpha)\tfrac{k^2}{\mu^2} + r\right] + \ln(4\pi) + C\right)$$

(note how μ has entered under the logarithm). The difference between the two renormalizations may be absorbed into a finite redefinitions of the renormalized parameters λ, r, see Problem 2 below.

Now the RG functions $\beta(\lambda, r)$, $\gamma_2(\lambda, r)$ are obtained by differentiating eqs. (32) with respect to $\ln\mu$ while keeping g and m^2 fixed:

$$\begin{aligned} 0 \;&=\; (3-\tfrac{D}{2})\mu^{3-D/2}\lambda + \mu^{4-D/2}\tfrac{d\lambda}{d\mu}\,, \\ 0 \;&=\; \mu\tfrac{d}{d\mu}\left[\mu^2 r + \hbar\,\tfrac{\mu^2\lambda^2}{16\pi^2(4-D)}\right] \;=\; \mu\tfrac{d}{d\mu}\left[\mu^2 r + \hbar\,\tfrac{\mu^{D-4}g^2}{16\pi^2(4-D)}\right] \\ &=\; 2\mu^2 r + \mu^3\tfrac{dr}{d\mu} - \hbar\,\mu^2\,\tfrac{\lambda^2}{16\pi^2} \end{aligned}$$

so that

$$\beta(\lambda, r) \;=\; (\tfrac{D}{2} - 3)\lambda\,, \qquad \gamma_2(\lambda, r) \;=\; -2r + \hbar\,\tfrac{\lambda^2}{16\pi^2}\,. \tag{33}$$

The gain is that we have obtained formulae for β and γ_2 in general dimension. They may serve as an indication of how the field theory behave in smaller or larger dimension then the one considered. For example, the vanishing of the linear contribution to $\beta(\lambda)$ in 6 dimensions signals that the theory becomes only renormalizable there. Note that eqs. (33) reduce to (30) at $D = 4$. This did not have to be the case since we have changed the parametrization of the limiting theories and what is geometrically defined is the vector field $\beta\partial_\lambda + \gamma_2\partial_r$.

Problem 2. Find to the 1 loop order the transformation between the coordinates (λ, r) of the renormalized 4-dimensional ϕ^3 theory corresponding to the passage between the two renormalization schemes discussed above. Show that it preserves the form of the vector field $\beta\partial_\lambda + \gamma_2\partial_r$.

Problem 3. Find the running couplings $\lambda(\mu)$, $r(\mu)$ using the 1 loop approximations to the β, γ_2 functions. What can one tell about the effect of higher loop corrections to the large μ (UV) behavior of the running couplings?

The four-dimensional ϕ^3 theory, in spite of its super-renormalizability (only a finite number of divergent 1PI graphs) and self-consistency of its perturbative calculations, has a non-perturbative stability problem related to the lack of a lower bound for the cubic polynomial. This should serve as a warning that even the RG improved perturbative analysis is not enough to assure existence of a renormalized QFT.

7. Renormalization of the sigma models to 1 loop

As mentioned above, the background field effective action $?_b(\phi)$ of the sigma model is equal to the $\zeta = 0$ value of the effective action of the ξ-theory with the action $S_\phi(\xi)$ given by the relation

$$\mathrm{e}^{-S_{g,u}(\mathrm{e}^\xi\phi)}\; D_g(\mathrm{e}^\xi\phi) \;=\; \mathrm{e}^{-S_\phi(\xi)}\; D\xi\,.$$

The 1^{st} of eqs. (9) implies then that in the perturbation expansion

$$?_b(\phi) \;=\; S_\phi(0) - \sum_{\substack{\text{1PI vacuum}\\ \text{graphs } G}} I_G(S_\phi) \,-\, \ln\big(\int \mathrm{e}^{-\frac{1}{2}S''_\phi(0)(\xi^2)}\, D\xi\big)\,. \tag{34}$$

One of the simplifying features of the dimensional regularization is that we may disregard the terms in $S_\phi(\xi)$ coming from the logarithm of the "Radon-Nikodym derivative" $\frac{D_g(\mathrm{e}^\xi\phi)}{D\xi}$. Formally, they are proportional to $\delta^{(2)}(0) = \int đ\, q$ which vanishes in the dimensional regularization.

Problem 4. (a). (for Pasha). Show by calculating the D-dimensional integral $\int \mathrm{e}^{-q^2}\, dq$ in two ways that the volume of the unit sphere in D dimensions is equal to $2\pi^{\frac{D}{2}}/?\,(\frac{D}{2})$.

(b). Show that in the radial variables the D-dimensional integral $\int\limits_{|q|\geq\epsilon} dq$ converges for $\mathrm{Re}\, D < 0$:

$$\int\limits_{|q|\geq\epsilon} đ\, q \;=\; -\,\frac{2^{1-D}\pi^{-D/2}}{D\,\Gamma(D/2)}\;\epsilon^D\,.$$

Defining the value of the integral by analytic continuation for integer $D \geq 0$ and taking ϵ to zero we infer that $\int đ\, q$ vanishes in positive dimensions in the dimensional regularization.

Expanding in local coordinates

$$(\mathrm{e}^\xi\phi)^i \;=\; \phi^i + \xi^i \,-\, \tfrac{1}{2}\,?^{\,i}_{\,jk}(\phi)\,\xi^j\xi^k \,+\, \mathcal{O}(\xi^3)$$

where $?^{\,i}_{\,jk} = \{{}^{\,i}_{jk}\}$ is the Levi-Civita symbol, we obtain after a little calculation (you may do it!)

$$\begin{aligned} S_{g,u}(\mathrm{e}^\xi\phi) \;=\; \tfrac{1}{4\pi}\int\Big(& g_{ij}(\phi)\,\partial_\nu\phi^i\,\partial_\nu\phi^j \;+\; u(\phi) \;+\; 2g_{ij}(\phi)\,\partial_\nu\phi^i\,\nabla_\nu\xi^j \;+\; \partial_i u(\phi)\,\xi^i \\ &+ g_{ij}(\phi)\,\nabla_\nu\xi^i\,\nabla_\nu\xi^j \;-\; R_{ijkl}(\phi)\,\partial_\nu\phi^i\,\partial_\nu\phi^k\,\xi^j\,\xi^l \;+\; \tfrac{1}{2}\nabla_i\partial_j u(\phi)\,\xi^i\,\xi^j\Big)\,dx \;+\; \mathcal{O}(\xi^3)\,. \end{aligned} \tag{35}$$

Above, ∇_i denotes the covariant derivative over the i^{th} coordinate and

$$\nabla_\nu \xi^i \;=\; \partial_\nu \xi^i \,+\, ?^{\,i}_{\,jk}(\phi)\,\partial_\nu \phi^j\;\xi^k\,.$$

The vector bundle ϕ^*TM has a natural metric given by ϕ^*g. It will be convenient to choose a global orthonormal frame $(e_a)_{a=1}^d$ of ϕ^*TM (d denotes the dimension on M). In coordinates, $e_a = e^i_a \partial_{\phi^i}$. Different choices of e_a are related by gauge transformations $e'_a = \lambda^b_a e_b$ with λ an $SO(d)$-valued function on the plane $\mathbf{E}^2$. Of course e^i_a depend also on field ϕ. It will be more convenient to rewrite

$$\xi = \xi^a\,e_a \qquad \text{or} \qquad \xi^i = \xi^a\,e^i_a$$

with (ξ^a) a sequence of functions on $\mathbf{E}^2$. The expansion (35) becomes then

$$\begin{aligned} S_{g,u}(\mathrm{e}^\xi\phi) \;&=\; \tfrac{1}{4\pi}\int\Big(g_{ij}(\phi)\,\partial_\nu\phi^i\,\partial_\nu\phi^j \;+\; u(\phi) \;+\; 2\,e^a_i\,\partial_\nu\phi^i\nabla_\nu\xi^a \;+\; e^i_a\,\partial_i u(\phi)\,\xi^a \\ &+\; (\nabla_\nu\xi^a)^2 \;-\; R_{iakb}(\phi)\,\partial_\nu\phi^i\,\partial_\nu\phi^k\;\xi^a\;\xi^b \;+\; \tfrac{1}{2}\,e^i_a\,e^j_b\,\nabla_i\partial_j u(\phi)\,\xi^a\,\xi^b\Big)\,dx \;+\; \mathcal{O}(\xi^3). \qquad (36)\end{aligned}$$

where (e^a_i) is the matrix inverse to (e^i_a), $R_{iakb} = e^j_a e^l_b R_{ijkl}$ and

$$\nabla_\nu\xi^a \;=\; \partial_\nu\xi^a \,+\, A^a_{b\nu}\,\xi^b \qquad \text{with} \qquad A^a_{b\nu} \;=\; e^a_i\left(\partial_\nu e^i_b + ?^{\,i}_{\,jk}\,\partial_\nu\phi^j\,e^k_b\right).$$

Clearly, $A^a_{b\nu}\,dx^\nu$ transforms as an $SO(d)$ connection form under the gauge transformations $e_a \mapsto \lambda^b_a e_b$. The perturbation expansion (34) for $?_b(\phi)$ becomes now to the 1 loop order:

$$\begin{aligned} ?_b(\phi) \;&=\; \tfrac{1}{4\pi\hbar}\int\Big(g_{ij}(\phi)\,\partial_\nu\phi^i\,\partial_\nu\phi^j \;+\; u(\phi)\Big)\,dx \\ &-\; \ln\left(\int \mathrm{e}^{-\frac{1}{4\pi}\int\left((\nabla_\nu\xi^a)^2 \;-\; R_{iakb}(\phi)\,\partial_\nu\phi^i\,\partial_\nu\phi^k\;\xi^a\;\xi^b \;+\; \frac{1}{2}\,e^i_a e^j_b\nabla_i\partial_j u(\phi)\,\xi^a\xi^b\right)dx}\;D\xi\right) \;+\; \mathcal{O}(\hbar) \qquad (37)\end{aligned}$$

(the terms linear in ξ on the right hand side of (35) do not contribute to $?_b$). The functional integral gives a determinant and we could use the zeta-function prescription to make sense out of it in an $SO(d)$-gauge-invariant way. We shall be, however, more interested in the divergent part of the determinant than in its renormalized value. The dimensional regularization will allow to extract the divergence in a convenient ($SO(d)$-gauge-invariant) manner and, besides, it works to all orders.

We shall obtain the expression for the 1 loop contribution to $?_b(\phi)$ expanded around a constant value ϕ_0 of ϕ in the form (in coordinates around ϕ_0)

$$[?_b(\phi)]^{1\,\text{loop}} \;=\; \sum_n\int K_{n,D}(x_1,\dots,x_n;\phi_0)\,\prod_{j=1}^n(\phi(x_j)-\phi_0)\,dx_j \qquad (38)$$

with the translationally invariant kernels $K_{n,D}$ regularized dimensionally, i.e. meromorphically dependent on D, with possible poles at $D=2$ and with $\phi-\phi_0$ vanishing fast at infinity. In order to generate the expansion (38) for the 1 loop contribution to $?_b(\phi)$, we shall separate the term

$$\tfrac{1}{4\pi}\int\Big((\partial_\nu\xi^a)^2 \,+\, \tfrac{1}{2}\,(e^i_a e^j_b\nabla_i\partial_j u)|_{\phi_0}\,\xi^a\,\xi^b\Big)\,dx$$

in the action as producing the Gaussian measure[19] from

$$\frac{1}{4\pi}\int\Big(2\,A^a_{b\nu}\,\partial_\nu\xi^a\,\xi^b \;+\; A^a_{b\nu}\,A^a_{c\nu}\,\xi^b\,\xi^c + \tfrac{1}{2}\left(e^i_a e^j_b\nabla_i\partial_j u(\phi) - (e^i_a e^j_b\nabla_i\partial_j u)|_{\phi_0}\right)\xi^a\xi^b \\ - R_{iakb}(\phi)\,\partial_\nu\phi^i\,\partial_\nu\phi^k\,\xi^a\,\xi^b\Big)\,dx$$

treated as an interaction. Now it is easy to enumerate the divergent graphs. First there are logarithmically divergent contributions coming from the graphs

.

They cancel each other (there are no divergences in 2D gauge theory). The only divergent terms we are left with are

,


and

$$\tfrac{1}{2}\,\ln\,\det\,\tfrac{1}{2\pi}\left(-\delta_{ab}\,\Delta + \tfrac{1}{2}(e^i_a e^j_b\nabla_i\partial_j u)|_{\phi_0}\right).\qquad(39)$$

All other contributions are easily checked to have finite limits at $D=2$. Since in the dimensional regularization

$$\int\frac{d\!\!\!{}^-q}{q^2+m^2} \;=\; \int_0^\infty d\sigma\int \mathrm{e}^{-\sigma\,(q^2+m^2)}\,d\!\!\!{}^-q \;=\; 2^{-D}\,\pi^{-D/2}\int_0^\infty d\sigma\,\sigma^{-D/2}\,\mathrm{e}^{-\sigma\,m^2}$$
$$=\; 2^{-D}\,\pi^{-D/2}\,m^{D-2}\,?\,(1-D/2) \;=\; \tfrac{1}{2\pi}\,\tfrac{1}{2-D} \;+\; \text{part regular at } D=2,$$

the pole part of the loops in (39) is equal to

$$\frac{1}{4\pi}\,\frac{1}{2-D}\int\Big(-R_{iaka}(\phi)\,\partial_\nu\phi^i\,\partial_\nu\phi^k + \tfrac{1}{2}(\Delta_g u(\phi) - \Delta_g u(\phi_0))\Big)\,dx$$

(since $e^i_a e^j_a\nabla_i\partial_j u \;=\; g^{ij}\nabla_i\partial_j u \;=\; \Delta_g u$ where Δ_g denotes the Laplacian on M). Similarly,

$$\tfrac{1}{2}\,\ln\,\det\,\tfrac{1}{2\pi}\left(-\Delta\delta_{ab} + \tfrac{1}{2}(e^i_a e^j_b\,\nabla_i\partial_j u)|_{\phi_0}\right) \;=\; \tfrac{1}{2}\int dx\int \mathrm{tr}\;\ln\,\tfrac{1}{2\pi}\left(q^2 + \tfrac{1}{2}\,e\,e\nabla\partial u|_{\phi_0}\right)d\!\!\!{}^-q$$
$$=\;\text{const.} + \tfrac{1}{2}\int dx\int_0^1 dt\;\mathrm{tr}\,\frac{1}{q^2+\frac{t}{2}\,e\,e\nabla\partial u|_{\phi_0}}\;\tfrac{1}{2}\,e\,e\nabla\partial u|_{\phi_0}\,d\!\!\!{}^-q$$
$$=\;\text{const.} + \tfrac{1}{8\pi}\,\tfrac{1}{2-D}\int\Delta_g u(\phi_0)\,dx\,.$$

We infer that the pole part of $?_g(\phi)$ to the 1 loop order is

$$[?_b(\phi)]^{1\ \text{loop}}_{\text{div}} \;=\; -\frac{1}{4\pi}\,\frac{1}{2-D}\int R_{ij}(\phi)\,\partial_\nu\phi^i\,\partial_\nu\phi^j\,dx \;+\; \frac{1}{8\pi}\,\frac{1}{2-D}\int\Delta_g u(\phi)\,dx$$

where $R_{ij} \;=\; R_{iaja}$ is the Ricci tensor on M. $[?_b(\phi)]^{1\ \text{loop}}_{\text{div}}$ is an integral of dimension 2 and dimension 0 operators and this result, in accord with a simple power counting, remains true at higher orders.

[19] we assume that the matrix $((e^i_a e^j_b\,\nabla_i\partial_j u)|_{\phi_0})$ is positive which is the case, for example, in the vicinity of a minimum of u

The minimal subtraction renormalization scheme adds counterterms to bare metric g and bare potential u which cancel the above poles. More exactly, one substitutes in the initial action $S_{g_0,u_0}(\phi)$ of the model with the bare metric g_0 and bare tachyon potential u_0,

$$g_0 \;=\; \mu^{D-2}(g + \tfrac{\hbar}{2-D}\,\delta g_1 \;+\; \mathcal{O}(\hbar^2))\,, \tag{40}$$

$$u_0 \;=\; \mu^{D}(u + \tfrac{\hbar}{2-D}\,\delta u_1 \;+\; \mathcal{O}(\hbar^2))\,. \tag{41}$$

The added 1 loop counterterms change the effective action by

$$\delta ?_b(\phi) \;=\; \tfrac{1}{4\pi}\,\tfrac{\mu^{D-2}}{2-D}\int\Big(\delta g_{1\,ij}(\phi)\,\partial_\nu\phi^i\,\partial_\nu\phi^j \;+\; \mu^2\,\delta u_1(\phi)\Big)\,dx \;+\; \mathcal{O}(\hbar)$$

and we put

$$\delta g_{1\,ij} \;=\; R_{ij}\,, \qquad\qquad \delta u_1 \;=\; -\tfrac{1}{2}\,\Delta_g u\,.$$

canceling the poles at $D=2$. This proves the renormalizability of the sigma model (background field effective action) to 1 loop.

8. Renormalization group analysis of sigma models

Let us compute the vector field $\beta\partial_g + \gamma\partial_u$ on the space of metrics and potentials as given by the minimal subtraction version of eqs. (20):

$$\beta(g,u) \;=\; \mu\tfrac{\partial}{d\mu}\,g\Big|_{\substack{g_0=\text{const.}\\ u_0=\text{const.}}}\,, \qquad\qquad \gamma(g,u) \;=\; \mu\tfrac{\partial}{d\mu}\,u\Big|_{\substack{g_0=\text{const.}\\ u_0=\text{const.}}}\,. \tag{42}$$

Applying the derivative $\frac{d}{d\ln\mu}$ to eq. (40), we obtain

$$\begin{aligned} 0 \;&=\; \mu\tfrac{d}{d\mu}\Big[\mu^{D-2}\Big(g_{ij} + \tfrac{\hbar}{2-D}\,R_{ij} \;+\; \mathcal{O}(\hbar^2)\Big)\Big] \;=\; \mu\tfrac{d}{d\mu}\Big[\mu^{D-2}\Big(g_{ij} + \tfrac{\hbar}{2-D}\,R_{0ij} \;+\; \mathcal{O}(\hbar^2)\Big)\Big] \\ &=\; \mu^{D-2}\Big[\beta_{ij}(g) - (2-D)g_{ij} - \hbar\,R_{ij} \;+\; \mathcal{O}(\hbar^2)\Big] \end{aligned}$$

from which we infer that

$$\beta_{ij}(g) \;=\; (2-D)g_{ij} + \hbar\,R_{ij} \;+\; \mathcal{O}(\hbar^2)\,. \tag{43}$$

Similarly,

$$\begin{aligned} \mu\tfrac{d}{d\mu}\Big[\mu^{D}\Big(u - \tfrac{\hbar}{2(2-D)}\,\Delta_g u \;+\; \mathcal{O}(\hbar^2)\Big)\Big] \;&=\; \mu\tfrac{d}{d\mu}\Big[\mu^{D}u - \mu^{D-2}\,\tfrac{\hbar}{2(2-D)}\,\Delta_{g_0}u_0 \;+\; \mathcal{O}(\hbar^2)\Big] \\ &=\; \mu^{D}\Big[\gamma(u) + Du + \tfrac{\hbar}{2}\,\Delta_g u \;+\; \mathcal{O}(\hbar^2)\Big] \end{aligned}$$

so that

$$\gamma(u) \;=\; -\,Du \;-\; \tfrac{1}{2}\,\hbar\,\Delta_g u \;+\; \mathcal{O}(\hbar^2) \tag{44}$$

The vector field β in the space of metrics may be used to find out in which situations we may expect the perturbative calculation to be self-consistent. The condition is that the running metric $g(\mu)$ satisfying the RG equation

$$\mu\frac{d}{d\mu}g \;=\; \beta(g) \tag{45}$$

stays on all scales $\mu \geq \mu_0$ in the perturbative regime. Let us illustrate this on the example where $M = S^{N-1}$ with the metric $\frac{1}{\alpha'}$ times the induced metric g of the unit sphere in $\mathbf{R}^N$. Due to the rotational symmetry, the renormalized metric is $\frac{1}{\alpha'(\mu)}g$ and the eq. (45) for its μ-dependence reduces in $D = 2$ to

$$\mu\frac{d}{d\mu}\alpha' \;=\; \hbar\,(2-N)\,(\alpha')^2 + \;\mathcal{O}(\hbar^2)\,.$$

Clearly, for $N > 2$, α' is driven to zero for large μ approximately as $\mathcal{O}(\frac{1}{\ln\mu})$. The perturbative regime corresponds to small α' so that the perturbative expansion is self-consistent for the sigma model with $M = S^{N-1}$ for $N > 2$. The phenomenon is called the **asymptotic freedom** of the spherical sigma model since $\alpha' = 0$ corresponds to a free theory. It permits to expect that the theory may be constructed non-perturbatively, at least in finite volume. Such a non-perturbative theory would break the conformal invariance of the classical sigma model. In fact, there are strong reasons to believe that its infinite volume version is massive (an exact expression for its S-matrix is, conjecturally, known).

The property of asymptotic freedom is shared by all the sigma models which have compact symmetric spaces as their targets (and also, more importantly, by the non-abelian 4-dimensional gauge theories with not too many fermion species, like Quantum Chromodynamics (QCD) describing the strong interactions of quarks, mediated by SU_3 gauge fields).

Problem 5. Consider the flow

$$\mu\frac{d}{d\mu}\alpha \;=\; -\alpha^2\,, \qquad \mu\frac{d}{d\mu}u \;=\; -2u \,+\, u\alpha \tag{46}$$

in $\mathbf{R}^2$. Show that there exist only one perturbative solution

$$u \;=\; \sum_{n=0}^{\infty} a_n\alpha^n \tag{47}$$

for the invariant manifold of the flow. Study the (non-perturbative) invariant manifolds.

Specially interesting cases correspond to manifolds with vanishing Ricci curvature. The $N =$ 2 spherical sigma model is the simplest example (coinciding with free field with values in S^1). As we know, it corresponds to a CFT. One may then read from the γ function the dimensions equal to $d_q = \frac{1}{2}q^2$ of the composite operators given by the exponential functions $\mathrm{e}^{iq\phi}$ on S^1. d_g are equal to the eigenvalues of $-\frac{1}{2}\Delta_g$ (the tree contribution to the γ comes from the fact that we have considered integrated insertions of the composite operator into the action). For Ricci flat targets there is no renormalization of the metric in the 1 loop order and, in the supersymmetric versions, up to 4 loops (4 loops excluded). No renormalization of the metric means that the beta function vanishes and the scale invariance is preserved (to 4 loops). One may then argue in perturbation theory (studying the Hessian of the 1 loop β) that in the Kählerian case, the Ricci flat metric may be perturbed as to give rise to a scale invariant quantum sigma model with $N=2$ superconformal symmetry (as discussed by Ed Witten during the lecture). Thus Calabi-Yau ($\cong$ Kähler, Ricci flat) manifolds should correspond to superconformal $N=2$ field theories. This observation resulted in a conjectured mirror symmetry between Calabi-Yau manifolds, now established in many instances.

In the case of SUSY sigma models with hyper-Kähler targets (i.e. with the $N=4$ supersymmetry), β vanishes to all orders of the loop expansion.

The inclusion of the 2-form term (11) into the action of the sigma models modifies the above results. In the 1 loop order, the beta function $\mu\frac{d}{d\mu}(g_{ij}+b_{ij})$ is given by the Ricci curvature of the metric connection with torsion $?^{i}_{jk} = \{{}^{\,i}_{jk}\} + \frac{3}{2}g^{il}H_{jkl}$ where the antisymmetric tensor H_{jkl} corresponds to $\pm d\omega$. In models in which the connections with torsion are globally flat, the beta function vanishes to all orders (even without supersymmetry). The WZW model of CFT, which we shall discuss in the next lecture, corresponds to such a situation. Addition of the 2-form which is closed does not modify the β function but may change the long-distance behavior of the model (that seems to happen for the sigma model with S^2 target where the addition of the term with ω equal to π times the volume form of the unit sphere should render the model massless).

As for the renormalization of the potentials whose scale-dependence is governed by the RG equation

$$\mu\frac{d}{d\mu}u \;=\; -2u-\tfrac{1}{2}\hbar\,\Delta_g u \;+\; \mathcal{O}(\hbar^2)$$

note that, on a symmetric space, u (approximately) reproduces itself up to a normalization if it belongs to an eigen-subspace of the Laplacian. The RG analysis allows then to predict the short distance behavior of the correlation functions involving insertions of the corresponding composite operators (somewhat similarly as for $M=S^1$).

References

For the rudiments of the perturbative approach to functional integrals, Feynman graphs etc. see again the book by Zinn-Justin, Sects. 5.1-5.3 and Kazhdan's, Witten's and Gross' lectures in the present series.

The original reference to the renormalization of geometric sigma models is Fiedan's thesis published with few years delay in Ann. Phys. 163 (1985), p. 318. The SUSY case is discussed in Alvarez-Gaume-Freedman-Mukhi, Ann. Phys. 134 (1981) p. 85, with further developments in Alvarez-Gaumé-Ginsparg, Commun. Math. Phys. 102 (1985), p. 311, Alvarez-Gaumé-S. Coleman-Ginsparg, Commun. Math. Phys. 103 (1986), p. 423 and Grisaru-Van De Ven-Zanon, Nucl. Phys. B 277 (1986), p. 388 and p. 409 (the last papers discovered $4^{\rm th}$ order contributions to the supesymmetric beta function). For the case of sigma models with a 2-form in the action see Braaten-Curtright-Zachos, Nucl. Phys. B 260 (1985), p. 630 and also Callan-Friedan-Martinec-Perry, Nucl. Phys. B 262 (1985), p. 593.

Lecture 4. Constructive conformal field theory

Contents:

Let us recall the logical structure of this course. In the first lecture we studied the free field examples of CFT's. In the second one, we analyzed the scheme of (two-dimensional) CFT from a more abstract, axiomatic point of view. In the third one, we searched perturbatively among geometric sigma models for non-free examples of CFT's. Finally, in the present lecture compressed due to lack of time, we shall analyze a specially important sigma model, the Wess-Zumino-Witten (WZW) one, whose correlation functions may be constructed non-perturbatively, with a degree of explicitness comparable to that attained for toroidal compactifications of free fields (constituting the simplest examples of WZW theories). The WZW model appears to be a generating theory of a vast family of CFT's whose correlations can be expressed in terms of the WZW ones. The comparison of the non-perturbative models obtained this way with the perturbative constructions of sigma models allows for highly non-trivial tests of differences between the geometry of Ricci flat (Einstein) spaces and that of CFT's, replacing the Einstein geometry in the stringy approach to gravity.

1. WZW model

The target space of the WZW sigma model is a compact Lie group manifold G and the two-dimensional theory may be considered as a generalization of quantum mechanics of a particle moving on G. In the latter case the (Euclidean) action functional is

$$S(g) = -\tfrac{1}{2}\int \mathrm{tr}\,(g^{-1}\partial_x g)^2\, dx \tag{2}$$

where "tr" denotes the Killing form[20]. Let R denote an (irreducible) unitary representation $g \mapsto g_R$ of G in a (finite dimensional) Hilbert space V_R. The path integral for the quantum-mechanical particle on G, corresponding to the Wiener measure on G, may be solved with the use of the Feynman-Kac formula taking the form

$$\int\limits_{Map([0,L]_{per},G)} \bigotimes_{i=1}^{n} g_{R_i}(x_i)\ \mathrm{e}^{-kS(g)} \prod_x dg(x) \Big/ \int\limits_{Map([0,L]_{per},G)} \mathrm{e}^{-kS(g)} \prod_x dg(x)$$

[20] normalized so that the long roots have length squared 2

$$= \ \mathrm{Tr}\ \mathrm{e}^{-x_1 H} g_{R_1}\, \mathrm{e}^{(x_2-x_1)H} g_{R_2} \ldots g_{R_n} \mathrm{e}^{(L-x_n)H} \Big/ \mathrm{Tr}\ \mathrm{e}^{-LH} \tag{3}$$

where $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq L$, $2kH$ is the Laplacian on G and g_R is viewed as a matrix of multiplication operators, both acting in $L^2(G, dg)$ (dg is the Haar measure). Compare Problem 3 in Lecture 1 dealing with the case $G = S^1$. The theory possesses the $G \times G$ symmetry which may be used to solve it: the right hand side of (3) is calculable in terms of the harmonic analysis on G.

Problem 1. *Compute explicitly the 1-,2- and 3-point functions in* (3).

The Euclidean action of the WZW model is a functional on $Map(\Sigma, G)$ where Σ is a compact Riemann surface. If, for simplicity, we assume G to be connected and simply connected then

$$S(g) \ = \ -\tfrac{i}{4\pi} \int_\Sigma \mathrm{tr}\, g^{-1}\partial g \wedge g^{-1}\bar\partial g \ + \ \tfrac{i}{4\pi} \int_\Sigma g^*\omega \tag{4}$$

where the 2-form ω is defined on open subsets $\mathcal{O} \subset G$ with $H_2(\mathcal{O}) = 0$ and satisfies there $d\omega = -\frac{1}{3}\mathrm{tr}\,(g^{-1}dg)^{\wedge 3}$. The dependence of the second term of $S(g)$ on the choice of ω makes $S(g)$ defined modulo $2\pi i\mathbf{Z}$ so that $\mathrm{e}^{-kS(g)}$ is well defined for integer k. To have the energy bounded below, we shall take k, called the level of the WZW model, positive.

Problem 2. *Assuming, more generally, that g takes values in the complexified group $G^{\mathbf{C}}$ find the equations for stationary points of $S(g)$.*

The correlation functions of the WZW model are formally given by the functional integrals:

$$\langle \bigotimes_{i=1}^{n} g_{R_i}(x_i) \rangle \ = \int\limits_{Map(\Sigma,G)} \bigotimes_{i=1}^{n} g_{R_i}(x_i)\, \mathrm{e}^{-kS(g)}\ Dg \Big/ \int\limits_{Map(\Sigma,G)} \mathrm{e}^{-kS(g)}\ Dg \quad \in \quad \bigotimes_{i=1}^{n} \mathrm{End}(V_{R_i}) \tag{5}$$

where Dg stands for the formal product of the Haar measures $dg(x)$ over $x \in \Sigma$.

As we have mentioned at the end of Lecture 3, the renormalization group beta function computed for the WZW sigma model vanishes to all orders due to the flatness of the connections with torsion generated from the metric and the 2-form ω on G. Thus the model is conformally invariant and does not need renormalization of the action (4) in perturbation theory. The conformal invariance holds, in fact, also non-perturbatively due to the $LG \times LG$ symmetry of the theory where LG denotes the loop group $Map(S^1, G)$ of G. The WZW model may be solved exactly by

1. harmonic analysis on LG

or by

2. exact functional integration.

As we shall see, the (matrix-valued) composite operators $g_R(x)$ need multiplicative renormalization and acquire scaling dimensions $\frac{2c_R}{k+h^\vee}$ where c_R denotes the quadratic Casimir of R and $h^\vee$ stands for the dual Coxeter number of G (the quadratic Casimir of the adjoint representation).

2. Gauge symmetry Ward identities

We shall sketch here the functional integral approach to the WZW theory. It will be convenient to extend a little the model by coupling it to an external gauge field[21] $A=A^{10}+A^{01}$, a 1-form with values in the complexified Lie algebra $\mathbf{g}^{\mathbf{C}}$ of G. Define[22]

$$S(g,A) \;=\; S(g) + \frac{ik}{2\pi}\int\limits_\Sigma \mathrm{tr}\,[A^{10}\wedge g^{-1}\bar\partial g + g\partial g^{-1}\wedge A^{01} + gA^{10}g^{-1}\wedge A^{01}]. \tag{6}$$

Under the ("chiral") gauge transformations corresponding to maps $h_{1,2}:\Sigma\to G^{\mathbf{C}}$ the action $S(g,A)$ transforms according to the Polyakov-Wiegmann formula

$$S(h_2gh_1^{-1},\,{}^{h_1}\!A^{10}+{}^{h_2}\!A^{01}) \;=\; S(g,\,A^{10}+A^{01}) - S(h_1,\,A^{10}) - S(h_2^{-1},\,A^{01}) \tag{7}$$

where ${}^{h_1}\!A^{10}=h_1A^{10}h_1^{-1}+h_1\partial h_1^{-1}$ and ${}^{h_2}\!A^{01}=h_2A^{01}h_2^{-1}+h_2\bar\partial h_2^{-1}$.

Problem 3. *Prove the Polyakov-Wiegmann formula.*

In the presence of the external gauge field A, the partition function of the WZW theory will be formally defined as

$$Z_{_A} \;=\; \int\limits_{Map(\Sigma,G)} \mathrm{e}^{-kS(g,A)}\; Dg \tag{8}$$

and the correlation functions $\langle\otimes g_{_{R_i}}(x_i)\rangle_{_A}$ by eq. (5) with $S(g)$ replaced by $S(g,A)$ (no functional integration over A). Using the formal extension to functional integrals of the simple invariance property

$$\int\limits_G f(h_2gh_1^{-1})\,dg \;=\; \int\limits_G f(g)\,dg$$

holding for $h_{1,2}\in G^{\mathbf{C}}$ if f is an analytic function on $G^{\mathbf{C}}$, we obtain

$$\begin{aligned} Z_{_{{}^{h_1}\!A^{10}+{}^{h_2}\!A^{01}}}\;\langle\otimes g_{_{R_i}}(x_i)\rangle_{_{{}^{h_1}\!A^{10}+{}^{h_2}\!A^{01}}} \;&=\; \int \mathop{\otimes}_{i=1}^{n}(h_2gh_1^{-1})_{_{R_i}}(x_i)\;\mathrm{e}^{-kS(h_2gh_1^{-1},\,{}^{h_1}\!A^{10}+{}^{h_2}\!A^{01})}\;Dg\\ &=\;\mathrm{e}^{kS(h_1,A^{10})}\;\mathrm{e}^{kS(h_2^{-1},A^{01})}\;\mathop{\otimes}_i(h_2)_{_{R_i}}(x_i)\;Z_{_A}\;\langle\otimes g_{_{R_i}}(x_i)\rangle_{_A}\;\mathop{\otimes}_i(h_1)^{-1}_{_{R_i}}(x_i).\end{aligned} \tag{9}$$

This is the chiral gauge symmetry Ward identity for the correlation functions (recall the diffeomorphism group and the local rescaling Ward identities discussed in Lecture 2).

The identity (9) factorizes into a holomorphic (A^{01}-dependent) and an anti-holomorphic (A^{10}-dependent) ones. Hence in order to study the A^{01}-dependence of the correlation functions it is enough to look for holomorphic maps on a Sobolev space of 0,1-forms[23] A^{01} with values in $\mathbf{g}$

$$\Psi:\;\mathcal{A}^{01}\longrightarrow\mathop{\otimes}_{i=1}^{n}V_{_{R_i}}\;\equiv\;V_{_{\mathbf{R}}}$$

[21] in general, we shall not assume the unitarity $A=-A^*$ of the gauge field
[22] a more standard definition subtracts also $A^{10}\wedge A^{01}$ inside $[\ldots]$
[23] what follows does not depend on the assumed degree of smoothness of forms provided it is high enough

satisfying the "factorized" Ward identity

$$\Psi({}^{h}\!A^{01}) \;=\; \mathrm{e}^{kS(h^{-1},A^{01})}\ \bigotimes_{i=1}^{n} h_{R_i}(x_i)\ \Psi(A^{01}). \tag{10}$$

The relation (10) describes the behavior of Ψ along the orbits of the group $\mathcal{G}^{\mathbf{C}}$ of complex (Sobolev-class) gauge transformations in $\mathcal{A}^{01}$. The orbit space $\mathcal{A}^{01}/\mathcal{G}^{\mathbf{C}}$ is the moduli space of holomorphic $G^{\mathbf{C}}$ bundles which, upon restriction to semi-stable bundles and appropriate treatment of semi-stable but not stable ones, becomes a compact variety $\mathcal{N}$ of complex dimension 0, rank(G) and $\dim(G)(h_\Sigma - 1)$ for genus h_Σ equal to 0, 1 and > 1, respectively. The space $W(\Sigma, \mathbf{x}, \mathbf{R}, k)$ of Ψ's coincides with the space $H^0(\mathcal{V})$ of holomorphic sections of a vector bundle $\mathcal{V}$ over $\mathcal{N}$ with typical fiber $V_{\mathbf{R}}$ ($\mathcal{V} = \mathcal{A}^{01} \times_{\mathcal{G}^{\mathbf{C}}} V_{\mathbf{R}}$ essentially). In another description, $W(\Sigma, \mathbf{x}, \mathbf{R}, k) = H^0(\mathcal{L})$ where $\mathcal{L}$ is a line bundle over the moduli space of holomorphic $G^{\mathbf{C}}$-bundles with parabolic structures at points x_i and Ψ's may be interpreted as a non-abelian generalization of theta functions. The essential implication of these identifications is that $W(\Sigma, \mathbf{x}, \mathbf{R}, k)$ is a finite-dimensional space. Its dimension depends, in fact, only on h_Σ, k and $\mathbf{R}$ and is given by the celebrated Verlinde formula. $W(\Sigma, \mathbf{x}, \mathbf{R}, k)$ may be also identified with the space of quantum states of the Chern Simons theory.

Out of the global Ward identities (9) one may extract the infinitesimal ones by taking $h_i = \mathrm{e}^{\Lambda_i}$ and Taylor-expanding in Λ_i similarly as we analyzed the infinitesimal consequences of the diffeomorphism and rescaling Ward identities in Lecture 2. Define the insertions of currents into the correlations by

$$\begin{aligned}\langle J^a_z \ldots \rangle_A &= -\pi \frac{1}{Z_A} \frac{\delta}{\delta A^a_{\bar z}} Z_A \langle \ldots \rangle_A, \\ \langle J^a_{\bar z} \ldots \rangle_A &= -\pi \frac{1}{Z_A} \frac{\delta}{\delta A^a_z} Z_A \langle \ldots \rangle_A\end{aligned}$$

(the subscript "a" corresponds to a basis (t^a) of the Lie algebra $\mathbf{g}$ s.t. $\mathrm{tr}\, t^a t^b = \frac{1}{2}\delta^{ab}$). Denote by $J(z)$ $(\bar J(\bar z))$ the insertions of J_z $(\bar J_{\bar z})$ into correlations with A vanishing around the insertion point and the metric locally flat. $J(z)$ $(\bar J(\bar z))$ depends holomorphically (anti-holomorphically) on z away from other insertions. Expanding to the second order, one obtains from the Ward identities (9) the operator product expansions

$$J^a(z)\, J^b(w) \;=\; \frac{\delta^{ab}\, k/2}{(z-w)^2} + \frac{i f^{abc}}{z-w} J^c(w) \;+\; \ldots\,, \tag{11}$$

$$\bar J^a(\bar z)\, \bar J^b(\bar w) \;=\; \frac{\delta^{ab}\, k/2}{(\bar z-\bar w)^2} - \frac{i f^{abc}}{\bar z-\bar w} \bar J^c(\bar w) \;+\; \ldots\,, \tag{12}$$

$$J^a(z)\, \bar J^b(\bar w) \;=\; \ldots \tag{13}$$

which imply for the modes of the corresponding Hilbert space operators $\mathcal{J}(z) = \sum_n J_n z^{-n-1}$, $\bar{\mathcal{J}}(\bar z) = \sum_n \bar J_n \bar z^{-n-1}$ the Kac-Moody algebra relations

$$[J^a_n,\, J^b_m] = i f^{abc} J^c_{n+m} + \tfrac{1}{2} kn\, \delta^{ab} \delta_{n+m,0}$$

and similarly for $\bar J_n$ with the commutators between J_n and $\bar J_m$ vanishing.

Problem 4. *Prove the operator product expansions* (11-13).

Subtraction of the singular part from the expression $\mathrm{tr}\, J(z)\, J(w)$ gives the Sugawara construction of the energy-momentum tensor of the WZW theory:

$$T(w) \;=\; \tfrac{2}{k+h^\vee}\, \lim_{z\to w} \Big(\mathrm{tr}\; J(z)\, J(w) \,-\, \tfrac{\dim(G)\, k}{4(z-w)^2}\Big)$$

and similarly for $\bar T(\bar w)$. In modes, this becomes

$$L_n \;=\; \tfrac{2}{k+h^\vee} \sum_{m=-\infty}^{\infty} \mathrm{tr}\; :J_{n-m}\, J_m:$$

where the normal ordering puts J_p with positive p to the right of the ones with negative p.

3. Scalar product of non-abelian theta functions

Since the A^{10}-dependence of the unnormalized correlation functions $Z_{_A}\,\langle \otimes g_{_{R_i}}(x_i)\rangle_{_A}$ coincides with that of $\overline{\Psi(-(A^{10})^*)}$ (recall that the complex conjugate space $\overline{V_{_R}} \cong V_{_R}^*$), we must have

$$Z_{_A}\,\langle \otimes g_{_{R_i}}(x_i)\rangle_{_A} \quad\in\quad W(\Sigma,\mathbf{x},\mathbf{R},k)\otimes\overline{W(\Sigma,\mathbf{x},\mathbf{R},k)}$$

as a function of A or, more explicitly,

$$Z_{_A}\,\langle \otimes g_{_{R_i}}(x_i)\rangle_{_A} \;=\; H^{\alpha\beta}\;\Psi_\alpha(A^{01})\otimes\overline{\Psi_\beta(-(A^{10})^*)} \tag{14}$$

where (Ψ_α) is a basis of $W(\Sigma,\mathbf{x},\mathbf{R},k)$, $(H^{\alpha\beta})$ is an $\mathbf{x}$-dependent matrix and the summation convention is assumed. From formal reality properties of the functional integral defining $Z_{_A}\,\langle \otimes g_{_{R_i}}(x_i)\rangle_{_A}$ one may see that $(H^{\alpha\beta})$ should be a hermitian matrix. In fact one may argue that the inverse matrix $(H_{\beta\alpha})$ corresponds to the scalar product on the space $W(\Sigma,\mathbf{x},\mathbf{R},k)$ of non-abelian theta functions:

$$H_{\beta\alpha} \;=\; (\Psi_\beta,\Psi_\alpha) \tag{15}$$

where $(\,\cdot\,,\,\cdot\,)$ is formally given by

$$(\Psi,\Psi') \;=\; \int (\Psi(A^{01}),\,\Psi'(A^{01}))_{V_{\mathbf{R}}}\;\mathrm{e}^{-\frac{k}{2\pi}\|A\|^2_{L^2}}\;DA \tag{16}$$

with the integration over the unitary gauge fields A with $A^{10}=-(A^{01})^*$. The scalar product (16) is exactly the one which gives the probability amplitudes between the states of the Chern-Simons theory. Expressions (14) and (15) for the correlation functions may be expressed in a basis-independent way as follows. Let $e_{A^{01}}$ denote the evaluation map

$$W(\Sigma,\mathbf{x},\mathbf{R},k)\,\ni\,\Psi\;\xrightarrow{e_{A^{01}}}\;\Psi(A^{01})\;\in\;V_{\mathbf{R}}\,.$$

$e_{A^{01}}$ may be considered as an element of $V_{\mathbf{R}} \otimes W(\Sigma, \mathbf{x}, \mathbf{R}, k)^*$ and using the scalar product dual to (16) on the second factor, we obtain the equality

$$Z_{_A} \left\langle \otimes g_{_{R_i}}(x_i) \right\rangle_{_A} \;=\; \overline{\left(e_{A^{01}},\, e_{-(A^{10})^*}\right)} \tag{17}$$

viewed as a relation between the $V_{\mathbf{R}} \otimes \overline{V_{\mathbf{R}}}$-valued functionals of A.

Let us present a functional integral proof of the relation (16). Denote $Z_{_A} \left\langle \otimes g_{_{R_i}}(x_i) \right\rangle_{_A} \equiv\, ?\,(A)$. Consider the integral over the unitary gauge fields B

$$\begin{aligned}
&\int ?\,(B^{10} + A^{01})\; ?\,(A^{10} + B^{01})\; \mathrm{e}^{-\frac{k}{2\pi}\|B\|^2_{L^2}}\; DB \\
&= \int \mathop{\otimes}_{i=1}^{n} (g_1 g_2)_{_{R_i}}(x_i)\; \mathrm{e}^{-kS(g_1) - kS(g_2)} \;\cdot\; \mathrm{e}^{-\frac{ik}{2\pi}\int \mathrm{tr}\,[A^{10}\wedge g_2^{-1}\bar\partial g_2 + g_1 \partial g_1^{-1}\wedge A^{01}]} \\
&\quad \cdot\, \mathrm{e}^{-\frac{ik}{2\pi}\int \mathrm{tr}\,[B^{10}\wedge g_1^{-1}\bar\partial g_1 + g_1 B^{10} g_1^{-1}\wedge A^{01} + g_2 \partial g_2^{-1}\wedge B^{01} + g_1 A^{10} g_2^{-1}\wedge B^{01} - B^{10}\wedge B^{01}]}\; Dg_1\; Dg_2\; DB \\
&= \int \mathop{\otimes}_{i=1}^{n} (g_1 g_2)_{_{R_i}}(x_i)\; \mathrm{e}^{-kS(g_1) - kS(g_2) - \frac{ik}{2\pi}\int \mathrm{tr}\,[A^{10}\wedge g_2^{-1}\bar\partial g_2 + g_1\partial g_1^{-1}\wedge A^{01}]} \\
&\qquad\qquad \cdot\, \mathrm{e}^{-\frac{ik}{2\pi}\int \mathrm{tr}\,[(g_2\partial g_2^{-1} + g_2 A^{10} g_2^{-1})\wedge(g_1^{-1}\bar\partial g_1 + g_1^{-1} A^{01} g_1)]}\; Dg_1\; Dg_2 \\
&\qquad\qquad = \int \mathop{\otimes}_{i=1}^{n} (g_1 g_2)_{_{R_i}}(x_i)\; \mathrm{e}^{-kS(g_1 g_2, A)}\; Dg_1\; Dg_2 \;=\; ?\,(A)
\end{aligned}$$

where the 2[nd] equality is obtained by a straightforward Gaussian integration over B. Upon the substitution of relations (14) and (16), the last identity becomes

$$H^{\alpha\beta}\, H^{\gamma\delta}\, (\Psi_\beta, \Psi_\gamma)\; \Psi_\alpha(A^{01}) \otimes \overline{\Psi_\delta(-(A^{10})^*)} \;=\; H^{\alpha\delta}\; \Psi_\alpha(A^{01}) \otimes \overline{\Psi_\delta(-(A^{10})^*)}$$

or $H^{\alpha\beta}\, H^{\gamma\delta}\, (\Psi_\beta, \Psi_\gamma) \;=\; H^{\alpha\delta}$ from which the relation (15) follows if we also assume that $(H^{\alpha\beta})$ is an invertible matrix.

The above expressions reduce the calculation of the correlation functions of the WZW model to that of the functional integral (16). The latter appears easier to calculate then the original functional integral (5). In the first step, the integral (16) may be rewritten by a trick resembling the Faddeev-Popov treatment of gauge theory functional integrals. The reparametrization of the gauge fields

$$A^{01} \;=\; {}^{h^{-1}}\!A^{01}(n) \tag{18}$$

by chiral gauge transforms of a (local) slice $n \mapsto A^{01}(n)$ in $\mathcal{A}^{01}$ cutting each $\mathcal{G}^{\mathbf{C}}$-orbit once[24] gives

$$\|\Psi\|^2 \;=\; \int \left(\Psi(A^{01}(n)), \otimes (hh^*)^{-1}_{R_i} \Psi(A^{01}(n))\right)_{V_{\mathbf{R}}}\; \mathrm{e}^{(k+2h^\vee)S(hh^*, A(n))}\; D(hh^*)\; d\mu_{_Q}(n)\,. \tag{19}$$

The term $2h^\vee S(hh^*)$ in the action comes from the Jacobian of the change of variables (18) contributing also to the measure $d\mu_{_Q}(n)$. The latter is defined as follows. Denote by S the composition of the derivative of the map $n \mapsto A^{01}(n)$ with the canonical projection of $\mathcal{A}^{01}$ onto the cokernel of $\bar\partial + [A^{01}(n), \cdot\,]$. Then the volume form $\mu_{_Q}(n)$ on the slice is the composition of the

[24]in genus 0 and 1, $h \in \mathcal{G}^{\mathbf{C}}$ should be additionally restricted

determinant ($\equiv$ the maximal exterior power) of S with the Quillen metric on the determinant bundle of the family $(\bar\partial + [A^{01}(n), \cdot\,])$ of $\bar\partial$-operators[25].

Unlike in the standard Faddeev-Popov setup, the integral over the group of gauge transformations did not drop out since the integrand in (16) is invariant only under the G-valued gauge transformations. Instead we are left with a functional integral (19) similar to the one (5) for the original correlation functions, except that it is over fields hh^* which may be considered as taking values in the hyperbolic space $G^{\mathbf{C}}/G$. $D(hh^*)$ is the formal product of $G^{\mathbf{C}}$-invariant measures on $G^{\mathbf{C}}/G$. The gain is that the functional integral (19) may be reduced to an explicitly doable iterative Gaussian integral. For example for $G = SU(2)$ and at genus 0 where we may take $A^{01}(n) \equiv 0$,

$$S(hh^*) \ = \ -\tfrac{i}{2\pi}\int \partial\phi \wedge \bar\partial\phi \ - \ \tfrac{i}{2\pi}\int(\partial + \partial\phi)\bar v \wedge (\bar\partial + \bar\partial\phi)v$$

in the Iwasawa parametrization $h = \left(\begin{smallmatrix} e^{\phi/2} & 0 \\ 0 & e^{-\phi/2}\end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & v \\ 0 & 1\end{smallmatrix}\right)u$ of the 3-dimensional hyperboloid $SL_2(\mathbf{C})/SU_2$ by $\phi \in \mathbf{R}$ and $v \in \mathbf{C}$ ($u \in SU_2$). Field v enters quadratically into the action and polynomially into insertions. Hence the v-integral is Gaussian and its explicit calculation requires the knowledge of the determinant of the operator $(-\partial + \partial\phi)(\bar\partial + \bar\partial\phi) = -e^{\phi}\partial e^{-2\phi}\bar\partial e^{\phi}$ and of the propagator

$$((-\partial + \partial\phi)(\bar\partial + \bar\partial\phi))^{-1}(z_1, z_2) \quad \sim \quad e^{-\phi(z_1)-\phi(z_2)}\int \frac{e^{2\phi(y)}\, d^2y}{(\bar z_1 - \bar y)(y - z_2)}\,. \tag{20}$$

The ϕ-dependence of $\ln\det((-\partial + \partial\phi)(\bar\partial + \bar\partial\phi))$ is given by the chiral anomaly (or local index theorem) and is the sum of a local quadratic and a linear term. The resulting ϕ-field integral appears to be also Gaussian (of the type encountered in functional-integral representations of a 2-dimensional Coulomb gas correlation functions in statistical mechanics). Similar iterative procedure based on the Iwasawa parametrization of $G^{\mathbf{C}}/G$ works for arbitrary G and also at higher genera. A result becomes a finite-dimensional integral over parameters $y_a \in \Sigma$ in the expressions of the type (20) for the v-field propagators (positions of the "screening charges" in the Coulomb gas interpretation) and, at genus $h_\Sigma > 0$, over (a part of) the moduli parameters n.

At genus 0, the $\mathcal{G}^{\mathbf{C}}$-orbit of $A^{01} = 0$ is dense in $\mathcal{A}^{01}$. As a result $\Psi \in W(\mathbf{C}P^1, \mathbf{x}, \mathbf{R}, k)$ is fully determined by $\Psi(0) \in (V_{\mathbf{R}})^G$, the G-invariant subspace of $V_{\mathbf{R}}$. Hence

$$W(\mathbf{C}P^1, \mathbf{x}, \mathbf{R}, k) \ \subset \ (V_{\mathbf{R}})^G$$

canonically. For $G = SU_2$ the representations R_i are labeled by integer or half-integer spins j_i and the representation spaces V_{j_i} are spanned by vectors $(f^l v_{j_i})_{l=0,1,\dots,2j_i}$ where v_{j_i} is the highest weight (HW) vector annihilated by e, with (e, f, h) the usual basis of sl_2. One has, using the standard complex variable z on $\mathbf{C}P^1$ to label the insertion points,

$$W(\mathbf{C}P^1, \mathbf{z}, \mathbf{j}, k) \ = \ \{\, v \in (V_{\mathbf{j}})^{SU_2} \mid (\otimes v_{j_i},\, \textstyle\prod_i e_i^{n_i}\, e^{z_i e_i}\, v\,) = 0 \ \text{if} \ N \le J - k - 1 \,\}$$

[25] again, the cases of genus 0 or 1 require minor modifications

where $e_i = 1\otimes 1\cdots\otimes 1\otimes \underset{i}{e}\otimes 1\otimes\cdots\otimes 1\,,\ \ N\equiv\sum_i n_i$ and $J\equiv\sum_i j_i$. In particular, for 2 or 3 points,

$$W(\mathbf{C}P^1,\mathbf{z},\mathbf{j},k) \;=\; \begin{cases} (V_{\mathbf{j}})^{SU_2} & \text{if } \ J\leq k\,,\\ \quad\{0\} & \text{if } \ J>k \end{cases}$$

and does not depend on $\mathbf{z}$. The scalar product (16) is given by

$$\|v\|^2 \;=\; f(\sigma,\mathbf{z},\mathbf{j},k)\int\limits_{\mathbf{C}^J} |\,(\,v\,,\omega(\mathbf{z},\mathbf{y})\,)\,\mathrm{e}^{-\frac{1}{k+2}U(\mathbf{z},\mathbf{y})}|^2\,\prod_{a=1}^{J} d^2y_a \tag{21}$$

where

$$f(\sigma,\mathbf{z},\mathbf{j},k) \;=\; \mathrm{e}^{\sum_i\frac{j_i(j_i+1)}{k+2}\sigma(z_i)+\frac{1}{16\pi(k+2)}\|d\sigma\|^2_{L^2}}\left(\frac{\det'(-\Delta)}{\mathrm{area}_{\mathbf{C}P^1}}\right)^{3/2}$$

carries the dependence on the metric $\mathrm{e}^\sigma|dz|^2$ on $\mathbf{C}P^1$, $\mathbf{y}=(y_1,\dots,y_J)$, $\omega(\mathbf{z},\mathbf{y})$ is a meromorphic $V_{\mathbf{j}}$-valued function

$$\omega(\mathbf{z},\mathbf{y}) \;=\; \prod_{a=1}^{J}\sum_{i=1}^{n}\frac{1}{y_a-z_i}\,f_i\,\bigotimes_{i=1}^{n}v_{j_i}$$

and $U(\mathbf{z},\mathbf{y})$ is a multivalued function

$$\tfrac{1}{2}U(\mathbf{z},\mathbf{y}) \;=\; \sum_{i<i'}j_ij_{i'}\,\ln(z_i-z_{i'}) - \sum_{i,a}j_i\,\ln(z_i-y_a) + \sum_{a<a'}\ln(y_a-y_{a'})\,.$$

Integral (21) is over a positive density with singularities at coinciding y_a and the question arises as to whether it does converge. A natural conjecture is that the integral is convergent if and only if $v\in W(\mathbf{C}P^1,\mathbf{x},\mathbf{R},k)\subset(V_{\mathbf{R}})^G$ (the only if part is easy). For 2- or 3-point functions the integrals can indeed be computed explicitly confirming the conjecture. Numerous other special cases have been checked. However, the general case of the conjecture remains to be verified. Note that the dependence of the scalar product (21) on the conformal factor σ agrees with the value $\frac{3k}{k+2}$ of the central charge of the SU_2 WZW theory and with the values $\Delta_j=\bar\Delta_j=\frac{j(j+1)}{k+2}$ of the conformal dimensions of field $g_j(x)$ (it is the inverse of $f(\sigma,\mathbf{x},\mathbf{j},k)$ which enters the WZW correlation functions).

Explicit finite-dimensional integral formulae for the scalar product (16) have been also obtained for general groups and at genus 1 and, for $G=SU_2$, for higher genera[26]. The proof of the convergence of the corresponding integrals is the only missing element in the explicit construction of all correlation functions of the WZW theory although several special cases have been settled completely.

[26] it is clear that the case of general group and genus >1 could be treated along the same lines

4. KZB connection

The spaces $W(\Sigma,\mathbf{x},\mathbf{R},k)$ of non-abelian theta functions depend on the complex structures of the surface Σ and on the insertion points. The complex structures $J\in\,?(\mathrm{End}T\Sigma)$, $J^2=-1$, form a complex (infinite dimensional Fréchet) manifold on which the group of Diffeomorphisms of Σ acts naturally. The holomorphic tangent vectors to the quotient moduli space $\delta J=\delta\mu$ correspond to sections of $\mathrm{End}T^{\mathbf{C}}\Sigma$ satisfying $J\delta\mu=-\delta\mu J=i\delta\mu$. Locally, $\delta\mu$ may be represented as $\delta\mu^z_{\bar z}\partial_z\otimes d\bar z$ in J-complex coordinates. The family of spaces $W(\Sigma,\mathbf{x},\mathbf{R},k)$ forms a complex finite-dimensional bundle $\mathcal{W}(\mathbf{R},k)$ over the space of complex structures and n-tuples of noncoincident points $\mathbf{x}$ in Σ.

The bundle $\mathcal{W}(\mathbf{R},k)$ may be supplied with a natural (w.r.t. the action of diffeomorphisms of Σ) connection ∇ provided that we choose (smoothly) for each J a compatible metric on Σ. The connections for different choices of the metric are related by the conformal anomaly. If $(J,\mathbf{x},A)\mapsto\Psi(J,\mathbf{x},A)$ depending holomorphically on $A^{01}=A(1+iJ)/2$ ($A=-A^*$ is assumed) represents a local section of $\mathcal{W}(\mathbf{R},k)$, then

$$\nabla_{\overline{\delta\mu}}\Psi \;=\; d_{\overline{\delta\mu}}\Psi+\tfrac{k}{8\pi}\left(\int \mathrm{tr}\,A^{01}\wedge A^{01}\overline{\delta\mu}\right)\Psi\,, \tag{22}$$

$$\nabla_{\bar z_i}\Psi \;=\; \bar\partial_{\bar z_i}\Psi+(A_{\bar z_i})_i\,\Psi\,, \tag{23}$$

$$\nabla_{\delta\mu}\Psi \;=\; d_{\delta\mu}\Psi-\tfrac{1}{2\pi i}\left(\int T(z)\,\delta\mu^z_{\bar z}\,d^2z\right)\Psi\,, \tag{24}$$

$$\nabla_{z_i}\Psi \;=\; \partial_{z_i}+\lim_{z\to z_i}\tfrac{2t^a_i}{k+h^\vee}\left(J^a(z)+\tfrac{t^a_i}{z-z_i}\right)\Psi\,. \tag{25}$$

Above z denotes a J-complex coordinate on Σ and $d_{\delta\mu}\Psi$ or $d_{\overline{\delta\mu}}\Psi$ stands for the directional derivatives of Ψ when the points $\mathbf{x}$ and A are kept constant. The first two equations equip $\mathcal{W}(\mathbf{R},k)$ with a structure of a holomorphic vector bundle. In the last 2 equations, the metric on Σ is assumed for simplicity to satisfy $\gamma^{z\bar z}=2$, $\delta\gamma^{zz}=\frac{2}{i}\delta\mu^z_{\bar z}$ and A is taken vanishing around the support of $\delta\mu$ or around the insertion point x_i and $\delta\mu=\mathcal{O}((z-z_i)^2)$.

In the genus 0 case, $\mathcal{W}(\mathbf{R},k)$ is a subbundle of the trivial bundle with the fiber $(V_{\mathbf{R}})^G$ and the connection ∇ extends to the bigger bundle and is given by

$$\nabla_{\bar z_i}\;=\;\partial_{\bar z_i}\,,\qquad \nabla_{z_i}\;=\;\partial_{z_i}+\tfrac{2}{k+h^\vee}\sum_{i'\neq i}\tfrac{t^a_i t^a_{i'}}{z_{i'}-z_i}\;\equiv\;\partial_{z_i}+\tfrac{1}{k+h^\vee}H_i(\mathbf{z})$$

for the metric flat around the insertions. The commuting operators $H_i(\mathbf{z})\in\mathrm{End}(V_{\mathbf{R}})$ are known as the Gaudin Hamiltonians. The corresponding flat connection appeared (implicitly) in the work of Knizhnik-Zamolodchikov on the WZW theory. The higher genus generalizations of the KZ connection were first studied by Bernard. We shall call the connection defined by eqs. (22-25) the KZB connection. In general, it is only projectively flat.

One of the basic open questions concerning the KZB connection is whether it is unitarizable. In other words, whether there exists a hermitian structure on the bundle $\mathcal{W}(\mathbf{R},k)$ preserved by ∇. It was conjectured that the answer to this question is positive and that it is exactly the scalar product on spaces $W(\Sigma,\mathbf{x},\mathbf{R},k)$ discussed above that provides the required hermitian structure. Note that a 0,1 unitary connection on a holomorphic hermitian vector bundle is uniquely determined. Recall that the scalar product, given formally by the gauge field functional integral (16), may be reduced to a finite-dimensional integral which, if convergent, defines a positive hermitian form on $W(\Sigma, \mathbf{x}, \mathbf{R}, k)$ and determines the unitary connection (and the energy momenstum tensor of the WZW theory). For genus 0 and $G = SU_2$, where the scalar product is given by integral (21), the unitarity of the KZ connection requires that

$$\partial_{z_i}(v\,,\,v) \;=\; (v\,,\,(\partial_{z_i} - \tfrac{1}{k+2}H_i)v) \tag{26}$$

for a holomorphic family $\mathbf{z} \mapsto v(\mathbf{z}) \in W(\mathbf{C}P^1, \mathbf{z}, \mathbf{j}, k) \subset V_{\mathbf{j}}^{SU_2}$. Assuming the convergence of the integrals permitting to differentiate under the integral and to integrate by parts, the above is a consequence of the relation

$$\Big(\partial_{z_i} + \tfrac{1}{k+2}H_i(\mathbf{z})\Big)\,\Big(\omega(\mathbf{z},\mathbf{y})\,\mathrm{e}^{-\frac{1}{k+2}U(\mathbf{z},\mathbf{y})}\Big) \;=\; \partial_{y_a}\,\Big(\eta_{i,a}\,\mathrm{e}^{-\frac{1}{k+2}U(\mathbf{z},\mathbf{y})}\Big) \tag{27}$$

where

$$\eta_{i,a}(\mathbf{z},\mathbf{y}) \;=\; \tfrac{1}{z_i-y_a}\,f_i\,\prod_{a'\neq a}\sum_{i'=1}^{n}\tfrac{1}{y_{a'}-z_{i'}}\,f_{i'}\,\mathop{\otimes}_{i=1}^{n}v_{j_i}\,.$$

Identity (27) is equivalent to two relations:

$$\begin{aligned}&\partial_{z_i}\omega(\mathbf{z},\mathbf{y}) \,=\, \partial_{y_a}\eta_{i,a}(\mathbf{z},\mathbf{y})\,,\\ &\partial_{z_i}U(\mathbf{z},\mathbf{y})\,\omega(\mathbf{z},\mathbf{y}) \,-\, \partial_{y_a}U(\mathbf{z},\mathbf{y})\,\eta_{i,a}(\mathbf{z},\mathbf{y}) \,-\, H_i(\mathbf{z})\,\omega(\mathbf{z},\mathbf{y}) \,=\, 0\,.\end{aligned} \tag{28}$$

The first one is immediate whereas the second, more involved one implies that

$$H_i(\mathbf{z})\,\omega(\mathbf{z},\mathbf{y}) \;=\; \partial_{z_i}U(\mathbf{x},\mathbf{y})\,\omega(\mathbf{z},\mathbf{y}) \qquad \text{if} \qquad \partial_{y_a}U(\mathbf{x},\mathbf{y}) \,=\, 0$$

i.e. the **Bethe Ansatz** diagonalization of the Gaudin Hamiltonians $H_i(\mathbf{z})$: vectors $\omega(\mathbf{z},\mathbf{y})$ are common eigenvectors of $H_i(\mathbf{z})$, $i = 1,\dots,n$ with eigenvalues $\partial_{z_i}U(\mathbf{z},\mathbf{y})$ provided that $\mathbf{y}$ satisfies the Bethe Ansatz equations $\partial_{y_a}U(\mathbf{x},\mathbf{y}) = 0$. The relations between the Bethe Ansatz and the limit of the KZB connection when $k \to -h^\vee$ appear in the context of Langlands geometric correspondence. These relations seem also to be at the heart of the question about the unitarity of the KZB connection at positive integer k.

5. Coset theories

There is a rich family of CFT's which may be obtained from the WZW models by a simple procedure known under the name of a **coset construction**. On the functional integral level, the procedure consists of coupling the G-group WZW theory to a subgroup $H \subset G$ unitary gauge field B which is also integrated over with gauge-invariant insertions. Let us assume, for simplicity, that H is connected and simply connected, as G. Let $t_i \in (\mathrm{Hom}(V_{R_i}, V_{r_i}))^H$ be intertwiners of the action of H in the irreducible G- and H-representation spaces, respectively. The simplest correlation functions of the G/H coset theory take the form

$$\langle\,\prod_{i=1}^{n}\mathrm{tr}\,t_i g_{R_i}(x_i)t_i^*\,\rangle \;=\; \int\,\prod_{i=1}^{n}\mathrm{tr}_{V_{r_i}}t_i g_{R_i}(x_i)\,t_i^*\;\mathrm{e}^{-kS(g,B)}\,Dg\,DB\Big/\!\!\Big/\int\mathrm{e}^{-kS(g,B)}\,Dg\,DB\,. \tag{29}$$

Note that the g-field integrals are the ones of the WZW theory and are given by eq. (14). Denoting $Z_{G/H} = \int \mathrm{e}^{-kS(g,B)}\, Dg\, DB$, we obtain

$$Z_{G/H} \, \langle \prod_{i=1}^{n} \mathrm{tr}\, t_i g_{R_i}(x_i) t_i^* \rangle \;=\; H^{\alpha\beta} \int \big(\otimes t_i \Psi_\beta(B^{01}), \otimes t_i \Psi_\alpha(B^{01}) \big)_{V_{\mathrm{r}}} \; \mathrm{e}^{-\frac{k}{2\pi}\|B\|^2_{L^2}} \, DB \,. \tag{30}$$

For $\Psi \in W(\Sigma, \mathbf{x}, \mathbf{R}, k)$, the map $B^{01} \mapsto \otimes t_i \Psi_\alpha(B^{01}) \in V_{\mathrm{r}}$ is a group H non-abelian theta function belonging to $W(\Sigma, \mathbf{x}, \mathbf{r}, \tilde{k})$ (the normalization of the Killing forms of G and H may differ, hence the replacement $k \to \tilde{k}$). Denote by T the corresponding map from $W(\Sigma, \mathbf{x}, \mathbf{R}, k)$ to $W(\Sigma, \mathbf{x}, \mathbf{r}, \tilde{k})$. Eq. (30) may be rewritten as

$$Z_{G/H} \, \langle \prod_{i=1}^{n} \mathrm{tr}\, t_i g_{R_i}(x_i) t_i^* \rangle \;=\; H^{\alpha\beta} \, (T\Psi_\beta, T\Psi_\alpha) \;=\; \mathrm{Tr}\, T^* T \,, \tag{31}$$

or choosing a basis (ψ_λ) of $W(\Sigma, \mathbf{x}, \mathbf{r}, \tilde{k})$,

$$Z_{G/H} \, \langle \prod_{i=1}^{n} \mathrm{tr}\, t_i g_{R_i}(x_i) t_i^* \rangle \;=\; H^{\alpha\beta} \, \overline{T^\lambda_\beta} \, h_{\lambda\nu} \, T^\nu_\alpha \tag{32}$$

where (T^λ_α) is the ("branching") matrix of the linear map T in bases (Ψ_α), (ψ_λ) and $h_{\lambda,\nu} = (\psi_\lambda, \psi_\nu)$. Since the above formula holds also for the partition function itself, it follows that the calculation of the coset theory correlation functions (29) reduces to that of the scalar products of group G and group H non-abelian theta functions, both given by explicit, finite-dimensional integrals.

Among the simplest examples of the coset theories is the case with $G = SU_2 \times SU_2$ with level $(k, 1)$ (for product groups, the levels may be taken independently for each group) and with H being the diagonal SU_2 subgroup. The resulting theories coincide with the unitary "minimal" series of CFT's with (Virasoro) central charges $c = 1 - \frac{6}{(k+2)(k+3)}$ first considered by Belavin-Polyakov-Zamolodchikov. The Hilbert spaces of these theories are built from the unitary heighest weight representations of the Virasoro algebras with $0 < c < 1$ discussed in Lecture 2. The simpliest one of them with $k = 1$ and $c = \frac{1}{2}$ is believed to describe the continuum limit of the Ising model at critical temperature or the scaling limit of the massless ϕ^4_2 theory. In particular, in the continuum limit the spins in the critical Ising model are represented by fields $\mathrm{tr}\, g_{1/2}(x)$ where g takes values in the first SU_2. The corresponding correlation functions may be computed as above. One obtains this way for the 4-point function an explicit expression in terms of hypergeometric functions.

Similar coset theories but at level $(k, 2)$ give rise to the supersymmetric $N = 1$ minimal unitary series of CFT's, the simplest one with $k = 1$ (appearing also at $k = 2$ in the previous series) corresponds to the so called 3-critical Ising model.

The G/H coset theory with $H = G$ is a prototype of a two-dimensional topological field theory. As follows from eq. (31), the correlation functions of fields $\mathrm{tr}\, g_R(x)$ are equal to the dimension of spaces $W(\Sigma, \mathbf{x}, \mathbf{R}, k)$, normalized by the dimension of $W(\Sigma, \emptyset, \emptyset, k)$ (and are given by the Verlinde formula). In particular, they do not depend on the position of the insertion points.

6. WZW factory

As we have seen above, the coset construction allows to obtain new soluble CFT's from the WZW models. Let us briefly discuss further refinements which permit a chain production of conformal models whose partition functions and correlation functions may be computed exactly, at least in principle. The most interesting cases of such models correspond to situations when two different constructions give rise to the same CFT, as in T-duality, mirror symmetry and other numerous instances.

1. If the group G is **not simply connected**, the original definition (4) of the action of the WZW model requires a modification. The result is possible further restrictions on the levels and the appearance, in some cases, of different quantizations of the same classical theory ("θ-vacua" or "discrete torsion"). The models are still exactly soluble although only the partition functions and the correlations of "untwisted" fields have been worked out in detail for general G.

2. Let $H \subset G$ and $Z \subset H$ be a subgroup of the center $Z_{_G}$ of G. Let $P_{_{H'}}$ be a principal H'-bundle where $H' = H/Z$ and $Q_{_G} = P_{_{H'}} \times_{\mathrm{Ad}_{H'}} G$ be the G-bundle associated to $P_{_{H'}}$ via the adjoint action of H' on G. For appropriate k, and for a section g of $Q_{_G}$ and a connection B on $P_{_{H'}}$ one may define the amplitude $\mathrm{e}^{-kS(g,B)}$. The (unnormalized) correlation functions of the coset G/H'-model may then be obtained by integrating gauge invariant insertions, weighted with $\mathrm{e}^{-kS(g,B)}$, over g and B and summing the result over inequivalent H'-bundles $P_{_{H'}}$. Hence, for given $H \subset G$, there are as many coset theories as subgroups of $H \cap Z_{_G}$ (some of them might have a non-unique vacuum).

3. If H is a discrete subgroup of G, then $P_{_H}$ carries a unique canonical flat connection and is given by a conjugation class of homomorphisms of the fundamental group of is Σ into H. The construction from the preceding point gives rise to the **orbifolds** of the WZW models.

4. **Supersymmetric WZW models**. One adds to the G-valued field g the (anticommuting) Majorana Fermi fields $\psi, \tilde{\psi}$ in the adjoint representation (i.e. sections of $L \otimes \mathbf{g}$ and $\bar{L} \otimes \mathbf{g}$, respectively, where L is a square root of the canonical bundle of Σ) and one considers the action

$$S(g, \psi, \tilde{\psi}, A) \;=\; kS(g, A) - \tfrac{2}{\pi} \int \mathrm{tr}\, \Big(\psi(\bar{\partial}_L + [A^{01}, \cdot\,])\psi \,+\, \tilde{\psi}(\partial_{\bar{L}} + [A^{10}, \cdot\,])\tilde{\psi}\Big) \tag{33}$$

with the external, group G gauge field A. The fermionic part of the theory is free and the complete theory may be easily solved.

5. **Supersymmetric coset models**. The action is as in eq. (33) except that A is replaced by a group H gauge field B and the Majorana fields $\psi, \tilde{\psi}$ are taken with values in $\mathbf{g}/\mathbf{h}$ rather than in $\mathbf{g}$. Both the supersymmetric WZW models and the supersymmetric coset models possess the $N = 1$ superconformal symmetry.

6. $N = 2$ **coset models**. If G/H is a Kähler symmetric space then the supersymmetric G/H coset model possesses the $N = 2$ superconformal symmetry including the $U(1)$ loop group symmetry. The simplest examples are provided by the $SU(2)/U(1)$ models which, at level k, give rise to the minimal $N = 2$ superconformal theory with central charge $c = \frac{3k}{k+2}$.

7. **Orbifods of tensor products** of conformal field theories may give rise to essentially new models. The famous example are the $(\mathbf{Z}/5\mathbf{Z})^3$ orbifolds of product of five $k = 3$ minimal $N = 2$

superconformal models. Two different orbifolds may give equivalent conformal sigma models corresponding to a mirror pair of Calabi-Yau quintic targets.

8. The theories with $U(1)$ loop group symmetries like the $N = 2$ supersymmetric coset models may be **twisted** by considering their fields as taking values in bundles associated with the sphere subbundle of the spin bundle L and coupled to the spin connection. Such twisting of $N = 2$ superconformal models may be done in two essentially different ways (A- and B-twist) and it produces topological field theories. The genus 0 correlations of the A-twisted $N = 2$ sigma models compute the quantum cohomology of the target.

9. For **non-compact groups** G, the WZW action $S(g)$ is not bounded below but one may try to stabilize the Euclidean functional integral by analytic continuation or/and coset-type gauging of subgroups of G. Such stabilization procedures may however destroy the physical positivity (Hilbert-space picture) of the theory. The best studied models of the non-compact type correspond to finite coverings of $SL_2(\mathbf{R})$ with the $U(1)$ subgroup twisted and the nilpotent subgroup gauged away (the construction of minimal models *à la* Drinfeld-Sokolov), the $SL_N(\mathbf{R})$ generalizations thereof, the Liouville and Toda theories and the $SL_2(\mathbf{R})/U(1)$ black hole model. Our knowledge of non-compact WZW models is certainly much less complete than that of the compact case (note that this is true also on the level of quantum mechanics where we know everything about harmonic analysis of compact Lie groups but the harmonic analysis of non-compact ones has still open problems).

References

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The present lecture follows a geometric approach developed in author's paper in Nucl. Phys. B 328 (1989), p. 733 where the relations (14) between the scalar product of non-abelian theta functions and WZW correlations as well as the conjecture about the unitarity of the KZ connection were first formulated. See also "Functional Integration, Geometry and Strings", Haba, Sobczyk (eds.), Birkhäuser 1989, p. 277. The scalar product formulae for arbitrary Lie group were obtained at genus 0 in Falceto-Gawędzki-Kupiainen: Phys. Lett. B 260 (1991), p. 101 and at genus 1 in Falceto-Gawędzki: hep-th/9604094. The case $G = SU_2$, genus >1 was studied in Gawędzki: Commun. Math. Phys. 169 (1995), p. 329 and Lett. Math. Phys. 33 (1995), p. 335. The functional integral proof of eq. (14) was borrowed from Witten: Commun. Math. Phys. 144 (1992), p. 189.

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