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Axiomatic Conformal Field Theory

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Abstract: A new rigourous approach to conformal field theory is presented. The basic objects are families of complex-valued amplitudes, which define a meromorphic conformal field theory (or chiral algebra) and which lead naturally to the definition of topological vector spaces, between which vertex operators act as continuous operators. In fact, in order to develop the theory, Möbius invariance rather than full conformal invariance is required but it is shown that every Möbius theory can be extended to a conformal theory by the construction of a Virasoro field.

In this approach, a representation of a conformal field theory is naturally defined in terms of a family of amplitudes with appropriate analytic properties. It is shown that these amplitudes can also be derived from a suitable collection of states in the meromorphic theory. Zhu's algebra then appears naturally as the algebra of conditions which states defining highest weight representations must satisfy. The relationship of the representations of Zhu's algebra to the classification of highest weight representations is explained.

1. Introduction

Conformal field theory has been a subject which has attracted a great deal of attention in the last thirty years, much of the interest being motivated by its importance in string theory. Its role in string theory goes back to its very beginning when Veneziano [1] proposed a form for the scattering amplitude for four particles, quickly generalised to n-particle amplitudes, which could conveniently be expressed as integrals in the complex plane of meromorphic functions [2]. From these amplitudes, the space of states was obtained by factorisation.

The power of two-dimensional conformal field theory was conclusively demonstrated by the work of Belavin, Polyakov and Zamolodchikov [3]. They set a general framework for its study which was further developed by Moore and Seiberg [4], in particular. This approach is couched within the general language of quantum field theory. Not least because it is possible to establish very strong results in conformal field theory, it is very desirable to have a precise mathematical context within which they can be established. Rigorous approaches to conformal field theory have been developed, broadly speaking, from three different standpoints: a geometrical approach initiated by Segal [5]; an algebraic approach due to Borcherds [6, 7], Frenkel, Lepowsky and Meurman [8] and developed further by Frenkel, Huang and Lepowsky [9] and Kac [10]; and a functional analytic approach in which techniques from algebraic quantum field theory are employed and which has been pioneered by Wassermann [11] and Gabbiani and Fröhlich [12].

Each of these three approaches produces a different perspective on conformal field and each facilitates the appreciation of its deep connections with other parts of mathematics, different in the three cases. Here we present a rigorous approach closely related to the way conformal field theory arose at the birth of string theory. It is a development of earlier studies of meromorphic conformal field theory [13]. Starting from a family of "amplitudes", which are functions of n complex variables and describe the vacuum expectation values of n fields associated with certain basic states, the full space of states of the theory is obtained by factorising these amplitudes in a certain sense.

The process of reconstructing the space of states from the vacuum expectation values of fields is familiar from axiomatic quantum field theory. In the usual Osterwalder-Schrader framework of Euclidean quantum field theory [14], the reflection positivity axiom guarantees that the resulting space of states has the structure of a Hilbert space. (In the context of conformal field theory this approach has been developed by Felder, Fröhlich and Keller [15].) In the present approach, the construction of the space of states depends only on the meromorphicity of the given family of amplitudes, $\mathcal{A}$, and positivity is not required for the basic development of the theory.

The spaces of states that are naturally defined are not Hilbert spaces but topological vector spaces, their topology being determined by requirements designed to ensure meromorphic amplitudes. (Recently Huang has also introduced topological vector spaces, which are related to ours, but from a different point of view [16].) They are also such that one can introduce fields ("vertex operators") for the basic states which are continuous operators. The locality property of these vertex operators is a direct consequence of the locality assumption about the family of amplitudes, $\mathcal{A}$, and this is then sufficient to prove the duality property (or Jacobi identity) of the vertex operators [13].

To develop the theory further, we need to assume that the basic amplitudes, $\mathcal{A}$, are Möbius invariant. This enables us to define vertex operators for more general states, modes of vertex operators and a Fock space which contains the essential algebraic content of the theory. This Fock space also enables us to define the concept of the equivalence of conformal field theories. The assumptions made so far are very general but if we assume that the amplitudes satisfy a cluster decomposition property we place much more severe restrictions on the theory, enabling us, in particular, to prove the uniqueness of the vacuum state.

Nothing assumed so far implies that the theory has a conformal structure, only one of Möbius symmetry. However, we show that it is always possible to extend the theory in such a way that it acquires a conformal structure. (For theories with a conformal structure this leaves the theory unchanged.) A conformal structure is necessary if we want to be able to define the theory on higher genus Riemann surfaces (although this is not discussed in the present paper). For this purpose, we also need to introduce the concept of a representation of a conformal (or rather a Möbius) field theory. Developing an idea of Montague, we show that any representation corresponds to a state in the space of states of the theory [17]. This naturally poses the question of what conditions a state has to satisfy in order to define a representation. For the case of highest weight representations, the conditions define an associative algebra which is that originally introduced by Zhu [18]. It is the main content of Zhu's Theorem that this algebra can be defined in terms of the algebraic Fock space.

The plan of the paper is as follows. In Sect. 2, we introduce the basic assumptions about the family of amplitudes, $\mathcal{A}$, and construct the topological vector space of states and the vertex operators for the basic states. In Sect. 3, Möbius invariance and its consequences are discussed. In Sect. 4, we define modes of vertex operators and use them to construct Fock spaces and thus to define the equivalence of theories. Examples of conformal field theories are provided in Sect. 5: the $U(1)$ theory, affine Lie algebra theory, the Virasoro theory, lattice theories, and an example which does not have a conformal structure. In Sect. 6, the assumption of cluster decomposition is introduced and in Sect. 7 we show how to extend a Möbius invariant theory to make it conformally invariant. In Sect. 8, we define what is meant by a representation and show how any representation can be characterised by a state in the theory. In Sect. 9, we define the idea of a Möbius covariant representation and the notion of equivalence for representations. An example of a representation is given in Sect. 10. In Sect. 11, we introduce Zhu's algebra and explain the significance of Zhu's Theorem in our context. Further developments, which are to be the subject of a future paper [19], are surveyed in Sect. 12. There are seven appendices in which some of the more technical details are described.

2. Amplitudes, Spaces and Vertex Operators

The starting point for our approach is a collection of functions, which are eventually to be regarded as the vacuum expectation values of the fields associated with a certain basic set of states which generate the whole theory. We shall denote the space spanned by such states by V. In terms of the usual concepts of conformal field theory, V would be a subspace of the space of quasi-primary states. V can typically be taken to be finite-dimensional but this is not essential in what follows. (If it is infinite-dimensional, we shall at least assume that the algebraic dimension of V is countable, that is that the elements of V consist of finite linear combinations of a countable basis.)

We suppose that V can be regarded as the direct sum of a collection of subspaces, V_h, to each of which we can attach an integer, h, called the conformal weight of the states in that subspace, so that $V = \bigoplus_h V_h$. This is equivalent to saying that we have a diagonalisable operator $\delta : V \to V$, with eigenspaces $V_h = \{\psi \in V : \delta\psi = h\psi\}$.

We also suppose that for any positive integer n, and any finite collection of vectors $\psi_i \in V_{h_i}$, and $z_i \in \mathbb{P}$ (the Riemann Sphere), where $i = 1, \ldots, n$, we have a density

$$f(\psi_1, \ldots, \psi_n; z_1, \ldots, z_n) \equiv \langle V(\psi_1, z_1)V(\psi_2, z_2)\cdots V(\psi_n, z_n)\rangle \prod_{j=1}^{n} \left(dz_j\right)^{h_j} . \quad (1)$$

Here $\langle V(\psi_1, z_1)V(\psi_2, z_2)\cdots V(\psi_n, z_n)\rangle$ is merely a suggestive notation for what will in the end acquire an interpretation as the vacuum expectation value of a product of fields. These "amplitudes" are assumed to be multilinear in ψ_i, invariant under the exchange of (ψ_i, z_i) with (ψ_j, z_j), and analytic in z_i, save only for possible singularities occurring at $z_i = z_j$ for $i \neq j$, which we shall assume to be poles of finite order (although one could consider generalisations in which the amplitudes are allowed to have essential singularities). Because of the independence of order of the (ψ_j, z_j), we can use the notation

$$f(\psi_1, \dots, \psi_n; z_1, \dots, z_n) = \left\langle \prod_{j=1}^{n} V(\psi_j, z_j) \right\rangle \prod_{j=1}^{n} \left(dz_j\right)^{h_j} . \tag{2}$$

We denote the collection of these densities, and the theory we develop from them, by $\mathcal{A} = \{f\}$. We may assume that if all amplitudes in $\mathcal{A}$ involving a given $\psi \in V$ vanish then $\psi = 0$ (for, if this is not so, we may replace V by its quotient by the space of all vectors $\psi \in V$ which are such that all amplitudes involving ψ vanish).

We use these amplitudes to define spaces of states associated with certain subsets $\mathcal{C}$ of the Riemann Sphere $\mathbb{P}$. We can picture these spaces as consisting of states generated by fields acting at points of $\mathcal{C}$. First introduce the set, $\mathcal{B}_{\mathcal{C}}$, whose elements are labelled by finite collections of $\psi_i \in V_{h_i}$, $z_i \in \mathcal{C} \subset \mathbb{P}$, $i = 1, \dots, n$, $n \in \mathbb{N}$ and $z_i \neq z_j$ if $i \neq j$; we denote a typical element $\boldsymbol{\psi} \in \mathcal{B}_{\mathcal{C}}$ by

$$\boldsymbol{\psi} = V(\psi_1, z_1) V(\psi_2, z_2) \cdots V(\psi_n, z_n)\Omega \equiv \prod_{i=1}^{n} V(\psi_i, z_i)\Omega. \tag{3}$$

We shall immediately identify $\boldsymbol{\psi} \in \mathcal{B}_{\mathcal{C}}$ with the other elements of $\mathcal{B}_{\mathcal{C}}$ obtained by replacing each ψ_j in (3) by $\mu_j \psi_j$, $1 \leq j \leq n$, where $\mu_j \in \mathbb{C}$ and $\prod_{j=1}^{n} \mu_j = 1$.

Next we introduce the free (complex) vector space on $\mathcal{B}_{\mathcal{C}}$, i.e. the complex vector space with *basis* $\mathcal{B}_{\mathcal{C}}$ that is consisting of formal finite linear combinations $\Psi = \sum_j \lambda_j \boldsymbol{\psi}_j$, $\lambda_j \in \mathbb{C}$, $\boldsymbol{\psi}_j \in \mathcal{B}_{\mathcal{C}}$; we denote this space by $\mathcal{V}_{\mathcal{C}}$.

The vector space $\mathcal{V}_{\mathcal{C}}$ is enormous, and, intuitively, as we consider more and more complex combinations of the basis vectors, $\mathcal{B}_{\mathcal{C}}$, we generate vectors which are very close to one another. To measure this closeness, we need in essence to use suitably chosen amplitudes as test functions. To select a collection of linear functionals which we may use to construct from $\mathcal{V}_{\mathcal{C}}$ a space in which we have some suitable idea of topology, we select another subset $\mathcal{O} \subset \mathbb{P}$ with $\mathcal{O} \cap \mathcal{C} = \varnothing$, where $\mathcal{O}$ is open, and we suppose further that the interior of $\mathcal{C}$, $\mathcal{C}^o$, is not empty. Let

$$\boldsymbol{\phi} = V(\phi_1, \zeta_1) V(\phi_2, \zeta_2) \cdots V(\phi_m, \zeta_m)\Omega \in \mathcal{B}_{\mathcal{O}}, \tag{4}$$

where $\phi_j \in V_{k_j}$, $j = 1, \dots, m$. Each $\boldsymbol{\phi} \in \mathcal{B}_{\mathcal{O}}$ defines a map on $\boldsymbol{\psi} \in \mathcal{B}_{\mathcal{C}}$ by

$$\eta_{\boldsymbol{\phi}}(\boldsymbol{\psi}) = (\boldsymbol{\phi}, \boldsymbol{\psi}) = \left\langle \prod_{i=1}^{m} V(\phi_i, \zeta_i) \prod_{j=1}^{n} V(\psi_j, z_j) \right\rangle, \tag{5}$$

which we can use as a contribution to our measure of nearness of vectors in $\mathcal{V}_{\mathcal{C}}$. [Strictly speaking, this map defines a density rather than a function, so that we should really be considering $\eta_{\boldsymbol{\phi}}(\boldsymbol{\psi}) \prod_{i=1}^{m} (d\zeta_i)^{k_i} \prod_{j=1}^{n} \left(dz_j\right)^{h_j}$.]

For each $\boldsymbol{\phi} \in \mathcal{B}_{\mathcal{O}}$, $\eta_{\boldsymbol{\phi}}$ extends by linearity to a map $\mathcal{V}_{\mathcal{C}} \to \mathbb{C}$, provided that $\mathcal{O} \cap \mathcal{C} = \varnothing$. We use these linear functionals to define our concept of closeness or, more precisely, the topology of our space. To make sure that we end up with a space which is complete, we need to consider sequences of elements of $\mathcal{V}_{\mathcal{C}}$ which are convergent in a suitable sense. Let $\widetilde{\mathcal{V}}_{\mathcal{C}}$ be the space of sequences $\boldsymbol{\Psi} = (\Psi_1, \Psi_2, \dots)$, $\Psi_j \in \mathcal{V}_{\mathcal{C}}$. We consider the subset $\widetilde{\mathcal{V}}_{\mathcal{C}}^{\mathcal{O}}$ of such sequences $\boldsymbol{\Psi}$ for which $\eta_{\boldsymbol{\phi}}(\Psi_j)$ converges on subsets of $\boldsymbol{\phi}$ of the form

$$\{\boldsymbol{\phi} = V(\phi_1, \zeta_1) V(\phi_2, \zeta_2) \cdots V(\phi_m, \zeta_m)\Omega : \zeta_j \in K, |\zeta_i - \zeta_j| \geq \epsilon, i \neq j\}, \tag{6}$$

where for each collection of ϕ_j, $\epsilon > 0$ and a compact subset $K \subset \mathcal{O}$, the convergence is uniform in the (compact) set

$$\{(\zeta_1, \dots, \zeta_m) : \zeta_j \in K, |\zeta_i - \zeta_j| \geq \epsilon, i \neq j\}. \tag{7}$$

If $\boldsymbol{\Psi} \in \widetilde{\mathcal{V}}_{\mathcal{C}}^{\mathcal{O}}$, the limit

$$\lim_{j\to\infty} \eta_{\boldsymbol{\phi}}(\Psi_j) \tag{8}$$

is necessarily an analytic function of the ζ_j, for $\zeta_j \in \mathcal{O}$, with singularities only at $\zeta_i = \zeta_j$, $i \neq j$. (Again these could in principle be essential singularities, but the assumption of the cluster decompostion property, made in Sect. 6, will imply that these are only poles of finite order.) We denote this function by $\eta_{\boldsymbol{\phi}}(\boldsymbol{\Psi})$. [A necessary and sufficient condition for uniform convergence on the compact set (7) is that the functions $\eta_{\boldsymbol{\phi}}(\Psi_j)$ should be both convergent in the compect set and locally uniformly bounded, i.e. each point of (7) has a neighbourhood in which $\eta_{\boldsymbol{\phi}}(\Psi_j)$ is bounded independently of j; see Appendix A for further details.]

It is natural that we should regard two such sequences $\boldsymbol{\Psi}^1 = (\Psi_i^1)$ and $\boldsymbol{\Psi}^2 = (\Psi_i^2)$ as equivalent if

$$\lim_{j\to\infty} \eta_{\boldsymbol{\phi}}(\Psi_j^1) = \lim_{j\to\infty} \eta_{\boldsymbol{\phi}}(\Psi_j^2), \tag{9}$$

i.e. $\eta_{\boldsymbol{\phi}}(\boldsymbol{\Psi}^1) = \eta_{\boldsymbol{\phi}}(\boldsymbol{\Psi}^2)$, for each $\boldsymbol{\phi} \in \mathcal{B}_{\mathcal{O}}$. We identify such equivalent sequences, and denote the space of them by $\mathcal{V}_{\mathcal{C}}^{\mathcal{O}}$.

The space $\mathcal{V}_{\mathcal{C}}^{\mathcal{O}}$ has a natural topology: we define a sequence $\chi_j \in \mathcal{V}_{\mathcal{C}}^{\mathcal{O}}$, $j = 1, 2, \dots$, to be convergent if, for each $\boldsymbol{\phi} \in \mathcal{B}_{\mathcal{O}}$, $\eta_{\boldsymbol{\phi}}(\chi_j)$ converges uniformly on each (compact) subset of the form (7). The limit

$$\lim_{j\to\infty} \eta_{\boldsymbol{\phi}}(\chi_j) \tag{10}$$

is again necessarily a meromorphic function of the ζ_j, for $\zeta_j \in \mathcal{O}$, with poles only at $\zeta_i = \zeta_j$, $i \neq j$. Provided that the limits of such sequences are always in $\mathcal{V}_{\mathcal{C}}^{\mathcal{O}}$, i.e.

$$\lim_{j\to\infty} \eta_{\boldsymbol{\phi}}(\chi_j) = \eta_{\boldsymbol{\phi}}(\chi), \quad \text{for some } \chi \in \mathcal{V}_{\mathcal{C}}^{\mathcal{O}}, \tag{11}$$

we can define the topology by defining its closed subsets to be those for which the limit of each convergent sequence of elements in the subset is contained within it. In fact we do not have to incorporate the need for the limit to be in $\mathcal{V}_{\mathcal{C}}^{\mathcal{O}}$, because it is so necessarily; we show this in Appendix B. [As we note in this appendix, this topology on $\mathcal{V}_{\mathcal{C}}^{\mathcal{O}}$ can be induced by a countable family of seminorms of the form $||\chi||_n = \max_{1\leq i\leq n} \max_{\zeta_{i_j}} |\eta_{\boldsymbol{\phi}_i}(\chi)|$, where the ϕ_{i_j} in $\boldsymbol{\phi}_i$ are chosen from finite subsets of a countable basis and the ζ_{i_j} are in a compact set of the form (7).]

$\mathcal{B}_{\mathcal{C}}$ can be identified with a subset of $\widetilde{\mathcal{V}}_{\mathcal{C}}^{\mathcal{O}}$ (using constant sequences), and this has an image in $\mathcal{V}_{\mathcal{C}}^{\mathcal{O}}$. It can be shown that this image is necessarily faithful provided that we assume the cluster property introduced in Sect. 6. In any case, we shall assume that this is the case in what follows and identify $\mathcal{B}_{\mathcal{C}}$ with its image in $\mathcal{V}_{\mathcal{C}}^{\mathcal{O}}$. There is a common vector $\Omega \in \mathcal{B}_{\mathcal{C}} \subset \mathcal{V}_{\mathcal{C}}^{\mathcal{O}}$ for all $\mathcal{C}, \mathcal{O}$ which is called the *vacuum vector*. The linear span of $\mathcal{B}_{\mathcal{C}}$ is dense in $\mathcal{V}_{\mathcal{C}}^{\mathcal{O}}$, (i.e. it is what is called a *total space*). With this identification, the image of $\mathcal{B}_{\mathcal{C}}$ in $\mathcal{V}_{\mathcal{C}}^{\mathcal{O}}$, $\boldsymbol{\psi}$, defined as in (3), depends linearly on the vectors $\psi_j \in V$.

A key result in our approach is that, for suitable $\mathcal{O}$, $\mathcal{V}_{\mathcal{C}}^{\mathcal{O}}$ does not depend on $\mathcal{C}$. This is an analogue of the Reeh-Schlieder Theorem of Axiomatic Quantum Field Theory. In our context it is basically a consequence of the fact that any meromorphic function is determined by its values in an arbitrary open set. Precisely, we have the result:

Theorem 1. *$\mathcal{V}_{\mathcal{C}}^{\mathcal{O}}$ is independent of $\mathcal{C}$ if the complement of $\mathcal{O}$ is path connected.*

The proof is given in Appendix C. In the following we shall mainly consider the case where the complement of $\mathcal{O}$ is path-connected and, in this case, we denote $\mathcal{V}_{\mathcal{C}}^{\mathcal{O}}$ by $\mathcal{V}^{\mathcal{O}}$.

The definition of $\eta_{\boldsymbol{\phi}} : \mathcal{V}_{\mathcal{C}} \to \mathbb{C}$, $\widetilde{\mathcal{V}}_{\mathcal{C}}$ and, in particular, $\mathcal{V}_{\mathcal{C}}^{\mathcal{O}}$ all depend, at least superficially, on the particular coordinate chosen on $\mathbb{P}$, that is the particular identification of $\mathbb{P}$ with $\mathbb{C}\cup\{\infty\}$. However the coefficients with which elements of $\mathcal{B}_{\mathcal{C}}$ are combined to constitute elements of $\mathcal{V}_{\mathcal{C}}$ should be regarded as densities and then a change of coordinate on $\mathbb{P}$ induces an endomorphism of $\mathcal{V}_{\mathcal{C}}$ which relates the definitions of the space $\mathcal{V}_{\mathcal{C}}^{\mathcal{O}}$ which we would get with the different choices of coordinates, because $\eta_{\boldsymbol{\phi}}$ only changes by an overall factor (albeit a function of the ζ_i). In this way $\mathcal{V}_{\mathcal{C}}^{\mathcal{O}}$, *etc.* can be regarded as coordinate independent.

Suppose that $\mathcal{O} \subset \mathcal{O}'$ and $\mathcal{C}\cap\mathcal{O}' = \varnothing$ with $\mathcal{C}^o \neq \varnothing$. Then if a sequence $\boldsymbol{\Psi} = (\Psi_j) \in \widetilde{\mathcal{V}}_{\mathcal{C}}$ is such that $\eta_{\boldsymbol{\phi}}(\Psi_j)$ is convergent for all $\boldsymbol{\phi} \in \mathcal{B}_{\mathcal{O}'}$ it follows that it is convergent for all $\boldsymbol{\phi} \in \mathcal{B}_{\mathcal{O}} \subset \mathcal{B}_{\mathcal{O}'}$. In these circumstances, if $\eta_{\boldsymbol{\phi}}(\boldsymbol{\Psi})$ vanishes for all $\boldsymbol{\phi} \in \mathcal{B}_{\mathcal{O}}$, it follows that $\eta_{\boldsymbol{\phi}'}(\boldsymbol{\Psi})$ will vanish for all $\boldsymbol{\phi}' \in \mathcal{B}_{\mathcal{O}'}$, because each $\eta_{\boldsymbol{\phi}'}(\boldsymbol{\Psi})$ is the analytic continuation of $\eta_{\boldsymbol{\phi}}(\boldsymbol{\Psi})$, for some $\boldsymbol{\phi} \in \mathcal{B}_{\mathcal{O}}$; the converse is also immediate because $\mathcal{B}_{\mathcal{O}} \subset \mathcal{B}_{\mathcal{O}'}$. Thus members of an equivalence class in $\mathcal{V}_{\mathcal{C}}^{\mathcal{O}'}$ are also in the same equivalence class in $\mathcal{V}_{\mathcal{C}}^{\mathcal{O}}$. We thus have an injection $\mathcal{V}^{\mathcal{O}'} \to \mathcal{V}^{\mathcal{O}}$, and we can regard $\mathcal{V}^{\mathcal{O}'} \subset \mathcal{V}^{\mathcal{O}}$. Since $\mathcal{B}_{\mathcal{C}}$ is dense in $\mathcal{V}^{\mathcal{O}}$, it follows that $\mathcal{V}^{\mathcal{O}'}$ is also.

Given a subset $\mathcal{C} \subset \mathbb{P}$ with $\mathcal{C}^o \neq \varnothing$, $\mathcal{B}_{\mathcal{C}}$ is dense in a collection of spaces $\mathcal{V}^{\mathcal{O}}$, with $\mathcal{C}\cap\mathcal{O} = \varnothing$. Given open sets $\mathcal{O}_1$ and $\mathcal{O}_2$ such that the complement of $\mathcal{O}_1 \cup \mathcal{O}_2$ contains an open set, $\mathcal{B}_{\mathcal{C}}$ will be dense in both $\mathcal{V}^{\mathcal{O}_1}$ and $\mathcal{V}^{\mathcal{O}_2}$ if $\mathcal{C}$ is contained in the complement of $\mathcal{O}_1 \cup \mathcal{O}_2$ and $\mathcal{C}^o \neq \varnothing$. The collection of topological vector spaces $\mathcal{V}^{\mathcal{O}}$, where $\mathcal{O}$ is an open subset of the Riemann sphere whose complement is path-connected, forms in some sense the space of states of the meromorphic field theory we are considering.

A *vertex operator* can be defined for $\psi \in V$ as an operator $V(\psi, z) : \mathcal{V}^{\mathcal{O}} \to \mathcal{V}^{\mathcal{O}'}$, where $z \in \mathcal{O}$ but $z \notin \mathcal{O}' \subset \mathcal{O}$, by defining its action on the dense subset $\mathcal{B}_{\mathcal{C}}$, where $\mathcal{C}\cap\mathcal{O} = \varnothing$,

$$V(\psi, z)\boldsymbol{\psi} = V(\psi, z)V(\psi_1, z_1)V(\psi_2, z_2)\cdots V(\psi_n, z_n)\Omega, \tag{12}$$

and $\boldsymbol{\psi} \in \mathcal{B}_{\mathcal{C}}$. The image is in $\mathcal{V}_{\mathcal{C}'}$ for any $\mathcal{C}' \supset \mathcal{C}$ which contains z, and we can choose $\mathcal{C}'$ such that $\mathcal{C}' \cap \mathcal{O}' = \varnothing$. This then extends by linearity to a map $\mathcal{V}_{\mathcal{C}} \to \mathcal{V}_{\mathcal{C}'}$. To show that it induces a map $\mathcal{V}_{\mathcal{C}}^{\mathcal{O}} \to \mathcal{V}_{\mathcal{C}'}^{\mathcal{O}'}$, we need to show that if $\Psi^j, \in \mathcal{V}_{\mathcal{C}}^{\mathcal{O}}, \to 0$ as $j \to \infty$, then $V(\psi, z)\Psi^j \to 0$ as $j \to \infty$; i.e. if $\eta_{\boldsymbol{\phi}}(\Psi^j) \to 0$ for all $\boldsymbol{\phi} \in \mathcal{B}_{\mathcal{O}}$, then $\eta_{\boldsymbol{\phi}'}(V(\psi, z)\Psi^j) \to 0$ for all $\boldsymbol{\phi}' \in \mathcal{B}_{\mathcal{O}'}$. But $\eta_{\boldsymbol{\phi}'}(V(\psi, z)\Psi^j) = \eta_{\boldsymbol{\phi}}(\Psi^j)$, where $\boldsymbol{\phi} = V(\psi, z)\boldsymbol{\phi}' \in \mathcal{B}_{\mathcal{O}}$ and so tends to zero as required. It is straightforward to show that the vertex operator $V(\psi, z)$ is continuous. We shall refer to these vertex operators also as *meromorphic fields*.

It follows directly from the invariance of the amplitudes under permutations that

Proposition 2. *If* $z, \zeta \in \mathcal{O}$, $z \neq \zeta$, *and* $\phi, \psi \in V$, *then*

$$V(\phi, z)V(\psi, \zeta) = V(\psi, \zeta)V(\phi, z) \tag{13}$$

as an identity on $\mathcal{V}^{\mathcal{O}}$.

This result, that the vertex operators, $V(\psi, z)$, commute at different z in a (bosonic) meromorphic conformal field theory, is one which should hold morally, but normally one has to attach a meaning to it in some other sense, such as analytic continuation (compare for example [13]).

3. Möbius Invariance

In order to proceed much further, without being dependent in some essential way on how the Riemann sphere is identified with the complex plane (and infinity), we shall need to assume that the amplitudes $\mathcal{A}$ have some sort of Möbius invariance. We shall say that the densities in $\mathcal{A}$ are invariant under the Möbius transformation γ, where

$$\gamma(z) = \frac{az + b}{cz + d}, \tag{14}$$

(and we can take $ad - bc = 1$), provided that the densities in (2) satisfy

$$\left\langle \prod_{j=1}^{n} V(\psi_j, z_j) \right\rangle \prod_{j=1}^{n} (dz_j)^{h_j} = \left\langle \prod_{j=1}^{n} V(\psi_j, \zeta_j) \right\rangle \prod_{j=1}^{n} (d\zeta_j)^{h_j}, \quad \text{where } \zeta_j = \gamma(z_j), \tag{15a}$$

i.e.

$$\left\langle \prod_{j=1}^{n} V(\psi_j, z_j) \right\rangle = \left\langle \prod_{j=1}^{n} V(\psi_j, \gamma(z_j)) \right\rangle \prod_{j=1}^{n} \left(\gamma'(z_j)\right)^{h_j}. \tag{15b}$$

Here $\psi_j \in V_{h_j}$. The Möbius transformations form the group $\mathcal{M} \cong \mathrm{SL}(2, \mathbb{C})/\mathbb{Z}_2$.

If $\mathcal{A}$ is invariant under the Möbius transformation γ, we can define an operator $U(\gamma) : \mathcal{V}^{\mathcal{O}} \to \mathcal{V}^{\mathcal{O}_\gamma}$, where $\mathcal{O}_\gamma = \{\gamma(z) : z \in \mathcal{O}\}$, by defining it on the dense subset $\mathcal{B}_{\mathcal{C}}$ for some $\mathcal{C}$ with $\mathcal{C} \cap \mathcal{O} = \varnothing$, by

$$U(\gamma)\boldsymbol{\psi} = \prod_{j=1}^{n} V(\psi_j, \gamma(z_j)) \prod_{j=1}^{n} \left(\gamma'(z_j)\right)^{h_j} \Omega, \tag{16}$$

where $\boldsymbol{\psi} = V(\psi_1, z_1) \cdots V(\psi_n, z_n)\Omega \in \mathcal{B}_{\mathcal{C}}$. Again, this extends by linearity to a map defined on $\mathcal{V}_{\mathcal{C}}$, and to show that it defines a map $\mathcal{V}_{\mathcal{C}}^{\mathcal{O}} \to \mathcal{V}_{\mathcal{C}_\gamma}^{\mathcal{O}_\gamma}$, where $\mathcal{C}_\gamma = \{\gamma(z) : z \in \mathcal{C}\}$, we again need to show that if $\eta_{\boldsymbol{\phi}}(\Psi_j) \to 0$ for all $\boldsymbol{\phi} \in \mathcal{B}_{\mathcal{O}}$, then $\eta_{\boldsymbol{\phi}'}(U(\gamma)\Psi_j) \to 0$ for all $\boldsymbol{\phi}' \in \mathcal{B}_{\mathcal{O}_\gamma}$. By the assumed invariance under γ, we have $\eta_{\boldsymbol{\phi}'}(U(\gamma)\Psi_j) = \eta_{\boldsymbol{\phi}}(\Psi_j)$, where $\boldsymbol{\phi} = U(\gamma^{-1})\boldsymbol{\phi}'$, and the result follows.

It follows immediately from the definition of $U(\gamma)$ that $U(\gamma)\Omega = \Omega$ (where we have identified $\Omega \in \mathcal{V}^{\mathcal{O}}$ with $\Omega \in \mathcal{V}^{\mathcal{O}_\gamma}$ as explained in Sect. 2). Furthermore,

$$U(\gamma)V(\psi, z)U(\gamma^{-1}) = V(\psi, \gamma(z))\gamma'(z)^h, \quad \text{for } \psi \in V_h. \tag{17}$$

By choosing a point $z_0 \notin \mathcal{O}$, we can identify V with a subspace of $\mathcal{V}^{\mathcal{O}}$ by the map $\psi \mapsto V(\psi, z_0)\Omega$; this map is an injection provided that $\mathcal{A}$ is invariant under an infinite subgroup of $\mathcal{M}$ which maps z_0 to an infinite number of distinct image points. For, if

$$\langle \prod_{i=1}^{n} V(\psi_i, z_i) V(\psi, \zeta) \rangle \tag{18}$$

vanishes for $\zeta = z_0$ for all ψ_i and z_i, then by the invariance property, the same holds for an infinite number of ζ's. Regarded as a function of ζ, (18) defines a meromorphic function with infinitely many zeros; it therefore vanishes identically, thus implying that $\psi = 0$.

In the following we shall use elements of SL(2, $\mathbb{C}$) to denote the corresponding elements of $\mathcal{M}$ where no confusion will result, so that

$$\text{if} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \gamma(z) = \frac{az+b}{cz+d}. \tag{19}$$

An element of $\mathcal{M}$ has either one or two fixed points or is the identity. The one-parameter complex subgroups of $\mathcal{M}$ are either conjugate to the translation group $z \mapsto z + \lambda$ (one fixed point) or the dilatation group $z \mapsto e^{\lambda} z$ (two fixed points).

Now, first, consider a theory which is invariant under the translation group $z \mapsto \tau_\lambda(z) = z + \lambda$. Then, if $\tau_\lambda = e^{\lambda L_{-1}}$, and we do not distinguish between $U(L_{-1})$ and L_{-1} in terms of notation, from (17) we have

$$e^{\lambda L_{-1}} V(\psi, z) e^{-\lambda L_{-1}} = V(\psi, z + \lambda). \tag{20}$$

[If, instead, we had a theory invariant under a subgroup of the Möbius group conjugate to the translation group, $\{\gamma_0^{-1} \tau_\lambda \gamma_0 : \lambda \in \mathbb{C}\}$ say, and if $\zeta = \gamma_0(z)$, $\zeta \mapsto \zeta' = \zeta + \lambda$ under $\gamma_0^{-1} \tau_\lambda \gamma_0$; then, if $\widehat{V}(\psi, \zeta) = V(\psi, z) \gamma_0'(z)^{-h}$ and $\hat{L}_{-1} = \gamma_0^{-1} L_{-1} \gamma_0$, then $e^{\lambda \hat{L}_{-1}} \widehat{V}(\psi, \zeta) e^{-\lambda \hat{L}_{-1}} = \widehat{V}(\psi, \zeta + \lambda)$.]

Consider now a theory which is invariant under the whole Möbius group. We can pick a group conjugate to the translation group, and we can change coordinates so that $z = \infty$ is the fixed point of the selected translation group. (In particular, this defines an identification of $\mathbb{P}$ with $\mathbb{C} \cup \{\infty\}$ up to a Euclidean or scaling transformation of $\mathbb{C}$.) If we select a point z_0 to define the injection $V \to \mathcal{V}^{\mathcal{O}}$, $z_0 \notin \mathcal{O}$, we have effectively selected two fixed points. Without loss of generality, we can choose $z_0 = 0$. Then

$$\psi = V(\psi, 0)\Omega \in \mathcal{V}^{\mathcal{O}}. \tag{21}$$

We can then introduce naturally two other one-parameter groups, one generated by L_0 which fixes both 0 and ∞ (the group of dilatations or scaling transformations), and another which fixes only 0, generated by L_1 (the group of special conformal transformations). Then

$$e^{\lambda L_{-1}}(z) = z + \lambda, \quad e^{\lambda L_0}(z) = e^{\lambda} z, \quad e^{\lambda L_1}(z) = \frac{z}{1 - \lambda z}, \tag{22a}$$

$$e^{\lambda L_{-1}} = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \quad e^{\lambda L_0} = \begin{pmatrix} e^{\frac{1}{2}\lambda} & 0 \\ 0 & e^{-\frac{1}{2}\lambda} \end{pmatrix}, \quad e^{\lambda L_1} = \begin{pmatrix} 1 & 0 \\ -\lambda & 1 \end{pmatrix}, \tag{22b}$$

and thus

$$L_{-1} = \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}, \quad L_0 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad L_1 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}. \tag{22c}$$

In particular, it then follows that

$$[L_m, L_n] = (m-n)L_{m+n}, \quad m, n = 0, \pm 1. \tag{23}$$

We also have that $L_n\Omega = 0$, $n = 0, \pm 1$. With this parametrisation, the operator corresponding to the Möbius transformation γ, defined in (19), is given as (see [13])

$$U(\gamma) = \exp\left(\frac{b}{d}L_{-1}\right)\left(\frac{\sqrt{ad-bc}}{d}\right)^{L_0} \exp\left(-\frac{c}{d}L_1\right). \tag{24}$$

For $\psi \in V_h$, by (17), $U(\gamma)V(\psi, z)U(\gamma^{-1}) = V(\psi, \gamma(z))\gamma'(z)^h$, and so, by (21), $U(\gamma)\psi = \lim_{z\to 0} V(\psi, \gamma(z))\Omega\gamma'(z)^h$. From this it follows that,

$$L_0\psi = h\psi, \quad L_1\psi = 0, \quad L_{-1}\psi = V'(\psi, 0)\Omega. \tag{25}$$

Thus $L_0 = \delta$ acting on V.

Henceforth we shall assume that our theory defined by $\mathcal{A}$ is Möbius invariant.

Having chosen an identification of $\mathbb{P}$ with $\mathbb{C} \cup \{\infty\}$ and of V with a subspace of $\mathcal{V}^{\mathcal{O}}$, we can now also define vertex operators for $\boldsymbol{\psi} = \prod_{j=1}^n V(\psi_j, z_j)\Omega \in \mathcal{B}_{\mathcal{C}}$ by

$$V(\boldsymbol{\psi}, z) = \prod_{j=1}^n V(\psi_j, z_j + z). \tag{26}$$

Then $V(\boldsymbol{\phi}, z)$ is a continuous operator $\mathcal{V}^{\mathcal{O}_1} \to \mathcal{V}^{\mathcal{O}_2}$, provided that $z_j + z \notin \mathcal{O}_2 \subset \mathcal{O}_1$ but $z_j + z \in \mathcal{O}_1$, $1 \le j \le n$.

We can further extend the definition of $V(\boldsymbol{\psi}, z)$ by linearity from $\boldsymbol{\psi} \in \mathcal{B}_{\mathcal{C}}$ to vectors $\Psi \in \mathsf{V}_{\mathcal{C}}^{\mathcal{O}}$, the image of $\mathcal{V}_{\mathcal{C}}$ in $\mathcal{V}_{\mathcal{C}}^{\mathcal{O}}$, to obtain a continuous linear operator $V(\Psi, z)$: $\mathcal{V}^{\mathcal{O}_1} \to \mathcal{V}^{\mathcal{O}_2}$, where $\mathcal{C}_z \cap \mathcal{O}_2 = \emptyset$, $\mathcal{O}_2 \subset \mathcal{O}_1$ and $\mathcal{C}_z \subset \mathcal{O}_1$ for $\mathcal{C}_z = \{\zeta + z : \zeta \in \mathcal{C}\}$. One might be tempted to try to extend the definition of the vertex operator even further to states in $\mathcal{V}_{\mathcal{C}}^{\mathcal{O}} \cong \mathcal{V}^{\mathcal{O}}$, but the corresponding operator will then only be well-defined on a suitable dense subspace of $\mathcal{V}^{\mathcal{O}_1}$.

For the vertex operator associated to $\Psi \in \mathsf{V}_{\mathcal{C}}^{\mathcal{O}}$, we again have

$$e^{\lambda L_{-1}}V(\Psi, z)e^{-\lambda L_{-1}} = V(\Psi, z+\lambda), \quad V(\Psi, 0)\Omega = \Psi. \tag{27}$$

Furthermore,

$$V(\Psi, z)V(\phi, \zeta) = V(\phi, \zeta)V(\Psi, z), \tag{28a}$$

$$V(\Psi, z)\Omega = e^{zL_{-1}}\Psi \tag{28b}$$

for any $\phi \in V$, $\zeta \notin \mathcal{C}_z$. [In (28a), the left-hand and rigth-hand sides are to be interpreted as maps $\mathcal{V}^{\mathcal{O}_1} \to \mathcal{V}^{\mathcal{O}_2}$, with $V(\phi, \zeta) : \mathcal{V}^{\mathcal{O}_1} \to \mathcal{V}^{\mathcal{O}_L}$ and $V(\Psi, z) : \mathcal{V}^{\mathcal{O}_L} \to \mathcal{V}^{\mathcal{O}_2}$ on the left-hand side and $V(\Psi, z) : \mathcal{V}^{\mathcal{O}_1} \to \mathcal{V}^{\mathcal{O}_R}$ and $V(\phi, \zeta) : \mathcal{V}^{\mathcal{O}_R} \to \mathcal{V}^{\mathcal{O}_2}$ on the rigth-hand side, where $\mathcal{O}_2 \subset \mathcal{O}_L \subset \mathcal{O}_1$, $\mathcal{O}_2 \subset \mathcal{O}_R \subset \mathcal{O}_1$, $\zeta \in \mathcal{O}_R \cap \mathcal{O}_L^c \cap \mathcal{O}_2^c$ and $\mathcal{C}_z \subset \mathcal{O}_L \cap \mathcal{O}_R^c \cap \mathcal{O}_2^c$ (where $\mathcal{O}_2^c$ denotes the complement of $\mathcal{O}_2$, *etc.*). Equation (28b) holds in $\mathcal{V}^{\mathcal{O}}$ with $\mathcal{C}_z \cap \mathcal{O} = \emptyset$.]

Actually, these two conditions characterise the vertex operator already uniquely:

Theorem 3 (Uniqueness). *For $\Psi \in \mathsf{V}_{\mathcal{C}}^{\mathcal{O}}$, the operator $V(\Psi, z)$ is uniquely characterised by the conditions (28a) and (28b).*

The proof is essentially that contained in ref. [13]: If $W(z)V(\phi,\zeta) = V(\phi,\zeta)W(z)$ for $\phi \in V$, $\zeta \notin \mathcal{C}_z$, and $W(z)\Omega = e^{zL_{-1}}\Psi$, it follows that, for $\Phi \in \mathsf{V}_{\mathcal{C}'}^{\mathcal{O}'}$, $W(z)V(\Phi,\zeta) = V(\Phi,\zeta)W(z)$ provided that $\mathcal{C}'_\zeta \cap \mathcal{C}_z = \varnothing$ and so

$$\begin{aligned} W(z)e^{\zeta L_{-1}}\Phi &= W(z)V(\Phi,\zeta)\Omega = V(\Phi,\zeta)W(z)\Omega = V(\Phi,\zeta)e^{zL_{-1}}\Psi \\ &= V(\Phi,\zeta)V(\Psi,z)\Omega = V(\Psi,z)V(\Phi,\zeta)\Omega = V(\Psi,z)e^{\zeta L_{-1}}\Phi \end{aligned}$$

for all $\Phi \in \mathsf{V}_{\mathcal{C}'}^{\mathcal{O}'}$, which is dense in $\mathcal{V}^{\mathcal{O}'}$, showing that $W(z) = V(\Psi, z)$.

From this uniqueness result and (17) we can deduce the commutators of vertex operators $V(\psi, z)$, $\psi \in V_h$, with L_{-1}, L_0, L_1:

$$[L_{-1}, V(\psi,z)] = \frac{d}{dz}V(\psi,z), \tag{29a}$$

$$[L_0, V(\psi,z)] = z\frac{d}{dz}V(\psi,z) + hV(\psi,z), \tag{29b}$$

$$[L_1, V(\psi,z)] = z^2\frac{d}{dz}V(\psi,z) + 2hzV(\psi,z). \tag{29c}$$

We recall from (25) that $L_1\psi = 0$ and $L_0\psi = h\psi$ if $\psi \in V_h$; if $L_1\psi = 0$, ψ is said to be *quasi-primary*.

The definition (26) immediately implies that, for states $\psi, \phi \in V$, $V(\psi,z)V(\phi,\zeta) = V(V(\psi, z-\zeta)\phi, \zeta)$. This statement generalises to the key duality result of Theorem 4, which can be seen to follow from the uniqueness theorem:

Theorem 4 (Duality). *If $\Psi \in \mathsf{V}_{\mathcal{C}}^{\mathcal{O}}$ and $\Phi \in \mathsf{V}_{\mathcal{C}'}^{\mathcal{O}'}$, where $\mathcal{C}_z \cap \mathcal{C}'_\zeta = \varnothing$, then*

$$V(\Psi,z)V(\Phi,\zeta) = V(V(\Psi,z-\zeta)\Phi,\zeta). \tag{30}$$

[In (30), the left-hand and rigth-hand sides are to be interpreted as maps $\mathcal{V}^{\mathcal{O}_1} \to \mathcal{V}^{\mathcal{O}_2}$, with $V(\Phi,\zeta) : \mathcal{V}^{\mathcal{O}_1} \to \mathcal{V}^{\mathcal{O}_L}$ and $V(\Psi, z) : \mathcal{V}^{\mathcal{O}_L} \to \mathcal{V}^{\mathcal{O}_2}$ on the left-hand side and $V(\Psi, z-\zeta)\Phi \in \mathsf{V}_{\mathcal{C}_{z-\zeta}\cup\mathcal{C}'}$ where $\mathcal{O}_2 \subset \mathcal{O}_L \subset \mathcal{O}_1$, $\mathcal{C}_z \subset \mathcal{O}_L \cap \mathcal{O}_2^c$ and $\mathcal{C}'_\zeta \subset \mathcal{O}_1 \cap \mathcal{O}_L^c$.]

The result follows from the uniqueness theorem on noting that

$$\begin{aligned} V(\Phi,z)V(\Psi,\zeta)\Omega &= V(\Phi,z)e^{\zeta L_{-1}}\Psi \\ &= e^{\zeta L_{-1}}V(\Phi,z-\zeta)\Psi = V(V(\Phi,z-\zeta)\Psi),\zeta)\Omega. \end{aligned}$$

4. Modes, Fock Spaces and the Equivalence of Theories

The concept of equivalence between two meromorphic field theories in our definition could be formulated in terms of the whole collection of spaces $\mathcal{V}^{\mathcal{O}}$, where $\mathcal{O}$ ranges over the open subsets of $\mathbb{P}$ with path-connected complement, but this would be very unwieldy. In fact, each meromorphic field theory has a Fock space at its heart and we can focus on this in order to define (and, in practice, test for) the equivalence of theories. To approach this we first need to introduce the concept of the modes of a vertex operator.

It is straightforward to see that we can construct contour integrals of vectors in $\mathcal{V}^{\mathcal{O}}$, *e.g.* of the form

$$\int_{\mathcal{C}_1} dz_1 \int_{\mathcal{C}_2} dz_2 \ldots \int_{\mathcal{C}_r} dz_r \mu(z_1, z_2, \ldots, z_r) \prod_{i=1}^{n} V(\psi_i, z_i)\Omega, \tag{31}$$

where $r \leq n$ and the weight function μ is analytic in some neighbourhood of $C_1 \times C_2 \times \cdots \times C_r$ and the distances $|z_i - z_j|$, $i \neq j$, are bounded away from 0 on this set. In this way we can define the modes

$$V_n(\psi) = \oint_C z^{h+n-1} V(\psi, z) dz, \quad \text{for } \psi \in V_h, \tag{32}$$

as linear operators on $\mathcal{V}_{\mathcal{C}}^{\mathcal{O}}$, where C encircles $\mathcal{C}$ and $C \subset \mathcal{O}$ with $\infty \in \mathcal{O}$ and $0 \in \mathcal{C}$, and we absorb a factor of $1/2\pi i$ into the definition of the symbol $\oint$. The meromorphicity of the amplitudes allows us to establish

$$V(\psi, z) = \sum_{n=-\infty}^{\infty} V_n(\psi) z^{-n-h} \tag{33}$$

with convergence with respect to the topology of $\mathcal{V}^{\mathcal{O}'}$ for an appropriate $\mathcal{O}'$.

The definition of $V_n(\psi)$ is independent of C if it is taken to be a simple contour encircling the origin once positively. Further, if $\mathcal{O}_2 \subset \mathcal{O}_1$, $\mathcal{V}^{\mathcal{O}_1} \subset \mathcal{V}^{\mathcal{O}_2}$ and if $\infty \in \mathcal{O}_2$, $0 \notin \mathcal{O}_1$, the definition of $V_n(\psi)$ on $\mathcal{V}^{\mathcal{O}_1}$, $\mathcal{V}^{\mathcal{O}_2}$, agrees on $\mathcal{V}^{\mathcal{O}_1}$, which is dense in $\mathcal{V}^{\mathcal{O}_2}$, so that we may regard the definition as independent of $\mathcal{O}$ also. $V_n(\psi)$ depends on our choice of 0 and ∞ but different choices can be related by Möbius transformations.

We define the Fock space $\mathcal{H}^{\mathcal{O}} \subset \mathcal{V}^{\mathcal{O}}$ to be the space spanned by finite linear combinations of vectors of the form

$$\Psi = V_{n_1}(\psi_1) V_{n_2}(\psi_2) \cdots V_{n_N}(\psi_N)\Omega, \tag{34}$$

where $\psi_j \in V$ and $n_j \in \mathbb{Z}$, $1 \leq j \leq N$. Then, by construction, $\mathcal{H}^{\mathcal{O}}$ has a countable basis. It is easy to see that $\mathcal{H}^{\mathcal{O}}$ is dense in $\mathcal{V}^{\mathcal{O}}$. Further it is clear that $\mathcal{H}^{\mathcal{O}}$ is independent of $\mathcal{O}$, and, where there is no ambiguity, we shall denote it simply by $\mathcal{H}$. It does however depend on the choice of 0 and ∞, but different choices will be related by the action of the Möbius group again.

It follows from (28b) that

$$V(\psi, 0)\Omega = \psi \tag{35}$$

which implies that

$$V_n(\psi)\Omega = 0 \quad \text{if } n > -h \tag{36}$$

and

$$V_{-h}(\psi)\Omega = \psi. \tag{37}$$

Thus $V \subset \mathcal{H}$.

Since ∞ and 0 play a special role, it is not surprising that L_0, the generator of the subgroup of $\mathcal{M}$ preserving them, does as well. From (29b) it follows that

$$[L_0, V_n(\psi)] = -n V_n(\psi), \tag{38}$$

so that for Ψ defined by (34),

$$L_0\Psi = h\Psi, \quad \text{where } h = -\sum_{j=1}^{N} n_j. \tag{39}$$

Thus

$$\mathcal{H} = \bigoplus_{h\in\mathbb{Z}} \mathcal{H}_h, \quad \text{where } V_h \subset \mathcal{H}_h, \tag{40}$$

where $\mathcal{H}_h$ is the subspace spanned by vectors of the form (34) for which $h = \sum_j n_j$.

Thus L_0 has a spectral decomposition and the $\mathcal{H}_h$, $h \in \mathbb{Z}$, are the eigenspaces of L_0. They have countable dimensions but here we shall only consider theories for which their dimensions are finite. (This is not guaranteed by the finite-dimensionality of V; in fact, in practice, it is not easy to determine whether these spaces are finite-dimensional or not, although it is rather obvious in many examples.)

We can define vertex operators for the vectors (34) by

$$V(\Psi, z) = \oint_{\mathcal{C}_1} z_1^{h_1+n_1-1} V(\psi_1, z+z_1)dz_1 \cdots \oint_{\mathcal{C}_N} z_N^{h_N+n_N-1} V(\psi_N, z+z_N)dz_N, \tag{41}$$

where the $\mathcal{C}_j$ are contours about 0 with $|z_i| > |z_j|$ if $i < j$. We can then replace the densities (1) by the larger class $\mathcal{A}'$ of densities

$$\langle V(\Psi_1, z_1)V(\Psi_2, z_2)\cdots V(\Psi_n, z_n)\rangle \prod_{j=1}^{n} \left(dz_j\right)^{h_j}, \tag{42}$$

where $\Psi_j \in \mathcal{H}_{h_j}$, $1 \le j \le n$. It is not difficult to see that replacing $\mathcal{A}$ with $\mathcal{A}'$, *i.e.* replacing V with $\mathcal{H}$, does not change the definition of the spaces $\mathcal{V}^{\mathcal{O}}$. Theorem 3 (Uniqueness) and Theorem 4 (Duality) will still hold if we replace $\mathsf{V}_{\mathcal{C}}^{\mathcal{O}}$ with $\mathsf{H}_{\mathcal{C}}^{\mathcal{O}}$, the space we would obtain if we started with $\mathcal{H}$ rather than V, *etc.* These theorems enable the Möbius transformation properties of vertex operators to be determined (see Appendix D).

However, if we use the whole of $\mathcal{H}$ as a starting point, the Möbius properties of the densities $\mathcal{A}'$ can not be as simple as in Sect. 3 because not all $\psi \in \mathcal{H}$ have the *quasi-primary* property $L_1\psi = 0$. But we can introduce the subspaces of quasi-primary vectors within $\mathcal{H}$ and $\mathcal{H}_h$,

$$\mathcal{H}^Q = \{\Psi \in \mathcal{H} : L_1\Psi = 0\}, \quad \mathcal{H}_h^Q = \{\Psi \in \mathcal{H}_h : L_1\Psi = 0\}, \quad \mathcal{H}^Q = \bigoplus_h \mathcal{H}_h^Q. \tag{43}$$

$V \subset \mathcal{H}^Q$ and $\mathcal{H}^Q$ is the maximal V which will generate the theory with the same spaces $\mathcal{V}^{\mathcal{O}}$ and with agreement of the densities. [Under the cluster decomposition assumption of Sect. 6, $\mathcal{H}$ is generated from $\mathcal{H}^Q$ by the action of the Möbius group or, more particularly, L_{-1}. See Appendix D.]

We are now in a position to define the equivalence of two theories. A theory specified by a space V and amplitudes $\mathcal{A} = \{f\}$, leading to a quasi-primary space $\mathcal{H}^Q$, is said to be equivalent to the theory specified by a space $\hat{V}$ and amplitudes $\hat{\mathcal{A}} = \{\hat{f}\}$, leading to a quasi-primary space $\hat{\mathcal{H}}^Q$, if there are graded injections $\iota : V \to \hat{\mathcal{H}}^Q$ (*i.e.* $\iota(V_h) \subset \hat{\mathcal{H}}_h^Q$) and $\hat{\iota} : \hat{V} \to \mathcal{H}^Q$ which map amplitudes to amplitudes.

Many calculations in conformal field theory are most easily performed in terms of modes of vertex operators which capture in essence the algebraic structure of the theory. In particular, the modes of the vertex operators define what is usually called a W-algebra; this can be seen as follows.

The duality property of the vertex operators can be rewritten in terms of modes as

$$\begin{aligned} V(\Phi, z)V(\Psi, \zeta) &= V(V(\Phi, z-\zeta)\Psi, \zeta) \\ &= \sum_n V(V_n(\Phi)\Psi, \zeta)(z-\zeta)^{-n-h_\phi}, \end{aligned} \tag{44}$$

where $L_0\Psi = h_\Psi\Psi$ and $L_0\Phi = h_\Phi\Phi$, and $\Psi, \Phi \in \mathcal{H}$. We can then use the usual contour techniques of conformal field theory to derive from this formula commutation relations for the respective modes. Indeed, the commutator of two modes $V_m(\Phi)$ and $V_n(\Psi)$ acting on $\mathcal{B}_\mathcal{C}$ is defined by

$$\begin{aligned} &= \oint_{|z|>|\zeta|} dz \oint d\zeta \;\; z^{m+h_\Phi-1}\zeta^{n+h_\Psi-1}V(\Phi, z)V(\Psi, \zeta) \\ &\quad - \oint_{|\zeta|>|z|} dz \oint d\zeta \;\; z^{m+h_\Phi-1}\zeta^{n+h_\Psi-1}V(\Phi, z)V(\Psi, \zeta), \end{aligned} \tag{45}$$

where the contours on the right-hand side encircle $\mathcal{C}$ anti-clockwise. We can then deform the two contours so as to rewrite (45) as

$$[V_m(\Phi),\ V_n(\Psi)] = \oint_0 \zeta^{n+h_\Psi-1}d\zeta \oint_\zeta z^{m+h_\Phi-1}dz \sum_l V(V_l(\Phi)\Psi, \zeta)(z-\zeta)^{-l-h_\Phi}, \tag{46}$$

where the z contour is a small positive circle about ζ and the ζ contour is a positive circle about $\mathcal{C}$. Only terms with $l \geq 1 - h_\Phi$ contribute, and the integral becomes

$$[V_m(\Phi), V_n(\Psi)] = \sum_{N=-h_\Phi+1}^{\infty} \binom{m+h_\Phi-1}{m-N} V_{m+n}(V_N(\Phi)\Psi). \tag{47}$$

In particular, if $m \geq -h_\Phi + 1$, $n \geq -h_\Psi + 1$, then $m - N \geq 0$ in the sum, and $m + n \geq N + n \geq N - h_\Psi + 1$. This implies that the modes $\{V_m(\Psi) : m \geq -h_\Psi + 1\}$ close as a Lie algebra; the same also holds for $\{V_m(\Psi) : 0 \geq m \geq -h_\Psi + 1\}$.

As we shall discuss below in Sect. 6, in conformal field theory it is usually assumed that the amplitudes satisfy another property which guarantees that the spectrum of L_0 is bounded below by 0. If this is the case then the sum in (47) is also bounded above by h_Ψ.

5. Some Examples

Before proceeding further, we shall give a number of examples of theories that satisfy the axioms that we have specified so far.

5.1. The $U(1)$ theory. The simplest example is the case where V is a one-dimensional vector space, spanned by a vector J of weight 1, in which case we write $J(z) \equiv V(J, z)$. The amplitude of an odd number of J-fields is defined to vanish, and in the case of an even number it is given by

$$\langle J(z_1) \cdots J(z_{2n}) \rangle = \frac{k^n}{2^n n!} \sum_{\pi \in S_{2n}} \prod_{j=1}^{n} \frac{1}{(z_{\pi(j)} - z_{\pi(j+n)})^2}, \tag{48a}$$

$$= k^n \sum_{\pi \in S'_{2n}} \prod_{j=1}^{n} \frac{1}{(z_{\pi(j)} - z_{\pi(j+n)})^2}, \tag{48b}$$

where k is an arbitrary (real) constant and, in (48a), S_{2n} is the permutation group on $2n$ object, whilst, in (48b), the sum is restricted to the subset S'_{2n} of permutations $\pi \in S_{2n}$ such that $\pi(i) < \pi(i+n)$ and $\pi(i) < \pi(j)$ if $1 \leq i < j \leq n$. (This defines the amplitudes on a basis of V and we extend the definition by multilinearity.) It is clear that the amplitudes are meromorphic in z_j, and that they satisfy the locality condition. It is also easy to check that they are Möbius covariant, with the weight of J being 1.

From the amplitudes we can directly read off the operator product expansion of the field J with itself as

$$J(z)J(w) \sim \frac{k}{(z-w)^2} + O(1). \tag{49}$$

Comparing this with (44), and using (47) we then obtain

$$[J_n, J_m] = nk\delta_{n,-m}. \tag{50}$$

This defines (a representation of) the affine algebra $\hat{u}(1)$.

5.2. Affine Lie algebra theory. Following Frenkel and Zhu [20], we can generalise this example to the case of an arbitrary finite-dimensional Lie algebra g. Suppose that the matrices t^a, $1 \leq a \leq \dim g$, provide a finite-dimensional representation of g so that $[t^a, t^b] = f^{ab}{}_c t^c$, where $f^{ab}{}_c$ are the structure constants of g. In this case, the space V is of dimension $\dim g$ and has a basis consisting of weight one states J^a, $1 \leq a \leq \dim g$. Again, we write $J^a(z) = V(J^a, z)$.

If K is any matrix which commutes with all the t^a, define

$$\kappa^{a_1 a_2 \ldots a_m} = \mathrm{tr}(K t^{a_1} t^{a_2} \cdots t^{a_m}). \tag{51}$$

The $\kappa^{a_1 a_2 \ldots a_m}$ have the properties that

$$\kappa^{a_1 a_2 a_3 \ldots a_{m-1} a_m} = \kappa^{a_2 a_3 \ldots a_{m-1} a_m a_1} \tag{52}$$

and

$$\kappa^{a_1 a_2 a_3 \ldots a_{m-1} a_m} - \kappa^{a_2 a_1 a_3 \ldots a_{m-1} a_m} = f^{a_1 a_2}{}_b \kappa^{b a_3 \ldots a_{m-1} a_m}. \tag{53}$$

With a cycle $\sigma = (i_1, i_2, \ldots, i_m) \equiv (i_2, \ldots, i_m, i_1)$ we associate the function

$$f_\sigma^{a_{i_1} a_{i_2} \ldots a_{i_m}}(z_{i_1}, z_{i_2}, \ldots, z_{i_m}) = \frac{\kappa^{a_{i_1} a_{i_2} \ldots a_{i_m}}}{(z_{i_1} - z_{i_2})(z_{i_2} - z_{i_3}) \cdots (z_{i_{m-1}} - z_{i_m})(z_{i_m} - z_{i_1})}. \tag{54}$$

If the permutation $\rho \in S_n$ has no fixed points, it can be written as the product of cycles of length at least 2, $\rho = \sigma_1\sigma_2 \dots \sigma_M$. We associate to ρ the product f_ρ of functions $f_{\sigma_1} f_{\sigma_2} \dots f_{\sigma_M}$ and define $\langle J^{a_1}(z_1) J^{a_2}(z_2) \dots J^{a_n}(z_n)\rangle$ to be the sum of such functions f_ρ over permutations $\rho \in S_n$ with no fixed point. Graphically, we can construct these amplitudes by summing over all graphs with n vertices where the vertices carry labels a_j, $1 \le j \le n$, and each vertex is connected by two directed lines (propagators) to other vertices, one of the lines at each vertex pointing towards it and one away. Thus, in a given graph, the vertices are divided into directed loops or cycles, each loop containing at least two vertices. To each loop, we associate a function as in (55) and to each graph we associate the product of functions associated to the loops of which it is composed.

Again, this defines the amplitudes on a basis of V and we extend the definition by multilinearity. The amplitudes are evidently local and meromorphic, and one can verify that they satisfy the Möbius covariance property with the weight of J^a being 1.

The amplitudes determine the operator product expansion to be of the form

$$J^a(z) J^b(w) \sim \frac{\kappa^{ab}}{(z-w)^2} + \frac{f^{ab}{}_c J^c(w)}{(z-w)} + O(1), \tag{55}$$

and the algebra therefore becomes

$$[J^a_m, J^b_n] = f^{ab}{}_c J^c_{m+n} + m\kappa^{ab}\delta_{m,-n}. \tag{56}$$

This is (a representation of) the affine algebra $\hat{g}$. In the particular case where g is simple, $\kappa^{ab} = \mathrm{tr}(K t^a t^b) = k\delta^{ab}$, for some k, if we choose a suitable basis.

5.3. The Virasoro theory. Again following Frenkel and Zhu [20], we can construct the Virasoro theory in a similar way. In this case, the space V is one-dimensional, spanned by a vector L of weight 2 and we write $L(z) = V(L, z)$. We can again construct the amplitudes graphically by summing over all graphs with n vertices, where the vertices are labelled by the integers $1 \le j \le n$, and each vertex is connected by two lines (propagators) to other vertices. In a given graph, the vertices are now divided into loops, each loop containing at least two vertices. To each loop $\ell = (i_1, i_2, \dots, i_m)$, we associate a function

$$f_\ell(z_{i_1}, z_{i_2}, \dots, z_{i_m}) = \frac{c/2}{(z_{i_1} - z_{i_2})^2 (z_{i_2} - z_{i_3})^2 \cdots (z_{i_{m-1}} - z_{i_m})^2 (z_{i_m} - z_{i_1})^2}, \tag{57}$$

where c is a real number, and, to a graph, the product of the functions associated to its loops. [Since it corresponds to a factor of the form $(z_i - z_j)^{-2}$ rather than $(z_i - z_j)^{-1}$, each line or propagator might appropriately be represented by a double line.] The amplitudes $\langle L(z_1)L(z_2)\dots L(z_n)\rangle$ are then obtained by summing the functions associated with the various graphs with n vertices. [Note graphs related by reversing the direction of any loop contribute equally to this sum.]

These amplitudes determine the operator product expansion to be

$$L(z)L(\zeta) \sim \frac{c/2}{(z-\zeta)^4} + \frac{2L(\zeta)}{(z-\zeta)^2} + \frac{L'(\zeta)}{z-\zeta} + O(1) \tag{58}$$

which leads to the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m,-n}. \tag{59}$$

5.4. Lattice theories. Suppose that Λ is an even n-dimensional Euclidean lattice, so that, if $k \in \Lambda$, k^2 is an even integer. We introduce a basis $e_1, e_2, \ldots, e_n$ for Λ, so that any element k of Λ is an integral linear combination of these basis elements. We can introduce an algebra consisting of matrices γ_j, $1 \leq j \leq n$, such that $\gamma_j^2 = 1$ and $\gamma_i \gamma_j = (-1)^{e_i \cdot e_j} \gamma_j \gamma_i$. If we define $\gamma_k = \gamma_1^{m_1} \gamma_2^{m_2} \ldots \gamma_n^{m_n}$ for $k = m_1 e_1 + m_2 e_2 + \ldots + m_n e_n$, we can define quantities $\epsilon(k_1, k_2, \ldots, k_N)$, taking the values ± 1, by

$$\gamma_{k_1} \gamma_{k_2} \ldots \gamma_{k_N} = \epsilon(k_1, k_2, \ldots, k_N) \gamma_{k_1 + k_2 + \ldots + k_N}. \tag{60}$$

We define the theory associated to the lattice Λ by taking V to have a basis $\{\psi_k : k \in \Lambda\}$, where the weight of ψ_k is $\frac{1}{2}k^2$, and, writing $V(\psi_k, z) = V(k, z)$, the amplitudes to be

$$\langle V(k_1, z_1) V(k_2, z_2) \cdots V(k_N, z_n) \rangle = \epsilon(k_1, k_2, \ldots, k_N) \cdot \prod_{1 \leq i < j \leq N} (z_i - z_j)^{k_i \cdot k_j} \tag{61}$$

if $k_1 + k_2 + \ldots + k_N = 0$ and zero otherwise. The $\epsilon(k_1, k_2, \ldots, k_N)$ obey the conditions

$$\epsilon(k_1, k_2, \ldots, k_{j-1}, k_j, k_{j+1}, k_{j+2}, \ldots, k_N) = (-1)^{k_j \cdot k_{j+1}} \epsilon(k_1, k_2, \ldots, k_{j-1}, k_{j+1}, k_j, k_{j+2}, \ldots, k_N),$$

which guarantees locality, and

$$\epsilon(k_1, k_2, \ldots, k_j) \epsilon(k_{j+1}, \ldots, k_N) = \epsilon(k_1 + k_2 + \ldots + k_j, k_{j+1} + \ldots + k_N) \epsilon(k_1, k_2, \ldots, k_j, k_{j+1}, \ldots, k_N),$$

which implies the cluster decomposition property of Sect. 6.

5.5. A non-conformal example. The above examples actually define meromorphic *conformal* field theories, but since we have not yet defined what we mean by a theory to be conformal, it is instructive to consider also an example that satisfies the above axioms but is not conformal. The simplest such case is a slight modification of the $U(1)$ example described in 5.1: again we take V to be a one-dimensional vector space, spanned by a vector K, but now the grade of K is taken to be 2. Writing $K(z) \equiv V(K, z)$, the amplitudes of an odd number of K-fields vanishes, and in the case of an even number we have

$$\langle K(z_1) \cdots K(z_{2n}) \rangle = \frac{k^n}{2^n n!} \sum_{\pi \in S_{2n}} \prod_{j=1}^{n} \frac{1}{(z_{\pi(j)} - z_{\pi(j+n)})^4}. \tag{62}$$

It is not difficult to check that these amplitudes satisfy all the axioms we have considered so far. In this case the operator product expansion is of the form

$$K(z) K(w) \sim \frac{k}{(z-w)^4} + O(1), \tag{63}$$

and the algebra of modes is given by

$$[K_n, K_m] = \frac{k}{6} n(n^2 - 1) \delta_{n,-m}. \tag{64}$$

6. Cluster Decomposition

So far the axioms we have formulated do not impose any restrictions on the relative normalisation of amplitudes involving for example a different number of vectors in V, and the class of theories we are considering is therefore rather flexible. This is mirrored by the fact that it does not yet follow from our considerations that the spectrum of the operator L_0 is bounded from below, and since L_0 is in essence the energy of the corresponding physical theory, we may want to impose this constraint. In fact, we would like to impose the slightly stronger condition that the spectrum of L_0 is bounded by 0, and that there is precisely one state with eigenvalue equal to zero. This will follow (as we shall show momentarily) from the *cluster decomposition property*, which states that if we separate the variables of an amplitude into two sets and scale one set towards a fixed point (*e.g.* 0 or ∞) the behaviour of the amplitude is dominated by the product of two amplitudes, corresponding to the two sets of variables, multiplied by an appropriate power of the separation, specifically

$$\left\langle \prod_i V(\phi_i, \zeta_i) \prod_j V(\psi_j, \lambda z_j) \right\rangle \sim \left\langle \prod_i V(\phi_i, \zeta_i) \right\rangle \left\langle \prod_j V(\psi_j, z_j) \right\rangle \lambda^{-\Sigma h_j} \quad \text{as } \lambda \to 0, \tag{65}$$

where $\phi_i \in V_{h'_i}$, $\psi_j \in V_{h_j}$. It follows from Möbius invariance, that this is equivalent to

$$\left\langle \prod_i V(\phi_i, \lambda\zeta_i) \prod_j V(\psi_j, z_j) \right\rangle \sim \left\langle \prod_i V(\phi_i, \zeta_i) \right\rangle \left\langle \prod_j V(\psi_j, z_j) \right\rangle \lambda^{-\Sigma h'_i} \quad \text{as } \lambda \to \infty. \tag{66}$$

The cluster decomposition property extends also to vectors $\Phi_i, \Psi_j \in \mathcal{H}$. It is not difficult to check that the examples of the previous section satisfy this condition.

We can use the cluster decomposition property to show that the spectrum of L_0 is non-negative and that the vacuum is, in a sense, unique. To this end let us introduce the projection operators defined by

$$P_N = \oint_0 u^{L_0-N-1} du, \quad \text{for } N \in \mathbb{Z}. \tag{67}$$

In particular, we have

$$P_N \prod_j V(\psi_j, z_j)\Omega = \oint u^{h-N-1} V(\psi_j, u z_j)\Omega du, \tag{68}$$

where $h = \sum_j h_j$. It then follows that the P_N are projection operators

$$P_N P_M = 0, \text{ if } N \neq M, \quad P_N^2 = P_N, \quad \sum_N P_N = 1 \tag{69}$$

onto the eigenspaces of L_0,

$$L_0 P_N = N P_N. \tag{70}$$

For $N \leq 0$, we then have

$$\left\langle \prod_i V(\phi_i, \zeta_i) P_N \prod_j V(\psi_j, z_j) \right\rangle$$
$$= \oint_0 u^{\Sigma h_j - N - 1} \left\langle \prod_i V(\phi_i, \zeta_i) \prod_j V(\psi_j, u z_j) \right\rangle du$$
$$\sim \left\langle \prod_i V(\phi_i, \zeta_i) \right\rangle \left\langle \prod_j V(\psi_j, z_j) \right\rangle \oint_{|u|=\rho} u^{-N-1} du,$$

which, by taking $\rho \to 0$, is seen to vanish for $N < 0$ and, for $N = 0$, to give

$$P_0 \prod_j V(\psi_j, z_j) \Omega = \Omega \left\langle \prod_j V(\psi_j, z_j) \right\rangle, \tag{71}$$

and so $P_0 \Psi = \Omega \langle \Psi \rangle$. Thus the cluster decomposition property implies that $P_N = 0$ for $N < 0$, *i.e.* the spectrum of L_0 is non-negative, and that $\mathcal{H}_0$ is spanned by the vacuum Ω, which is thus the unique state with $L_0 = 0$.

As we have mentioned before the absence of negative eigenvalues of L_0 gives an upper bound on the order of the pole in the operator product expansion of two vertex operators, and thus to an upper bound in the sum in (47): if $\Phi, \Psi \in \mathcal{H}$ are of grade $L_0 \Phi = h_\Phi \Phi$, $L_0 \Psi = h_\Psi \Psi$, we have that $V_n(\Phi)\Psi = 0$ for $n > h_\Psi$ because otherwise $V_n(\Phi)\Psi$ would have a negative eigenvalue, $h_\Psi - n$, with respect to L_0. In particular, this shows that the leading singularity in $V(\Phi, z) V(\Psi, \zeta)$ is at most $(z - \zeta)^{-h_\Psi - h_\Phi}$.

The cluster property also implies that the space of states of the meromorphic field theory does not have any proper invariant subspaces in a suitable sense. To make this statement precise we must first give meaning to a subspace of the space of states of a conformal field theory. The space of states of the theory is really the collection of topological spaces $\mathcal{V}^{\mathcal{O}}$, where $\mathcal{O}$ is an open subset of $\mathbb{P}$ whose complement is path-connected. Recall that $\mathcal{V}^{\mathcal{O}} \subset \mathcal{V}^{\mathcal{O}'}$ if $\mathcal{O} \supset \mathcal{O}'$. By a subspace of the conformal field theory we shall mean subspaces $\mathcal{U}^{\mathcal{O}} \subset \mathcal{V}^{\mathcal{O}}$ specified for each open subset $\mathcal{O} \subset \mathbb{P}$ with path-connected complement, such that $\mathcal{U}^{\mathcal{O}} = \mathcal{U}^{\mathcal{O}'} \cap \mathcal{V}^{\mathcal{O}}$ if $\mathcal{O} \supset \mathcal{O}'$.

Proposition 5. *Suppose $\{\mathcal{U}^{\mathcal{O}}\}$ is an invariant closed subspace of $\{\mathcal{V}^{\mathcal{O}}\}$, i.e. $\mathcal{U}^{\mathcal{O}}$ is closed; $\mathcal{U}^{\mathcal{O}} = \mathcal{U}^{\mathcal{O}'} \cap \mathcal{V}^{\mathcal{O}}$ if $\mathcal{O} \supset \mathcal{O}'$; and $V(\psi, z)\mathcal{U}^{\mathcal{O}} \subset \mathcal{U}^{\mathcal{O}'}$ for all $\psi \in V$, where $z \in \mathcal{O}$, $z \notin \mathcal{O}' \subset \mathcal{O}$. Then $\{\mathcal{U}^{\mathcal{O}}\}$ is not a proper subspace, i.e. either $\mathcal{U}^{\mathcal{O}} = \mathcal{V}^{\mathcal{O}}$ for all $\mathcal{O}$, or $\mathcal{U}^{\mathcal{O}} = \{0\}$.*

Proof. Suppose that $\phi \in \mathcal{U}^{\mathcal{O}}$, $\psi_j \in V$, $z_j \in \mathcal{O}$, $z_j \notin \mathcal{O}' \subset \mathcal{O}$ and consider

$$\prod_{j=1}^{n} V(\psi_j, z_j)\phi \in \mathcal{U}^{\mathcal{O}'}. \tag{72}$$

Now, taking a suitable integral of the left-hand side,

$$P_0 \prod_{j=1}^{n} V(\psi_j, z_j)\phi = \lambda\Omega = \left\langle \prod_{j=1}^{n} V(\psi_j, z_j)\phi \right\rangle \Omega. \tag{73}$$

Thus either all the amplitudes involving ϕ vanish for all $\phi \in \mathcal{U}$, in which case $\mathcal{U} = \{0\}$, or $\Omega \in \mathcal{U}^{\mathcal{O}'}$ for some $\mathcal{O}'$, in which case it is easy to see that $\Omega \in \mathcal{U}^{\mathcal{O}}$ for all $\mathcal{O}$ and it follows that $\mathcal{U}^{\mathcal{O}} = \mathcal{V}^{\mathcal{O}}$ for all $\mathcal{O}$.

The cluster property also implies that the image of $\mathcal{B}_{\mathcal{C}}$ in $\mathcal{V}_{\mathcal{C}}^{\mathcal{O}}$ is faithful. To show that the images of the elements $\boldsymbol{\psi}, \boldsymbol{\psi}' \in \mathcal{B}_{\mathcal{C}}$ are distinct we note that otherwise $\eta_{\boldsymbol{\phi}}(\boldsymbol{\psi}) = \eta_{\boldsymbol{\phi}}(\boldsymbol{\psi}')$ for all $\boldsymbol{\phi} \in \mathcal{V}_{\mathcal{O}}$ with $\boldsymbol{\phi}$ as in (4). By taking m in (4) to be sufficiently large, dividing the ζ_i, $1 \leq i \leq m$, into n groups which we allow to approach the z_j, $1 \leq j \leq n$, successively. The cluster property then shows that these must be the same points as the z'_j, $1 \leq j \leq n'$ in $\boldsymbol{\psi}'$ and that $\psi_i = \mu_j \psi'_j$ for some $\mu_j \in \mathbb{C}$ with $\prod_{j=1}^{n} \mu_j = 1$, establishing that $\boldsymbol{\psi} = \boldsymbol{\psi}'$ as elements of $\mathcal{V}_{\mathcal{C}}^{\mathcal{O}}$. □

7. Conformal Symmetry

So far our axioms do not require that our amplitudes correspond to a conformal field theory, only that the theory have a Möbius invariance, and indeed, as we shall see, the example in 5.3 is not conformally invariant. Further, what we shall discuss in the sections which follow the present one will not depend on a conformal structure, except where we explicitly mention it; in this sense, the present section is somewhat of an interlude. On the other hand, the conformal symmetry is crucial for more sophisticated considerations, in particular the theory on higher genus Riemann surfaces, and therefore forms a very important part of the general framework.

Let us first describe a construction by means of which a potentially new theory can be associated to a given theory, and explain then in terms of this construction what it means for a theory to be conformal.

Suppose we are given a theory that is specified by a space V and amplitudes $\mathcal{A} = \{f\}$. Let us denote by $\hat{V}$ the vector space that is obtained from V by appending a vector L of grade two, and let us write $V(L, z) = L(z)$. The amplitudes involving only fields in V are given as before, and the amplitude

$$\langle \prod_{j=1}^{m} L(w_j) \prod_{i=1}^{n} V(\psi_i, z_i) \rangle, \tag{74}$$

where $\psi_i \in V_{h_i}$ is defined as follows: we associate to each of the $n+m$ fields a point, and then consider the (ordered) graphs consisting of loops where each loop contains at most one of the points associated to the ψ_i, and each point associated to an L is a vertex of precisely one loop. (The points associated to ψ_i may be vertices of an arbitrary number of loops.) To each loop whose vertices only consist of points corresponding to L we associate the same function as before in Sect. 5.3, and to the loop $(z_i, w_{\pi(1)}, \ldots, w_{\pi(l)})$ we associate the expression

$$\prod_{j=1}^{l-1} \frac{1}{(w_{\pi(j)} - w_{\pi(j+1)})^2} \left(\frac{h_i}{(w_{\pi(1)} - z_i)(w_{\pi(l)} - z_i)} + \frac{1}{2} \left[\frac{1}{(w_{\pi(1)} - z_i)} \frac{d}{dz_i} + \frac{1}{(w_{\pi(l)} - z_i)} \frac{d}{dz_i} \right] \right). \tag{75}$$

We then associate to each graph the product of the expressions associated to the different loops acting on the amplitude which is obtained from (74) upon removing $L(w_1) \cdots$

$L(w_m)$, and the total amplitude is the sum of the functions associated to all such (ordered) graphs. (The product of the expressions of the form (75) is taken to be "normal ordered" in the sense that all derivatives with respect to z_i only act on the amplitude that is obtained from (74) upon excising the Ls; in this way, the product is independent of the order in which the expressions of the form (75) are applied.)

We extend this definition by multilinearity to amplitudes defined for arbitrary states in $\hat{V}$. It follows immediately that the resulting amplitudes are local and meromorphic; in Appendix E we shall give a more explicit formula for the extended amplitudes, and use it to prove that the amplitudes also satisfy the Möbius covariance and the cluster property. In terms of conventional conformal field theory, the construction treats all quasiprimary states in V as primary with respect to the Virasoro algebra of the extended theory; this is apparent from the formula given in Appendix E.

We can generalise this definition further by considering in addition graphs which contain "double loops" of the form (z_i, w_j) for those points z_i which correspond to states in V of grade two, where in this case neither z_i nor w_j can be a vertex of any other loop. We associate the function

$$\frac{c_\psi/2}{(z_i - w_j)^4} \tag{76}$$

to each such loop (where c_ψ is an arbitrary linear functional on the states of weight two in V), and the product of the different expressions corresponding to the different loops in the graph act in this case on the amplitude (74), where in addition to all L-fields also the fields corresponding to $V(\psi_i, z_i)$ (for each i which appears in a double loop) have been removed. It is easy to see that this generalisation also satisfies all axioms.

This construction typically modifies the structure of the meromorphic field theory in the sense that it changes the operator product expansion (and thus the commutators of the corresponding modes) of vectors in V; this is for example the case for the "non-conformal" model described in Sect. 5.5. If we introduce the field L as described above, we find the commutation relations

$$[L_m, K_n] = (m-n)\, K_{m+n} + \frac{c_K}{12}\, m\, (m^2-1)\, \delta_{m,-n}. \tag{77}$$

However, this is incompatible with the original commutator in (64): the Jacobi identity requires that

$$\begin{aligned} 0 &= [L_m, [K_n, K_l]] + [K_n, [K_l, L_m]] + [K_l, [L_m, K_n]] \\ &= (l-m)\, [K_n, K_{l+m}] + (m-n)\, [K_l, K_{m+n}] \\ &= \frac{k}{6}\, \delta_{l+m+n,0} \Big[-(l-m)\, (l+m)\, \Big((l+m)^2 - 1 \Big) + (2m+l)\, l\, (l^2-1) \Big] \\ &= \frac{k}{6}\, \delta_{l+m+n,0}\, m\, (m^2-1)\, (2l+m) \end{aligned}$$

and this is not satisfied unless $k = 0$ (in which case the original theory is trivial). In fact, the introduction of L modifies (64) as

$$\begin{aligned} [K_m, K_n] &= \frac{k}{6} m(m^2-1)\delta_{m,-n} + \frac{k}{a}(m-n) Z_{m+n}, \\ [L_m, K_n] &= \frac{c_K}{12} m(m^2-1)\delta_{m,-n} + (m-n) K_{m+n}, \\ [Z_m, K_n] &= 0, \end{aligned}$$

$$[L_m, Z_n] = \frac{a}{12}m(m^2-1)\delta_{m,-n} + (m-n)Z_{m+n},$$
$$[L_m, L_n] = \frac{c}{12}m(m^2-1)\delta_{m,-n} + (m-n)L_{m+n},$$
$$[Z_m, Z_n] = 0,$$

where a is non-zero and can be set to equal k by rescaling Z, and the Z_n are the modes of a field of grade two. This set of commutators then satisfies the Jacobi identities. It also follows from the fact that the commutators of Z with K and Z vanish, that amplitudes that involve only K-fields and at least one Z-field vanish; in this way we recover the original amplitudes and commutators.

The construction actually depends on the choice of V (as well as the values of c_ψ and c), and therefore does not only depend on the equivalence class of meromorphic field theories. However, we can ask whether a given equivalence class of meromorphic field theories contains a representative $(V, \mathcal{A})$ (i.e. a choice of V that gives an equivalent description of the theory) for which $(\hat{V}, \hat{\mathcal{A}})$ is equivalent to $(V, \mathcal{A})$; if this is the case, we call the meromorphic field theory *conformal*. It follows directly from the definition of equivalence that a meromorphic field theory is conformal if and only if there exists a representative $(V, \mathcal{A})$ and a vector $L^0 \in V$ (of grade two) so that

$$\langle\Big(L(w) - L^0(w)\Big)\prod_{j=1}^{n} V(\psi_j, z_j)\rangle = 0 \tag{78}$$

for all $\psi_j \in V$, where L is defined as above. In this case, the linear functional c_ψ is defined by

$$c_\psi = 2(w-z)^4\langle L^0(w)V(\psi, z)\rangle.$$

In the case of the non-conformal example of Sect. 5.5, it is clear that (78) cannot be satisfied as the Fock space only contains one vector of grade two, $L^0 = \alpha K_{-2}\Omega$, and L^0 does not satisfy (78) for any value of α. On the other hand, for the example of Sect. 5.1, we can choose

$$L^0 = \frac{1}{2k} J_{-1}J_{-1}\Omega, \tag{79}$$

and this then satisfies (78). Similarly, in the case of the example of Sect. 5.2, we can choose

$$L^0 = \frac{1}{2(k+Q)}\sum_a J^a_{-1}J^a_{-1}\Omega,$$

where Q is the dual Coxeter number of g (i.e. the value of the quadratic Casimir in the adjoint representation), and again (78) is satisfied for this choice of L^0 (and the above choice of V). This construction is known as the "Sugawara construction".

For completeness it should be mentioned that the modes of the field L (that is contained in the theory in the conformal case) satisfy the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m,-n},$$

where c is the number that appears in the above definition of L. Furthermore, the modes L_m with $m = 0, \pm 1$ agree with the Möbius generators of the theory.

8. Representations

In order to introduce the concept of a representation of a meromorphic conformal field theory or conformal algebra, we consider a collection of densities more general than those used in Sect. 2 to define the meromorphic conformal field theory itself. The densities we now consider are typically defined on a cover of the Riemann sphere, $\mathbb{P}$, rather than $\mathbb{P}$ itself. We consider densities which are functions of variables u_i, $1 \le i \le N$, and z_j, $1 \le j \le n$, which are analytic if no two of these $N+n$ variables are equal, may have poles at $z_i = z_j$, $i \ne j$, or $z_i = u_j$, and may be branched about $u_i = u_j$, $i \ne j$. To define a representation, we need the case where $N = 2$, in which the densities are meromorphic in all but two of the variables.

Starting again with $V = \oplus_h V_h$, together with two finite-dimensional spaces W_α and W_β (which may be one-dimensional), we suppose that, for each integer $n \ge 0$, and $z_i \in \mathbb{P}$ and u_1, u_2 on some branched cover of $\mathbb{P}$, and for any collection of vectors $\psi_i \in V_{h_i}$ and $\chi_1 \in W_\alpha$, $\chi_2 \in W_\beta$, we have a density

$$\begin{aligned} &g(\psi_1, \ldots, \psi_n; z_1, \ldots, z_n; \chi_1, \chi_2; u_1, u_2) \\ &\quad\equiv \langle V(\psi_1, z_1) V(\psi_2, z_2) \cdots V(\psi_n, z_n) W_\alpha(\chi_1, u_1) W_\beta(\chi_2, u_2)\rangle \\ &\qquad \cdot \prod_{j=1}^{n} \left(dz_j\right)^{h_j} (du_1)^{r_1} (du_2)^{r_2}, \end{aligned} \tag{80}$$

where r_1, r_2 are real numbers, which we call the *conformal weights* of χ_1 and χ_2, respectively. The amplitudes

$$\langle V(\psi_1, z_1) V(\psi_2, z_2) \cdots V(\psi_n, z_n) W_\alpha(\chi_1, u_1) W_\beta(\chi_2, u_2)\rangle \tag{81}$$

are taken to be multilinear in the ψ_j and χ_1, χ_2, and invariant under the exchange of (ψ_i, z_i) with (ψ_j, z_j), and meromorphic as a function of the z_j, analytic except for possible poles at $z_i = z_j$, $i \ne j$, and $z_i = u_1$ or $z_i = u_2$. As functions of u_1, u_2, the amplitudes are analytic except for the possible poles at $u_1 = z_i$ or $u_2 = z_i$ and a possible branch cut at $u_1 = u_2$. We denote a collection of such densities by $\mathcal{R} = \{g\}$.

Just as before, given an open set $\mathcal{C} \subset \mathbb{P}$ we introduced spaces $\mathcal{B}_\mathcal{C}$, whose elements are of the form (3), so we can now introduce sets, $\mathcal{B}_{\mathcal{C}\alpha\beta}$, labelled by finite collections of $\psi_i \in V_{h_i}$, $z_i \in \mathcal{C} \subset \mathbb{P}$, $i = 1, \ldots, n$, $n \in \mathbb{N}$ and $z_i \ne z_j$ if $i \ne j$, together with $\chi_1 \in W_\alpha$, $\chi_2 \in W_\beta$ and $u_1, u_2 \in \mathcal{C}$, $u_1 \ne u_2$ and $z_i \ne u_j$, denoted by

$$\begin{aligned} \boldsymbol{\chi} &= V(\psi_1, z_1) V(\psi_2, z_2) \cdots V(\psi_n, z_n) W_\alpha(\chi_1, u_1) W_\beta(\chi_2, u_2)\Omega \\ &\equiv \prod_{i=1}^{n} V(\psi_i, z_i) W_\alpha(\chi_1, u_1) W_\beta(\chi_2, u_2)\Omega. \end{aligned} \tag{82}$$

We again immediately identify different $\boldsymbol{\chi} \in \mathcal{B}_{\mathcal{C}\alpha\beta}$ with the other elements of $\mathcal{B}_{\mathcal{C}\alpha\beta}$ obtained by replacing each ψ_j in (82) by $\mu_j \psi_j$, $1 \le j \le n$, χ_i by $\lambda_i \chi_i$, $i = 1, 2$, where $\lambda_1, \lambda_2, \mu_j \in \mathbb{C}$ and $\lambda_1 \lambda_2 \prod_{j=1}^n \mu_j = 1$.

Proceeding as before, we introduce the vector space $\mathcal{V}_{\mathcal{C}\alpha\beta}$ with basis $\mathcal{B}_{\mathcal{C}\alpha\beta}$ and we cut it down to size *exactly* as before, *i.e.* we note that if we introduce another open set $\mathcal{O} \subset \mathbb{P}$, with $\mathcal{O} \cap \mathcal{C} = \varnothing$, and, as in (4) write

$$\boldsymbol{\phi} = V(\phi_1, \zeta_1) V(\phi_2, \zeta_2) \cdots V(\phi_m, \zeta_m)\Omega \in \mathcal{B}_\mathcal{O}, \tag{83}$$

where $\phi_j \in V_{k_j}$, $j = 1, \dots m$, each $\boldsymbol{\phi} \in \mathcal{B}_{\mathcal{O}}$ defines a map on $\mathcal{B}_{\mathcal{C}\alpha\beta}$ by

$$\eta_{\boldsymbol{\phi}}(\boldsymbol{\chi}) = (\boldsymbol{\phi}, \boldsymbol{\chi}) = \left\langle \prod_{i=1}^{m} V(\phi_i, \zeta_i) \prod_{i=1}^{n} V(\psi_i, z_i) W_\alpha(\chi_1, u_1) W_\beta(\chi_2, u_2) \right\rangle . \tag{84}$$

Again $\eta_{\boldsymbol{\phi}}$ extends by linearity to a map $\mathcal{V}_{\mathcal{C}\alpha\beta} \to \mathbb{C}$ and we consider the space, $\widetilde{\mathcal{V}}_{\mathcal{C}\alpha\beta}$, of sequences $\boldsymbol{X} = (X_1, X_2, \dots)$, $X_j \in \mathcal{V}_{\mathcal{C}\alpha\beta}$, for which $\eta_{\boldsymbol{\phi}}(X_j)$ converges uniformly on each of the family of compact sets of the form (7). We write $\eta_{\boldsymbol{\phi}}(\boldsymbol{X}) = \lim_{j\to\infty} \eta_{\boldsymbol{\phi}}(X_j)$ and define the space $\mathcal{V}^{\mathcal{O}}_{\mathcal{C}\alpha\beta}$ as being composed of the equivalence classes of such sequences, identifying two sequences $\boldsymbol{X}_1, \boldsymbol{X}_2$, if $\eta_{\boldsymbol{\phi}}(\boldsymbol{X}_1) = \eta_{\boldsymbol{\phi}}(\boldsymbol{X}_2)$ for all $\boldsymbol{\phi} \in \mathcal{B}_{\mathcal{O}}$. Using the same arguments as in the proof of Theorem 1 (see Appendix C), it can be shown that the space $\mathcal{V}^{\mathcal{O}}_{\mathcal{C}\alpha\beta}$ is independent of $\mathcal{C}$, provided that the complement of $\mathcal{O}$ is path-connected; in this case we write $\mathcal{V}^{\mathcal{O}}_{\alpha\beta} \equiv \mathcal{V}^{\mathcal{O}}_{\mathcal{C}\alpha\beta}$. We can define a family of seminorms for $\mathcal{V}^{\mathcal{O}}_{\alpha\beta}$ by $||\boldsymbol{X}||_{\boldsymbol{\phi}} = |\eta_{\boldsymbol{\phi}}(\boldsymbol{X})|$, where $\boldsymbol{\phi}$ is an arbitrary element of $\mathcal{B}_{\mathcal{O}}$, and the natural topology on $\mathcal{V}^{\mathcal{O}}_{\alpha\beta}$ is the topology that is induced by this family of seminorms. (This is to say, that a sequence of states in $\boldsymbol{X}_j \in \mathcal{V}^{\mathcal{O}}_{\alpha\beta}$ converges if and only if $\eta_{\boldsymbol{\phi}}(\boldsymbol{X}_j)$ converges for every $\boldsymbol{\phi} \in \mathcal{B}_{\mathcal{O}}$.)

So far we have not specified a relationship between the spaces $\mathcal{V}^{\mathcal{O}}$, which define the conformal field theory, and the new spaces $\mathcal{V}^{\mathcal{O}}_{\alpha\beta}$, which we have now introduced to define a representation of it. Such a relation is an essential part of the definition of a representation; it has to express the idea that the two spaces define the same relations between combinations of vectors in the sets $\mathcal{B}_{\mathcal{C}}$. To do this consider the space of all continuous linear functionals on $\mathcal{V}^{\mathcal{O}}$, the *dual* space of $\mathcal{V}^{\mathcal{O}}$, which we will denote $(\mathcal{V}^{\mathcal{O}})'$, and also the dual, $(\mathcal{V}^{\mathcal{O}}_{\alpha\beta})'$, of $\mathcal{V}^{\mathcal{O}}_{\alpha\beta}$. It is natural to consider these dual spaces as topological vector spaces with the weak topology: for each $f \in (\mathcal{V}^{\mathcal{O}})'$, we can consider the (uncountable) family of seminorms defined by $||f||_\Psi \equiv |f(\Psi)|$, where Ψ is an arbitrary element of $\mathcal{V}^{\mathcal{O}}$ (and similarly for $(\mathcal{V}^{\mathcal{O}}_{\alpha\beta})'$). The weak topology is then the topology that is induced by this family of seminorms (so that $f_n \to f$ if and only if $f_n(\Psi) \to f(\Psi)$ for each $\Psi \in \mathcal{V}^{\mathcal{O}}$).

Every element of $\boldsymbol{\phi} \in \mathcal{B}_{\mathcal{O}}$ defines a continuous linear functional both on $\mathcal{V}^{\mathcal{O}}$ and on $\mathcal{V}^{\mathcal{O}}_{\alpha\beta}$, each of which we shall denote by $\eta_{\boldsymbol{\phi}}$, and the linear span of the set of all linear functionals that arise in this way is dense in both $(\mathcal{V}^{\mathcal{O}})'$ and $(\mathcal{V}^{\mathcal{O}}_{\alpha\beta})'$. We therefore have a map from a dense subspace of $(\mathcal{V}^{\mathcal{O}})'$ to a dense subspace of $(\mathcal{V}^{\mathcal{O}}_{\alpha\beta})'$, and the condition for the amplitudes (80) to define a *representation* of the meromorphic (conformal) field theory whose spaces of states are given by $\mathcal{V}^{\mathcal{O}}$ is that this map extends to a *continuous* map between the dual spaces, i.e. that there exists a continuous map

$$\iota : (\mathcal{V}^{\mathcal{O}})' \to (\mathcal{V}^{\mathcal{O}}_{\alpha\beta})' \quad \text{such that} \quad \iota(\eta_{\boldsymbol{\phi}}) = \eta_{\boldsymbol{\phi}}. \tag{85}$$

This in essence says that $\mathcal{V}^{\mathcal{O}}_{\alpha\beta}$ will not distinguish limits of linear combinations of $\mathcal{B}_{\mathcal{O}}$ not distinguished by $\mathcal{V}^{\mathcal{O}}$.

Given a collection of densities $\mathcal{R}$ we can construct (in a similar way as before for the collection of amplitudes $\mathcal{A}$) spaces of states $\mathcal{V}^{\mathcal{O}}_{\alpha}$ and $\mathcal{V}^{\mathcal{O}}_{\beta}$, on which the vertex operators of the meromorphic theory are well-defined operators. By the by now familiar scheme, let us introduce the set $\mathcal{B}_{\mathcal{C}\alpha}$ that is labelled by finite collections of $\psi_i \in V_{h_i}$, $z_i \in \mathcal{C} \subset \mathbb{P}$,

$i = 1, \ldots, n$, $n \in \mathbb{N}$ and $z_i \neq z_j$ if $i \neq j$, together with $\chi \in W_\alpha$ and $u \in \mathcal{C}$, $z_i \neq u$, denoted by

$$\begin{aligned} \boldsymbol{\chi} &= V(\psi_1, z_1)V(\psi_2, z_2)\cdots V(\psi_n, z_n)W_\alpha(\chi, u)\Omega \\ &\equiv \prod_{i=1}^{n} V(\psi_i, z_i)W_\alpha(\chi, u)\Omega. \end{aligned} \tag{86}$$

We again immediately identify different $\boldsymbol{\chi} \in \mathcal{B}_{\mathcal{C}\alpha}$ with the other elements of $\mathcal{B}_{\mathcal{C}\alpha}$ obtained by replacing each ψ_j in (86) by $\mu_j\psi_j$, $1 \leq j \leq n$, χ by $\lambda\chi$, where $\lambda, \mu_j \in \mathbb{C}$ and $\lambda\prod_{j=1}^{n}\mu_j = 1$. We also define $\mathcal{B}_{\mathcal{C}\beta}$ analogously (by replacing $\chi \in W_\alpha$ by $\chi \in W_\beta$).

We then introduce the vector space $\mathcal{V}_{\mathcal{C}\alpha}$ with basis $\mathcal{B}_{\mathcal{C}\alpha}$, and we cut it down to size exactly as before by considering the map analogous to (84), where now $\boldsymbol{\phi} \in \mathcal{B}_{\mathcal{O}\beta}$. The resulting space is denoted by $\mathcal{V}_{\mathcal{C}\alpha}^{\mathcal{O}\beta}$, and is again independent of $\mathcal{C}$ provided that the complement of $\mathcal{O}$ is path-connected; in this case we write $\mathcal{V}_{\alpha}^{\mathcal{O}\beta} \equiv \mathcal{V}_{\mathcal{C}\alpha}^{\mathcal{O}\beta}$. It also has a natural topology induced by the seminorms $|\eta_{\boldsymbol{\phi}}(\boldsymbol{X})|$, where now $\boldsymbol{\phi} \in \mathcal{B}_{\mathcal{O}\beta}$. With respect to this topology, the span of $\mathcal{B}_{\mathcal{C}\alpha}$ is dense in $\mathcal{V}_{\alpha}^{\mathcal{O}\beta}$. We can similarly consider the spaces $\mathcal{V}_{\beta}^{\mathcal{O}\alpha}$ by exchanging the rôles of W_α and W_β.

For $\psi \in V$, a vertex operator $V(\psi, z)$ can be defined as an operator $V(\psi, z) : \mathcal{V}_{\alpha}^{\mathcal{O}\beta} \to \mathcal{V}_{\alpha}^{\mathcal{O}'\beta}$, where $z \in \mathcal{O}$ but $z \notin \mathcal{O}' \subset \mathcal{O}$, by defining its action on the total subset $\mathcal{B}_{\mathcal{C}\alpha}$, where $\mathcal{C} \cap \mathcal{O} = \varnothing$

$$V(\psi, z)\boldsymbol{\chi} = V(\psi, z)V(\psi_1, z_1)\cdots V(\psi_n, z_n)W_\alpha(\chi, u)\Omega,$$

and $\boldsymbol{\chi} \in \mathcal{B}_{\mathcal{C}\alpha}$ is as in (86). The image is in $\mathcal{B}_{\mathcal{C}'\alpha}$ for any $\mathcal{C}' \supset \mathcal{C}$ which contains z, and we can choose $\mathcal{C}'$ such that $\mathcal{C}' \cap \mathcal{O}' = \varnothing$. This then extends by linearity to a map $\mathcal{V}_{\mathcal{C}\alpha} \to \mathcal{V}_{\mathcal{C}'\alpha}$, and we can show, by analogous arguments as before, that it induces a map $\mathcal{V}_{\mathcal{C}\alpha}^{\mathcal{O}\beta} \to \mathcal{V}_{\mathcal{C}'\alpha}^{\mathcal{O}'\beta}$.

By the same arguments as before in Sect. 4, this definition can be extended to vectors Ψ of the form (34) that span the Fock space of the meromorphic theory. The actual Fock space $\mathcal{H}^{\mathcal{O}'}$ (that is typically a quotient space of the free vector space spanned by the vectors of the form (34)) is a subspace of $(\mathcal{V}^{\mathcal{O}})'$ provided that $\mathcal{O}' \cup \mathcal{O} = \mathbb{P}$, and if the amplitudes define a representation, $\iota(\mathcal{H}^{\mathcal{O}'}) \subset (\mathcal{V}_{\alpha\beta}^{\mathcal{O}})'$ because of (85). In this case it is then possible to define vertex operators $V(\Psi, z) : \mathcal{V}_{\alpha}^{\mathcal{O}\beta} \to \mathcal{V}_{\alpha}^{\mathcal{O}'\beta}$ for arbitrary elements of the Fock space $\mathcal{H}$, and this is what is usually thought to be the defining property of a representation. By the same argument the vertex operators are also well-defined for elements in $\mathsf{V}_{\mathcal{C}}$ for suitable $\mathcal{C}$.

There exists an alternative criterion for a set of densities to define a representation, which is in essence due to Montague [17], and which throws considerable light on the nature of conformal field theories and their representations. (Indeed, we shall use it to construct an example of a representation for the $u(1)$-theory below.)

Theorem 6. *The densities (80) define a representation provided that, for each open set $\mathcal{O} \subset \mathbb{P}$ with path-connected complement and $u_1, u_2 \notin \mathcal{O}$, there is a state $\Sigma_{\alpha\beta}(u_1, u_2; \chi_1, \chi_2) \in \mathcal{V}^{\mathcal{O}}$ that is equivalent to $W_\alpha(u_1, \chi_1)W_\beta(u_2, \chi_2)$ in the sense that the amplitudes of the representation are given by $\eta_{\boldsymbol{\phi}}(\Sigma_{\alpha\beta})$:*

$$\left\langle \prod_{i=1}^{m} V(\phi_i, \zeta_i) W_\alpha(u_1, \chi_1) W_\beta(u_2, \chi_2) \right\rangle = \left\langle \prod_{i=1}^{m} V(\phi_i, \zeta_i) \Sigma_{\alpha\beta}(u_1, u_2; \chi_1, \chi_2) \right\rangle, \quad (87)$$

where $\zeta_i \in \mathcal{O}$.

The proof of this theorem depends on the following:

Lemma. *There exist sequences* $e_i \in \mathcal{V}_C^{\mathcal{O}}$, $f_i \in (\mathcal{V}_C^{\mathcal{O}})'$, *dense in the appropriate topologies, such that* $f_j(e_i) = \delta_{ij}$ *and such that* $\sum_{i=1}^{\infty} e_i f_i(\Psi)$ *converges to* Ψ *for all* $\Psi \in \mathcal{V}_C^{\mathcal{O}}$.

To prove the lemma, take the $\{e_i\}$ to be formed from the union of the bases of the eigenspaces $\mathcal{H}_N$ of L_0, which we have taken to be finite-dimensional, taken in order, $N = 0, 1, 2, \ldots$. Using the projection operators P_N defined by (67), we have that $\sum_{N=0}^{\infty} P_N \Psi = \Psi$ and $P_n \Psi$ can be written as a sum of the e_i which are basis elements of $\mathcal{H}_N$, with coefficients $f_i(\Psi)$ which depend continuously and linearly on Ψ. It is then clear that $\sum_{i=1}^{\infty} e_i f_i(\Psi) = \Psi$ and, if $\eta \in (\mathcal{V}^{\mathcal{O}})'$, $\eta = \sum_{i=1}^{\infty} f_i \eta(e_i)$, showing that $\{e_i\}$ is dense in $\mathcal{V}^{\mathcal{O}}$ and $\{f_i\}$ is dense in $(\mathcal{V}^{\mathcal{O}})'$.

Proof of Theorem 6. Assuming we have a continuous map $\iota : (\mathcal{V}^{\mathcal{O}})' \to (\mathcal{V}_{\alpha\beta}^{\mathcal{O}})'$, let us define $\Sigma_{\alpha\beta}(u_1, u_2; \chi_1, \chi_2)$ by

$$\Sigma_{\alpha\beta}(u_1, u_2; \chi_1, \chi_2) = \sum_i e_i \iota(f_i) \Big(W_\alpha(u_1, \chi_1) W_\beta(u_2, \chi_2) \Big). \quad (88)$$

Then, if $\eta_{\boldsymbol{\phi}} = \sum_j \lambda_j f_j$,

$$\begin{aligned} \left\langle \prod_{i=1}^{m} V(\phi_i, \zeta_i) W_\alpha(u_1, \chi_1) W_\beta(u_2, \chi_2) \right\rangle &= \eta_{\boldsymbol{\phi}} \Big(\Sigma_{\alpha\beta}(u_1, u_2; \chi_1, \chi_2) \Big) \\ &= \sum_{ji} \lambda_j f_j(e_i)\, \iota(f_i) \Big(W_\alpha(u_1, \chi_1) W_\beta(u_2, \chi_2) \Big) \\ &= \iota(\eta_{\boldsymbol{\phi}}) \Big(W_\alpha(u_1, \chi_1) W_\beta(u_2, \chi_2) \Big) \\ &= \left\langle \prod_{i=1}^{m} V(\phi_i, \zeta_i) \Sigma_{\alpha\beta}(u_1, u_2; \chi_1, \chi_2) \right\rangle, \end{aligned}$$

and the convergence of (88) can be deduced from this.

Conversely, suppose that (87) holds; then

$$\prod_{i=1}^{n} V(\psi_i, \zeta_i) W_\alpha(u_1, \chi_1) W_\beta(u_2, \chi_2) \to \prod_{i=1}^{n} V(\psi_i, \zeta_i) \Sigma_{\alpha\beta}(u_1, u_2; \chi_1, \chi_2) \quad (89)$$

defines a continuous map $\mathcal{V}_{\alpha\beta}^{\mathcal{O}} \to \mathcal{V}^{\mathcal{O}}$ (where $\zeta_i, u_1, u_2 \notin \mathcal{O}$), and this induces a dual map $\iota : (\mathcal{V}^{\mathcal{O}})' \to (\mathcal{V}_{\alpha\beta}^{\mathcal{O}})'$, continuous in the weak topology, satisfying $\iota(\eta_{\boldsymbol{\phi}}) = \eta_{\boldsymbol{\phi}}$, i.e. (85) holds.

The map (89) defines an isomorphism of $\mathcal{V}_{\alpha\beta}^{\mathcal{O}}$ onto $\mathcal{V}^{\mathcal{O}}$: it is onto for otherwise its image would define an invariant subspace of $\mathcal{V}^{\mathcal{O}}$ and the argument of Proposition 5 shows that this must be the whole space; and it is an injection because if it maps a vector $\boldsymbol{X}$ to zero, $\eta_{\boldsymbol{\phi}}(\boldsymbol{X})$ must vanish for all $\boldsymbol{\phi} \in \mathcal{B}_{\mathcal{O}}$, implying $\boldsymbol{X} = 0$. □

9. Möbius Covariance, Fock Spaces and the Equivalence of Representations

We shall now assume that each density in the collection $\mathcal{R}$ is invariant under the action of the Möbius transformations, *i.e.* that the amplitudes satisfy

$$\begin{aligned}&\langle\prod_{i=1}^{n} V(\psi_i, z_i) W_\alpha(\chi_1, u_1) W_\beta(\chi_2, u_2)\rangle \\ &= \langle\prod_{i=1}^{n} V(\psi_i, \gamma(z_i)) W_\alpha(\chi_1, \gamma(u_1)) W_\beta(\chi_2, \gamma(u_2))\rangle \\ &\quad \cdot \prod_{l=1}^{2}(\gamma'(u_l))^{r_l} \prod_{i=1}^{n}(\gamma'(z_i))^{h_i},\end{aligned} \tag{90}$$

where r_l are the real numbers which appear in the definition of the densities, and h_i is the grade of ψ_i.

In this case, we can define operators $U(\gamma)$, m apping $\mathcal{V}_\alpha^{\mathcal{O}\beta}$ to $\mathcal{V}_\alpha^{\mathcal{O}_\gamma \beta}$; on the total subset $\mathcal{B}_{\mathcal{C}\alpha}$, where $\mathcal{C} \cap \mathcal{O} = \varnothing$, these operators are defined by

$$U(\gamma)\boldsymbol{\chi} = V(\psi_1, \gamma(z_1)) \cdots V(\psi_n, \gamma(z_n)) W_\alpha(\chi, \gamma(u))\Omega \prod_{i=1}^{n}(\gamma'(z_i))^{h_i}\, \gamma'(u)^{r_1}, \tag{91}$$

where $\boldsymbol{\chi}$ is defined as in (86), and h_i is the grade of ψ_i, $i = 1, \ldots, n$. This definition extends by linearity to operators being defined on $\mathcal{V}_{\mathcal{C}\alpha}$, and by analogous arguments to those in Sect. 3, this extends to a well-defined map $\mathcal{V}_\alpha^{\mathcal{O}\beta} \to \mathcal{V}_\alpha^{\mathcal{O}_\gamma \beta}$. If we choose two points z_∞ and z_0 as before, we can introduce the Möbius generators L_0^M, $L_{\pm 1}^M$ which are well-defined on these spaces.

We define the Fock space $\mathcal{H}_\alpha^{\mathcal{O}} \subset \mathcal{V}_\alpha^{\mathcal{O}}$ to be the space spanned by finite linear combinations of vectors of the form

$$\Phi = V_{n_1}(\psi_1) \cdots V_{n_N}(\psi_N) W_\alpha(\chi, 0)\Omega, \tag{92}$$

where $\psi_j \in V$, $\chi \in W_\alpha$ and $n_j \in \mathbb{Z}$, $1 \leq j \leq N$. Here the modes $V_n(\psi)$ are defined as before in (32) where the contour encircles the point $0 \in \mathcal{C}$, and this still makes sense since the amplitudes $\mathcal{R}$ are not branched about $u_i = z_j$. It is clear that $\mathcal{H}_\alpha^{\mathcal{O}}$ is a dense subspace of $\mathcal{V}_\alpha^{\mathcal{O}}$, and that it is independent of $\mathcal{O}$; where no ambiguity arises we shall therefore denote it by $\mathcal{H}_\alpha$. By construction, $W_\alpha \subset \mathcal{H}_\alpha$. We can also define $W_\beta \subset \mathcal{H}_\beta$ in the same way.

As before it is then possible to extend the amplitudes $\mathcal{R}$ to amplitudes being defined for $\chi_1 \in \mathcal{H}_\alpha$ and $\chi_2 \in \mathcal{H}_\beta$ (rather than $\chi_1 \in W_\alpha$ and $\chi_2 \in W_\beta$), and for the subset of quasiprimary states in $\mathcal{H}_\alpha$ and $\mathcal{H}_\beta$ (i.e. for the states that are annihilated by L_1^M defined above), the Möbius properties are analogous to those in (90).

As in the case of the meromorphic theory we can then define the equivalence of two representations. Let us suppose that for a given meromorphic field theory specified by V and $\mathcal{A} = \{f\}$, we have two collections of densities, one specified by W_α, W_β with the amplitudes given by $\mathcal{R} = \{g\}$, and one specified by $\hat{W}_\alpha$, $\hat{W}_\beta$ and $\hat{\mathcal{R}}$. We denote the corresponding Fock spaces by $\mathcal{H}_\alpha$, $\mathcal{H}_\beta$ in the case of the former densities, and by

$\hat{\mathcal{H}}_\alpha$ and $\hat{\mathcal{H}}_\beta$ in the case of the latter. We say that the two densities define *equivalent* representations if there exist graded injections

$$\iota_\alpha : W_\alpha \to \hat{\mathcal{H}}_\alpha \quad \iota_\beta : W_\beta \to \hat{\mathcal{H}}_\beta, \tag{93}$$

and

$$\hat{\iota}_\alpha : \hat{W}_\alpha \to \mathcal{H}_\alpha \quad \hat{\iota}_\beta : \hat{W}_\beta \to \mathcal{H}_\beta, \tag{94}$$

that map amplitudes to amplitudes. We similarly define two representations to be *conjugate* to one another if instead of (93) and (94) the amplitudes are mapped to each other under

$$\iota_\alpha : W_\alpha \to \hat{\mathcal{H}}_\beta \quad \iota_\beta : W_\beta \to \hat{\mathcal{H}}_\alpha, \tag{93$'$}$$

and

$$\hat{\iota}_\alpha : \hat{W}_\alpha \to \mathcal{H}_\beta \quad \hat{\iota}_\beta : \hat{W}_\beta \to \mathcal{H}_\alpha. \tag{94$'$}$$

A representation is called *highest weight*, if the equivalence class of collections of densities contains a representative which has the *highest weight property*: for each density g and each choice of $\chi_1 \in W_\alpha$, $\chi_2 \in W_\beta$ and $\psi_i \in V_{h_i}$, the pole in $(z_i - u_l)$ is bounded by h_i. This definition is slightly more general than the definition which is often used, in that it is not assumed that the highest weight vectors transform in any way under the zero modes of the meromorphic fields.

In Sect. 6, we showed, using the cluster property, that the meromorphic conformal field theory does not have any proper ideals. This implies now

Proposition 7. *Every non-trivial representation is faithful.*

Proof. Suppose that $V(\Phi, z)$, where $\Phi \in \mathsf{V}_{\mathcal{C}}^{\mathcal{O}'}$, $\mathcal{C} \cap \mathcal{O}' = \varnothing$ and $\mathcal{C}_z \subset \mathcal{O}$, acts trivially on the representation $\mathcal{V}_\alpha^{\mathcal{O}}$, i.e. that

$$V(\Phi, z)\Psi = 0 \quad \text{for every} \quad \Psi \in \mathcal{V}_\alpha^{\mathcal{O}}. \tag{95}$$

Then, for any $\psi \in V$ and $\zeta \in \mathcal{O}'$ for which $\zeta + z \in \mathcal{O}$ we have

$$V\left(V(\psi, \zeta)\Phi, z\right)\Psi = V(\psi, \zeta + z)V(\Phi, z)\Psi = 0, \tag{96}$$

and thus $V(\psi, \zeta)\Phi$ also acts trivially on $\mathcal{V}_\alpha^{\mathcal{O}}$. This implies that the subspace of states in $\mathsf{V}_{\mathcal{C}}^{\mathcal{O}'}$ that act trivially on $\mathcal{V}_\alpha^{\mathcal{O}}$ is an ideal. Since there are no non-trivial ideals in $\mathcal{V}_{\mathcal{C}}^{\mathcal{O}}$, this implies that the representation is faithful. □

10. An Example of a Representation

Let us now consider the example of the $U(1)$ theory which was first introduced in Sect. 5.1. In this section we want to construct a family of representations for this meromorphic conformal field theory.

Let us first define the state

$$\Psi_n = \int_a^b dw_1 \cdots \int_a^b dw_n \, : J(w_1) \cdots J(w_n) :, \tag{97}$$

where $a, b \in \mathcal{C} \subset \mathbb{C}$, the integrals are chosen to lie in $\mathcal{C}$, and the normal ordering prescription $: \cdot :$ means that all poles in $w_i - w_j$ for $i \neq j$ are subtracted. We can deduce from the definition of the amplitudes (48) and (97) that the amplitudes involving Ψ_n are of the form

$$\left\langle \Psi_n \prod_{j=1}^{N} J(\zeta_j) \right\rangle = k^n \sum_{\substack{i_1,\ldots,i_n\in\{1,\ldots,N\} \\ i_j\neq i_l}} \prod_{l=1}^{n} \frac{(b-a)}{(a-\zeta_{i_l})(b-\zeta_{i_l})} \quad \langle \prod_{j\notin\{i_1,\ldots i_n\}} J(\zeta_j)\rangle, \tag{98}$$

where $\zeta_j \in \mathcal{O} \subset \mathbb{C}$ and $\mathcal{C} \cap \mathcal{O} = \varnothing$. By analytic continuation of (98) we can then calculate the contour integral $\oint_{C_a} J(z)dz\Psi_n$, where C_a is a contour in $\mathcal{C}$ encircling a but not b, and we find that

$$\oint_{C_a} J(z)dz\Psi_n = -nk\Psi_{n-1}, \tag{99}$$

and

$$\oint_{C_a} (z-a)^n J(z)dz\Psi_n = 0 \quad \text{for} \quad n \geq 1, \tag{100}$$

where the equality holds in $\mathcal{V}^{\mathcal{O}}$. Similar statements also hold for the contour integral around b,

$$\oint_{C_b} J(z)dz\Psi_n = nk\Psi_{n-1}, \tag{101}$$

and

$$\oint_{C_b} (z-b)^n J(z)dz\Psi_n = 0 \quad \text{for} \quad n \geq 1. \tag{102}$$

Next we define

$$\Psi_\alpha = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!\,k^n} \Psi_n =: \exp\left(\frac{\alpha}{k}\int_a^b J(w)dw\right):, \tag{103}$$

where α is any (real) number. This series converges in $\mathcal{V}^{\mathcal{O}}$, since for any amplitude of the form

$$\langle \Psi_\alpha\, J(\zeta_1)\cdots J(\zeta_N)\rangle$$

only the terms in (103) with $n \leq N$ contribute, as follows from (98).

We can use Ψ to define amplitudes as in (87), and in order to show that these form a representation, it suffices (because of Theorem 6) to demonstrate that the functions so defined have the appropriate analyticity properties. The only possible obstruction arises from the singularity for $\zeta_i \to a$ and $\zeta_i \to b$, but it follows from (99–102) that

$$J(\zeta)\Psi \sim \frac{-\alpha}{(\zeta-a)} + O(1) \quad \text{as} \quad \zeta \to a,$$

and

$$J(\zeta)\Psi \sim \frac{\alpha}{(\zeta-b)} + O(1) \quad \text{as} \quad \zeta \to b,$$

and thus that the singularities are only simple poles. This proves that the amplitudes defined by (87) give rise to a representation of the $U(1)$ theory. From the point of view of conventional conformal field theory, this representation (and its conjugate) is the highest weight representation that is generated from a state of $U(1)$-charge $\pm\alpha$.

It may be worthwhile to point out that we can rescale all amplitudes of a representation of a meromorphic field theory by

$$g \mapsto C(u_1-u_2)^{2\delta} g, \tag{104}$$

where C and δ are fixed constants (that are the same for all g), without actually violating any of the conditions we have considered so far. (The only effect is that r_1 and r_2 are replaced by $\hat{r}_l = r_l - \delta$, $l = 1, 2$.) For the representation of a meromorphic *conformal* field theory, the ambiguity in δ can however be canonically fixed: since the meromorphic fields contain the stress-energy field L (whose modes satisfy the Virasoro algebra L_n), we can require that

$$L_n = L_n^M \quad \text{for} \quad n = 0, \pm 1, \tag{105}$$

when acting on $\mathcal{H}_\alpha$. The action of L_0 on $\mathcal{H}_\alpha$ is not modified by (104), but since $\hat{r}_l = r_l - \delta$, the action of L_0^M is, and (105) therefore fixes the choice of δ in (104).

In the above example, in order to obtain a representation of the meromorphic conformal field theory (with L^0 being given by (79)), we have to modify the amplitudes as in (104) with $\delta = -\alpha^2/2k$. This can be easily checked using (99–102).

11. Zhu's Algebra

The description of representations in terms of collections of densities has a large redundancy in that many different collections of densities define the same representation. Typically we are only interested in highest weight representations, and for these we may restrict our attention to the representatives for which the highest weight property holds. In this section we want to analyse the conditions that characterise the corresponding states $\Sigma_{\alpha\beta}$; this approach is in essence due to Zhu [18].

Suppose we are given a highest weight representation, *i.e.* a collection of amplitudes that are described in terms of the states $\Sigma_{\alpha\beta}(u_1, u_2; \chi_1, \chi_2) \in \mathcal{V}^{\mathcal{O}}$, where $u_l \notin \mathcal{O}$. Each such state defines a linear functional on the Fock space $\mathcal{H}^{\mathcal{O}'}$, where $\mathcal{O} \cup \mathcal{O}' = \mathbb{P}$. But, for given u_1, u_2, the states $\Sigma_{\alpha\beta}(u_1, u_2; \chi_1, \chi_2)$, associated with the various possible representations, satisfy certain linear conditions: they vanish on a certain subspace $O_{u_1,u_2}(\mathcal{H}^{\mathcal{O}'})$. Thus they define, and are characterised by, linear functionals on the quotient space $\mathcal{H}^{\mathcal{O}'}/O_{u_1,u_2}(\mathcal{H}^{\mathcal{O}'})$. This is a crucial realisation, because it turns out that, in cases of interest, this quotient is finite-dimensional. Further the quotient has the structure of an algebra, first identified by Zhu [18], in terms of which the equivalence of representations, defined by these linear functionals, can be characterised.

Let us consider the case where $u_1 = \infty$ and $u_2 = -1$, for which we can choose $\mathcal{O}$ and $\mathcal{O}'$ so that $0 \in \mathcal{O}$ and $0 \notin \mathcal{O}'$. We want to characterise the subspace of $\mathcal{H} = \mathcal{H}^{\mathcal{O}'}$ on which the linear functional defined by $\Sigma_{\alpha\beta}(\infty, -1; \chi_1, \chi_2)$ vanishes identically. Given ψ and χ in $\mathcal{H}$, we define the state $V^{(N)}(\psi)\chi$ in $\mathcal{H}$ by

$$V^{(N)}(\psi)\chi = \oint_0 \frac{dw}{w^{N+1}} V\left[(w+1)^{L_0}\,\psi, w\right]\chi, \tag{106}$$

where N is an arbitrary integer, and the contour is a small circle (with radius less than one) around $w = 0$. If $\Sigma_{\alpha\beta}$ has the highest weight property then

$$\langle \Sigma_{\alpha\beta}(\infty, -1; \chi_1, \chi_2) V^{(N)}(\psi)\phi \rangle = 0 \quad \text{for } N > 0. \tag{107}$$

This follows directly from the observation that the integrand in (107) does not have any poles at $w = -1$ or $w = \infty$.

Let us denote by $O(\mathcal{H})$ the subspace of $\mathcal{H}$ that is generated by states of the form (106) with $N > 0$, and define the quotient space $A(\mathcal{H}) = \mathcal{H}/O(\mathcal{H})$. Then it follows that every highest weight representation defines a linear functional on $A(\mathcal{H})$. If two representations induce the same linear functional on $A(\mathcal{H})$, then they are actually equivalent representations, and thus the number of inequivalent representations is always bounded by the dimension of $A(\mathcal{H})$. In fact, as we shall show below, the vector space $A(\mathcal{H})$ has the structure of an associative algebra, where the product is defined by (106) with $N = 0$. In terms of the states $\Sigma_{\alpha\beta}$ this product corresponds to

$$\langle \Sigma_{\alpha\beta}(\infty, -1; \chi_1, \chi_2) V^{(0)}(\psi)\phi \rangle = (-1)^{h_\psi} \langle \Sigma_{\alpha\beta}(\infty, -1; V_0(\psi)\chi_1, \chi_2)\phi \rangle. \quad (108)$$

One may therefore expect that the different highest weight representations of the meromorphic conformal field theory are in one-to-one correspondence with the different representations of the algebra $A(\mathcal{H})$, and this is indeed true [18]. Most conformal field theories of interest have the property that $A(\mathcal{H})$ is a finite-dimensional algebra, and there exist therefore only finitely many inequivalent highest weight representations of the corresponding meromorphic conformal field theory; we shall call a meromorphic conformal field theory for which this is true *finite*.

In the above discussion the two points, $u_1 = \infty$ and $u_2 = -1$ were singled out, but the definition of the quotient space (and the algebra) is in fact independent of this choice. Let us consider the Möbius transformation γ which maps $\infty \mapsto u_1$, $-1 \mapsto u_2$ and $0 \mapsto 0$ (where $u_l \neq 0$); it is explicitly given as

$$\gamma(\zeta) = \frac{u_1 u_2 \zeta}{u_2(\zeta + 1) - u_1} \quad \leftrightarrow \quad \begin{pmatrix} u_1 u_2 & 0 \\ u_2 & u_2 - u_1 \end{pmatrix},$$

with inverse

$$\gamma^{-1}(z) = \frac{u_1 - u_2}{u_2} \frac{z}{(z - u_1)} \quad \leftrightarrow \quad \begin{pmatrix} u_1 - u_2 & 0 \\ u_2 & -u_1 u_2 \end{pmatrix}.$$

Writing $\psi' = U(\gamma)\psi$ and $\chi' = U(\gamma)\chi$ we then find (see Appendix F)

$$V^{(N)}_{u_1,u_2}(\psi')\chi' = U(\gamma) V^{(N)}(\psi)\,\chi, \quad (109)$$

where $V^{(N)}_{u_1,u_2}(\psi)$ is defined by

$$V^{(N)}_{u_1,u_2}(\psi)\chi = \oint_0 \frac{dw}{w} \frac{u_1}{(u_1 - w)} \left(\frac{u_2}{(u_2 - u_1)} \frac{(u_1 - w)}{w} \right)^N V\left[\left(\frac{(u_1 - w)(u_2 - w)}{u_1 u_2} \right)^{L_0} e^{\frac{w}{u_1 u_2} L_1} \psi, w \right] \chi, \quad (110)$$

and the contour encloses $w = 0$ but not $w = u_l$. We can then also define $O_{u_1,u_2}(\mathcal{H})$ to be the space that is generated by states of the form (110) with $N > 0$, and $A_{u_1,u_2}(\mathcal{H}) = \mathcal{H}/O_{u_1,u_2}(\mathcal{H})$.

As $z = 0$ is a fixed point of γ, $U(\gamma) : \mathcal{H} \to \mathcal{H}$, and because of (109), $U(\gamma) : O(\mathcal{H}) \to O_{u_1,u_2}(\mathcal{H})$. It also follows from (109) with $N = 0$ that the product is covariant, and this implies that the different algebras $A_{u_1,u_2}(\mathcal{H})$ for different choices of u_1 and u_2 are isomorphic. To establish that the algebra action is well-defined and associative, it is therefore sufficient to consider the case corresponding to $u_1 = \infty$ and $u_2 = -1$. In this case we write $V^{(0)}(\psi)\chi$ also as $\psi * \chi$.

Let us first show that $O_{u_1,u_2}(\mathcal{H}) = O_{u_2,u_1}(\mathcal{H})$. Because of the Möbius covariance it is sufficient to prove this for the special case, where $u_1 = \infty$ and $u_2 = -1$. For this case we have $V^{(N)}_{\infty,-1}(\psi) = V^{(N)}(\psi)$ as before, and

$$V^{(N)}_{-1,\infty}(\psi) \equiv V^{(N)}_c(\psi) = (-1)^N \oint \frac{dw}{w}\frac{1}{(w+1)}\left(\frac{w+1}{w}\right)^N V\left((w+1)^{L_0}\psi, w\right). \tag{111}$$

The result then follows from the observation that, for $N \geq 1$,

$$\frac{(w+1)^{N-1}}{w^N} = \sum_{l=1}^{N}\binom{N-1}{l-1}w^{-l},$$

and

$$\frac{1}{w^N} = \sum_{l=1}^{N}(-1)^{N-l}\binom{N-1}{l-1}\frac{(w+1)^{l-1}}{w^l}.$$

In particular, it follows from this calculation that (107) also holds if $V^{(N)}(\psi)$ is replaced by $V^{(N)}_c(\psi)$. Because of the definition of $V^{(N)}_c(\psi)$ it is clear that the analogue of (108) is now

$$\langle \Sigma_{\alpha\beta}(\infty, -1; \chi_1, \chi_2)V^{(0)}_c(\psi)\phi\rangle = \langle \Sigma_{\alpha\beta}(\infty, -1; \chi_1, V_0(\psi)\chi_2)\phi\rangle. \tag{112}$$

One should therefore expect that for $N \geq 0$, the action of $V^{(0)}(\psi)$ and $V^{(N)}_c(\chi)$ commute up to elements of the form $V^{(M)}_c(\phi)$, where $M > 0$ which generate states in $O(\mathcal{H})$. To prove this, it is sufficient to consider the case, where ψ and χ are eigenvectors of L_0 with eigenvalues h_ψ and h_χ, respectively; then the commutator $[V^{(0)}(\psi), V^{(N)}_c(\chi)]$ equals (up to the constant $(-1)^N$ in (111))

$$\begin{aligned}
&= \oint\oint_{|\zeta|>|w|}\frac{d\zeta}{\zeta}(\zeta+1)^{h_\psi}\frac{dw}{w(w+1)}\left(\frac{w+1}{w}\right)^N(w+1)^{h_\chi}V(\psi,\zeta)V(\chi,w)\\
&\quad-\oint\oint_{|z|>|\zeta|}\frac{dw}{w(w+1)}\left(\frac{w+1}{w}\right)^N(w+1)^{h_\chi}\frac{d\zeta}{\zeta}(\zeta+1)^{h_\psi}V(\chi,w)V(\psi,\zeta)\\
&= \oint_0\left\{\oint_w\frac{d\zeta}{\zeta}(\zeta+1)^{h_\psi}V(\psi,\zeta)V(\chi,w)\right\}\frac{dw}{w(w+1)}\left(\frac{w+1}{w}\right)^N(w+1)^{h_\chi}\\
&= \sum_n\oint_0\left\{\oint_w\frac{d\zeta}{\zeta}(\zeta+1)^{h_\psi}V(V_n(\psi)\chi,w)(\zeta-w)^{-n-h_\psi}\right\}\\
&\quad\cdot\frac{dw}{w(w+1)}\left(\frac{w+1}{w}\right)^N(w+1)^{h_\chi}\\
&= \sum_{n\leq h_\chi}\sum_{l=0}^{n+h_\psi-1}(-1)^l\binom{h_\psi}{l+1-n}\\
&\quad\cdot\oint_0\frac{dw}{w(w+1)}\left(\frac{w+1}{w}\right)^{N+l+1}(w+1)^{h_\chi-n}V(V_n(\psi)\chi,w)\approx 0,
\end{aligned}$$

where we denote by $\approx$ equality in $\mathcal{H}$ up to states in $O(\mathcal{H})$. Because of the Möbius covariance, it then also follows that $[V^{(N)}(\psi), V_c^{(0)}(\chi)] \approx 0$ for $N \geq 0$,

As $O_{u_1,u_2}(\mathcal{H}) = O_{u_2,u_1}(\mathcal{H})$, this calculation implies that the action of $V^{(0)}(\psi)$ is well-defined on the quotient space. To prove that the action defines an associative algebra, we observe that in the same way in which $V(\psi, z)$ is uniquely characterised by the two properties (28a) and (28b), $V^{(0)}(\psi)$ is uniquely determined by the two properties

$$\begin{aligned} V^{(0)}(\psi)\Omega &= \psi, \\ [V^{(0)}(\psi), V_c^{(N)}(\chi)] &\approx 0 \quad \text{for } N \geq 0. \end{aligned} \tag{113}$$

Indeed, if $V^{(0)}(\psi_1)$ and $V^{(0)}(\psi_2)$ both satisfy these properties for the same ψ, then

$$\begin{aligned} V^{(0)}(\psi_1)\phi &= V^{(0)}(\psi_1)V_c^{(0)}(\phi)\Omega \\ &\approx V_c^{(0)}(\phi)V^{(0)}(\psi_1)\Omega \\ &= V_c^{(0)}(\phi)\psi \\ &= V_c^{(0)}(\phi)V^{(0)}(\psi_2)\Omega \\ &\approx V^{(0)}(\psi_2)V_c^{(0)}(\phi)\Omega \\ &= V^{(0)}(\psi_2)\phi, \end{aligned} \tag{114}$$

where we have used that $V_c^{(0)}(\phi)\Omega = \phi$, as follows directly from the definition of $V_c^{(0)}(\phi)$. This therefore implies that $V^{(0)}(\psi_1) = V^{(0)}(\psi_2)$ on $A(\mathcal{H})$.

It is now immediate that

$$V^{(0)}(V^{(0)}(\psi)\chi) \approx V^{(0)}(\psi)V^{(0)}(\chi), \tag{115}$$

since both operators commute with $V_c^{(0)}(\phi)$ for arbitrary ϕ, and since

$$V^{(0)}(V^{(0)}(\psi)\chi)\Omega = V^{(0)}(\psi)\chi = V^{(0)}(\psi)V^{(0)}(\chi)\Omega. \tag{116}$$

We have thus shown that $(\psi * \chi) * \phi \approx \psi * (\chi * \phi)$. Similarly,

$$V^{(0)}(V^{(N)}(\psi)\chi) \approx V^{(N)}(\psi)V^{(0)}(\chi), \tag{117}$$

and this implies that $\psi_1 * \phi \approx 0$ if $\psi_1 \approx 0$. This proves that $A(\mathcal{H})$ forms an associative algebra.

The algebraic structures on $A_{u_1,u_2}(\mathcal{H})$ and $A_{u_2,u_1}(\mathcal{H})$ are related by

$$A_{u_1,u_2}(\mathcal{H}) = \left(A_{u_2,u_1}(\mathcal{H})\right)^o, \tag{118}$$

where A^o is the *reverse algebra* as explained in Appendix G. Indeed, it follows from (111) and (113) that

$$\begin{aligned} V^{(0)}(\psi_1)V^{(0)}(\psi_2)\Omega &= V^{(0)}(\psi_1)V_c^{(0)}(\psi_2)\Omega \\ &= V_c^{(0)}(\psi_2)V^{(0)}(\psi_1)\Omega \\ &= V_c^{(0)}(\psi_2)V_c^{(0)}(\psi_1)\Omega, \end{aligned}$$

and this implies (118).

By a similar calculation to the above, we can also deduce that for $h_\phi > 0$,

$$
\begin{aligned}
&\approx \sum_{m=0}^{h_\phi+h_\psi-1} V^{(0)}(V_{m+1-h_\phi}(\phi)\psi) \sum_{s=0}^{\min(h_\phi,m)} (-1)^{m+s}\binom{h_\phi}{s} \\
&= \sum_{m=0}^{h_\phi-1} V^{(0)}(V_{m+1-h_\phi}(\phi)\psi) \sum_{s=0}^{m}(-1)^{m+s}\binom{h_\phi}{s} \\
&= \sum_{m=0}^{h_\phi-1} \binom{h_\phi-1}{m} V^{(0)}(V_{m+1-h_\phi}(\phi)\psi) \\
&= \oint V^{(0)}(V(\phi,\zeta)\psi)(\zeta+1)^{h_\phi-1}d\zeta,
\end{aligned}
\tag{119}
$$

and

$$
\begin{aligned}
V^{(1)}(\psi)\Omega &= \oint V(\psi,\zeta)(\zeta+1)^{h_\psi}\frac{d\zeta}{\zeta^2}\Omega \\
&= \sum_{n=0}^{h_\psi}\binom{h_\psi}{n}V_{n-h_\psi-1}(\psi)\Omega \\
&= V_{-h_\psi-1}(\psi)\Omega + h_\psi V_{-h_\psi}(\psi)\Omega \\
&= (L_{-1}+L_0)V_{-h_\psi}(\psi)\Omega.
\end{aligned}
$$

In particular, this implies that $(L_{-1}+L_0)\psi \approx 0$ for every $\psi \in \mathcal{H}$.

For the Virasoro field $L(z) = V(L,z)$ (119) becomes

$$
\begin{aligned}
&= \sum_{m=0}^{1}\binom{1}{m}V^{(0)}(V_{m-1}(L)\psi) \\
&= (L_{-1}+L_0)\psi \approx 0,
\end{aligned}
$$

which thus implies that L is central in Zhu's algebra.

So far our considerations have been in essence algebraic, in that we have considered the conditions $\Sigma_{\alpha\beta}$ have to satisfy in terms of the linear functional it defines on the Fock space $\mathcal{H}$. If, however, we wish to reverse this process, and proceed from a linear functional on $A(\mathcal{H})$ to a representation of the conformal field theory, we need to be concerned about the analytic properties of the resulting amplitudes. To this end, we note that we can perform an analytic version of the construction as follows.

In fact, since $\Sigma_{\alpha\beta}$ is indeed an element of $\mathcal{V}^{\mathcal{O}}$ for a suitable $\mathcal{O}$, it actually defines a linear functional on the whole dual space $\overline{\mathcal{V}_{\mathcal{O}}} \equiv (\mathcal{V}^{\mathcal{O}})'$ (of which the Fock space is only a dense subspace). Let us denote by $\overline{O(\mathcal{V}_{\mathcal{O}})}$ the completion (in $\overline{\mathcal{V}_{\mathcal{O}}}$) of the space that is generated by states of the form (110) with $N > 0$, where now $\psi \in \mathcal{H}$ and $\chi \in \overline{\mathcal{V}_{\mathcal{O}}}$. By the same arguments as before, the linear functional associated to $\Sigma_{\alpha\beta}$ vanishes then on $\overline{O(\mathcal{V}_{\mathcal{O}})}$, and thus defines a linear functional on the quotient space

$$
A\left(\overline{\mathcal{V}_{\mathcal{O}}}\right) = \overline{\mathcal{V}_{\mathcal{O}}}/\overline{O(\mathcal{V}_{\mathcal{O}})}. \tag{120}
$$

It is not difficult to show (see [19] for further details) that a priori $A\left(\overline{\mathcal{V}_{\mathcal{O}}}\right)$ is a quotient space of $A(\mathcal{H})$; the main content of Zhu's Theorem [18] is equivalent to:

Theorem 8 (*Zhu's Theorem***).** *The two quotient spaces are isomorphic vector spaces*

$$A\left(\overline{\mathcal{V}_{\mathcal{O}}}\right) \simeq A(\mathcal{H}).$$

Proof. It follows from the proof in [18] that every non-trivial linear functional on $A(\mathcal{H})$ (that is defined by $\rho(a) = \langle w^*, aw\rangle$, where w is an element of a representation of $A(\mathcal{H})$, and w^* is an element of the corresponding dual space) defines a non-trivial state $\Sigma_{\alpha\beta} \in \mathcal{V}^{\mathcal{O}}$, and therefore a non-trivial element in the dual space of $A\left(\overline{\mathcal{V}_{\mathcal{O}}}\right)$. □

The main importance of this result is that it relates the analytic properties of correlation functions (which are in essence encoded in the definition of the space $\mathcal{V}^{\mathcal{O}}$, *etc.*) to the purely algebraic Fock space $\mathcal{H}$.

Every linear functional on $A(\overline{\mathcal{V}_{\mathcal{O}}})$ defines a highest weight representation of the meromorphic conformal field theory, and two such functionals define equivalent representations if they are related by the action of Zhu's algebra as in (108). Because of Zhu's Theorem there is therefore a one-to-one correspondence between highest weight representations of the meromorphic conformal field theory whose Fock space is $\mathcal{H}$, and representations of the algebra $A(\mathcal{H})$; this (or something closely related to it) is the form in which Zhu's Theorem is usually stated.

Much of the structure of the meromorphic conformal field theory (and its representations) can be read off from properties of $A(\mathcal{H})$. For example, it was shown in ref. [21] (see also Appendix G) that if $A(\mathcal{H})$ is semisimple, then it is necessarily finite-dimensional, and therefore there exist only finitely many irreducible representations of the meromorphic field theory.

12. Further Developments

In this paper we have introduced a rigorous approach to conformal field theory taking the amplitudes of meromorphic fields as a starting point. We have shown how the paradigm examples of conformal field theories, i.e. lattice theories, affine Lie algebra theories and the Virasoro theory, all fit within this approach. We have shown how to introduce the concept of a representation of such a meromorphic conformal field theory by using a collection of amplitudes which involve two non-meromorphic fields, so that the amplitudes may be branched at the corresponding points. We showed how this led naturally to the introduction of Zhu's algebra and why the condition that this algebra be finite-dimensional is a critical one in distinguishing interesting and tractable theories from those that appear to be less so.

To complete a treatment of the fundamental aspects of conformal field theory we should discuss subtheories, coset theories and orbifolds, all of which can be expressed naturally within the present approach [19]. It is also clear how to modify the axioms for theories involving fermions.

It is relatively straightforward to generalise the discussion of representations to correlation functions involving $N > 2$ non-meromorphic fields. The only difference is that in this case, there are more than two points $u_l, l = 1, \ldots, N$, at which the amplitudes are allowed to have branch cuts. The condition that a collection of such amplitudes defines an N-point correlation function of the meromorphic conformal field theory can then be described analogously to the case of $N = 2$: we consider the vector space $\mathcal{V}_{\mathcal{C}\boldsymbol{\alpha}}$ (where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_N)$ denotes the indices of the N vector spaces W_{α_i} that are associated to the N points $u_1, \ldots, u_N$), whose elements are finite linear combinations of vectors of the form

$$V(\psi_1, z_1) \cdots V(\psi_n, z_n)\, W_{\alpha_1}(\chi_1, u_1) \cdots W_{\alpha_N}(\chi_N, u_N)\Omega. \tag{121}$$

We complete this space (and cut it down to size) using the standard construction with respect to $\mathcal{B}_{\mathcal{O}}$ and the above set of amplitudes, and we denote the resulting space by $\mathcal{V}^{\mathcal{O}}_{\mathcal{C}\boldsymbol{\alpha}}$. The relevant condition is then that $\mathcal{B}_{\mathcal{O}}$ induces a continuous map

$$(\mathcal{V}^{\mathcal{O}})' \to (\mathcal{V}^{\mathcal{O}}_{\boldsymbol{\alpha}})'. \tag{122}$$

There also exists a formulation of this condition analogous to (87): a collection of amplitudes defines an N-point correlation function if there exists a family of states $\Sigma_{\boldsymbol{\alpha}}(u_1, \ldots, u_N; \chi_1, \ldots, \chi_N) \in \mathcal{V}^{\mathcal{O}}_{\mathcal{C}}$ for each $\mathcal{O}, \mathcal{C}$ with $\mathcal{O} \cap \mathcal{C} = \varnothing$ that is equivalent to $W_{\alpha_1}(\chi_1, u_1) \cdots W_{\alpha_N}(\chi_N, u_N)$ in the sense that

$$\Big\langle \prod_{i=1}^{n} V(\psi_i, z_i) W_{\alpha_1}(\chi_1, u_1) \cdots W_{\alpha_N}(\chi_N, u_N) \Big\rangle \\ = \Big\langle \prod_{i=1}^{n} V(\psi_i, z_i) \Sigma_{\boldsymbol{\alpha}}(u_1, \ldots, u_N; \chi_1, \ldots, \chi_N) \Big\rangle, \tag{123}$$

where $z_i \in \mathcal{O}$. An argument analogous to Theorem 6 then implies that (122) is equivalent to (123).

In the case of the two-point correlation functions (or representations) we introduced a quotient space (120) of the vector space $\overline{\mathcal{V}_{\mathcal{O}}} = (\mathcal{V}^{\mathcal{O}})'$ that classified the different highest weight representations. We can now perform an analogous construction. Let us consider the situation where the N highest weight states are at $u_1 = \infty, u_2, \ldots, u_N$, and define

$$V_N^{(M)}(\psi)\chi = \oint_0 \frac{d\zeta}{\zeta^{1+M}} \left(\frac{\prod_{j=2}^{N}(\zeta - u_j)}{\zeta^{N-2}} \right)^{h_\psi} V(\psi, \zeta)\chi, \tag{124}$$

where M is an integer, $\psi \in \mathcal{H}$, $\chi \in \overline{\mathcal{V}_{\mathcal{O}}}$, and the contour encircles 0, but does not encircle $u_1, \ldots, u_N$. We denote by $\overline{O_N(\mathcal{V}_{\mathcal{O}})}$ the completion of the space (in $\overline{\mathcal{V}_{\mathcal{O}}}$) that is spanned by states of the form (124) with $M > 0$, and we denote by $A_N(\overline{\mathcal{V}_{\mathcal{O}}})$ the quotient space $\overline{\mathcal{V}_{\mathcal{O}}}/\overline{O_N(\mathcal{V}_{\mathcal{O}})}$. (In this terminology, Zhu's algebra is the space $A_2(\overline{\mathcal{V}_{\mathcal{O}}})$.) By the same arguments as in (107) is is easy to see that every state $\Sigma_{\boldsymbol{\alpha}}$ that corresponds to an N-point function of N highest weights (where the highest weight property is defined as before) defines a functional on $\overline{\mathcal{V}_{\mathcal{O}}}$ that vanishes on $\overline{O_N(\mathcal{V}_{\mathcal{O}})}$, and thus defines a linear functional on $A_N(\overline{\mathcal{V}_{\mathcal{O}}})$.

One can show (see [19] for more details) that the space $A_N(\overline{\mathcal{V}_{\mathcal{O}}})$ carries N commuting actions of Zhu's algebra $A(\mathcal{H})$ which are naturally associated to the N non-meromorphic points $u_1, \ldots, u_N$. For example the action corresponding to u_1 is given by (124) with $M = 0$ and $\psi \in \mathcal{H}$, $\psi \circ \phi = V_N^{(0)}(\psi)\phi$. This action is actually well-defined for $\psi \in A(\mathcal{H})$ since we have for $L \geq 0$,

$$V_N^{(0)} \left(V^{(L)}(\psi_1)\, \psi_2 \right) \phi \approx V_N^{(L)}(\psi_1)\, V_N^{(0)}(\psi_2)\, \phi, \tag{125}$$

where we denote by $\approx$ equality in $\overline{\mathcal{V}_{\mathcal{O}}}$ up to states in $\overline{O_N(\mathcal{V}_{\mathcal{O}})}$. Applying (125) for $L = 0$ implies that the algebra relations of $A(\mathcal{H})$ are respected, *i.e.* that

$$(\psi_1 * \psi_2) \circ \phi = \psi_1 \circ (\psi_2 \circ \phi) \,, \tag{126}$$

where $*$ denotes the multiplication of $A(\mathcal{H})$. Every N-point correlation function determines therefore N representations of Zhu's algebra, and because of Zhu's Theorem, we can associate N representations of the meromorphic conformal field theory to it. Conversely, every linear functional on $A_N(\overline{\mathcal{V}_{\mathcal{O}}})$ defines an N-point correlation function, and two functionals define equivalent such functions if they are related by the actions of Zhu's algebra; in this way the different N-point correlation functions of the meromorphic conformal field theory are classified by $A_N(\overline{\mathcal{V}_{\mathcal{O}}})$.

There exists also an "algebraic" version of this quotient space, $A_N(\mathcal{H}) = \mathcal{H}/O_N(\mathcal{H})$, where $O_N(\mathcal{H})$ is generated by the states of the form (124), where now ψ and ϕ are in $\mathcal{H}$. This space is much more amenable for study, and one may therefore hope that in analogy to Zhu's Theorem, the two quotient spaces are isomorphic vector spaces,

$$A_N(\overline{\mathcal{V}_{\mathcal{O}}}) \simeq A_N(\mathcal{H}) \,;$$

it would be interesting if this could be established.

It is also rather straightforward to apply the above techniques to an analysis of correlation functions on higher genus Riemann surfaces. Again, it is easy to see that the correlation functions on a genus g surface can be described in terms of a state of the meromorphic conformal field theory on the sphere, in very much the same way in which N-point correlation functions can be defined by (87). The corresponding state induces a linear functional on $\overline{\mathcal{V}_{\mathcal{O}}}$ (or $\mathcal{H}$), and since it vanishes on a certain subspace thereof, defines a linear functional on a suitable quotient space. For the case of the genus $g = 1$ surface, the torus, the corresponding quotient space is very closely related to Zhu's algebra, and one may expect that similar relations hold more generally.

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Appendix A: Sequences of Holomorphic Functions

A sequence of functions $\{f_n\} = f_1, f_2, f_3, \ldots$, each defined on a domain $D \subset \mathbb{C}$, is said to be *uniformly bounded* on D if there exists a real number M such that $|f_n(z)| < M$ for all n and $z \in D$. The sequence is said to be *locally uniformly bounded* on D if, given $z_0 \in D$, $\{f_n\}$ is uniformly bounded on $N_\delta(z_0, D) = \{z \in D : |z - z_0| < \delta\}$ for some $\delta \equiv \delta(z_0) > 0$.

A sequence of functions $\{f_n\}$ defined on D is said to be *uniformly convergent* to $f : D \to \mathbb{C}$ if, given $\epsilon > 0$, $\exists\ N$ such that $|f_n(z) - f(z)| < \epsilon$ for all $z \in D$ and $n > N$. We write $f_n \to f$ uniformly in D. The sequence is said to be *locally uniformly convergent* to $f : D \to \mathbb{C}$ if, given $z_0 \in D$, $f_n \to f$ uniformly in $N_\delta(z_0, D)$ for some $\delta \equiv \delta(z_0) > 0$.

Clearly, by the Heine-Borel Theorem, a sequence is locally uniform bounded on D if and only if it is uniformly bounded on every compact subset of D, and locally uniformly convergent on D if and only if it is uniformly convergent on every compact subset of D. Local uniformity of the convergence of a sequence of continuous functions guarantees continuity of the limit and, similarly, analyticity of the limit of a sequence of analytic functions is guaranteed by local uniformity of the convergence [22].

The following result [23] is of importance in the approach to conformal field theory developed in this paper:

Theorem. *If D is an open domain and $f_n : D \to \mathbb{C}$ is analytic for each n, $f_n(z) \to f(z)$ at each $z \in D$, and the sequence $\{f_n\}$ is locally uniformly bounded in D, then $f_n \to f$ locally uniformly in D and f is analytic in D.*

Proof. Again, given $z_0 \in D$, $\overline{N_\delta(z_0)} = \{z \in \mathbb{C} : |z - z_0| \leq \delta\} \subset D$ for some $\delta > 0$ because D is open. Because $\overline{N_\delta(z_0)}$ is compact, $\exists M$, such that $|f_n(z)| < M$ for all n and all $z \in \overline{N_\delta(z_0)}$. First we show that the sequence $f_n'(z)$ is uniformly bounded on $N_\rho(z_0)$ for all $\rho < \delta$. For

$$|f_n'(z)| = \left| \frac{1}{2\pi i} \oint_{\mathcal{C}_\delta(z_0)} \frac{f_n(\zeta)d\zeta}{(\zeta - z)^2} \right| \leq \frac{M\delta}{(\delta - \rho)^2} = M_1(\rho), \quad \text{say.}$$

Thus for fixed ρ, given $\epsilon > 0$, $\exists\ \delta_1 > 0$ such that $|f_n(z) - f_n(z')| < \frac{1}{3}\epsilon$ for all values of n provided that $z, z' \in N_\rho(z_0)$ are such that $|z - z'| < \delta_1(\epsilon)$. Then also $|f(z) - f(z')| < \frac{1}{3}\epsilon$ for $|z - z'| < \delta_1(\epsilon)$. Now we can find a finite number K of points $z_j \in N_\rho(z_0)$, $1 \leq j \leq K$, such that given any point in $z \in N_\rho(z_0)$, $|z - z_j| < \delta_1$ for some j. Now each $f_n(z_j) \to f(z_j)$ and so we can find integers L_j such that $|f(z_j) - f_n(z_j)| < \frac{1}{3}\epsilon$ for $n > L_j$. Now if $n > L = \max_{1\leq j\leq K}\{L_j\}$ and $z \in N_\rho(z_0)$, $|f_n(z) - f(z)| \leq |f_n(z) - f_n(z_j)| + |f_n(z_j) - f(z_j)| + |f(z_j) - f(z)| < \epsilon$, establishing uniform convergence on $N_\rho(z_0)$ and so local uniform convergence. This is sufficient to deduce that f is analytic. □

Appendix B: Completeness of $\mathcal{V}_\mathcal{C}^\mathcal{O}$

We prove that, if a sequence $\chi_j \in \mathcal{V}_\mathcal{C}^\mathcal{O}$, $j = 1, 2, \ldots$, $\eta_{\boldsymbol{\phi}}(\chi_j)$ converges on each subset of $\boldsymbol{\phi}$ of the form (6) (where the convergence is uniform on (7)), the limit $\lim_{j\to\infty} \eta_{\boldsymbol{\phi}}(\chi_j)$ necessarily equals $\eta_{\boldsymbol{\phi}}(\chi)$ for some $\chi \in \mathcal{V}_\mathcal{C}^\mathcal{O}$.

To see this, note that uniform convergence in this sense is implied by the (uniform) convergence on a countable collection of such sets, taken by considering $\epsilon = 1/N$, N a positive integer, K one of a collection of compact subsets of $\mathcal{O}$ and the ϕ_j to be elements of some countable basis. Taken together we obtain in this way a countable number of conditions for the uniform convergence. Defining $||\Psi||_n = \max_{\boldsymbol{\phi}} \max_{\zeta_j} |\eta_{\boldsymbol{\phi}}(\Psi)|$, where $\boldsymbol{\phi}$ ranges over the first n of these countable conditions for each $\mathcal{O}$ and the maximum is taken over ζ_j within (7), we have a sequence of semi-norms, $||\Psi||_n$, on $\mathcal{V}_\mathcal{C}$, with $||\Psi||_n \leq ||\Psi||_{n+1}$. Given such a sequence of semi-norms, we can define a Cauchy sequence (Ψ_j), $\Psi_j \in \mathcal{V}_\mathcal{C}$, by the requirement that $||\Psi_i - \Psi_j||_n \to 0$ as $i, j \to \infty$ for each fixed n. This requirement is equivalent to uniform convergence on each set of the form (6). Moreover, the space $\mathcal{V}_\mathcal{C}^\mathcal{O}$ is obtained by adding in the limits of these Cauchy sequences (identifying points zero distance apart with respect to all of the semi-norms). This space is necessarily complete because if $\chi_j \in \mathcal{V}_\mathcal{C}^\mathcal{O}$ is Cauchy, *i.e.* $||\chi_i - \chi_j||_n \to 0$ as $i, j \to \infty$ for each fixed n, and $\Psi_i^m \to \chi_i$ as $m \to \infty$, $\Psi_i^m \in \mathcal{V}_\mathcal{C}$, then selecting I_N

so that $||\chi_i - \chi_j||_N < 1/3N$ for $i, j \geq I_N$, and $I_{N+1} \geq I_N$, we can find an integer m_N such that $||\psi_{I_N}^{m_N} - \chi_{I_N}||_N < 1/3N$, and if $\psi_N = \psi_{I_N}^{m_N}$,

$$\begin{aligned}||\psi_M - \psi_N||_p &\leq ||\psi_M - \chi_{I_M}||_p + ||\chi_{I_M} - \chi_{I_N}||_p + ||\chi_{I_N} - \psi_N||_p \\ &\leq ||\psi_M - \chi_{I_M}||_M + ||\chi_{I_M} - \chi_{I_N}||_N + ||\chi_{I_N} - \psi_N||_N \\ &\leq 1/N,\end{aligned}$$

provided that $M \geq N \geq p$, implying that ψ_M is Cauchy. It is easy to see that its limit is the limit of χ_j, showing that $\mathcal{V}_{\mathcal{C}}^{\mathcal{O}}$ is complete. The completeness of this space is equivalent to the condition (11).

Appendix C: Proof that $\mathcal{V}_{\mathcal{C}}^{\mathcal{O}}$ is Independent of $\mathcal{C}$ (Theorem 1)

Proof. To prove that $\widetilde{\mathcal{V}}_{\mathcal{C}}^{\mathcal{O}}$ is independent of $\mathcal{C}$, first note that we may identify a vector $\chi \in \widetilde{\mathcal{V}}_{\mathcal{C}}^{\mathcal{O}}$ with $\boldsymbol{\psi} = \prod_{i=1}^{n} V(\psi_i, z_i)\Omega$ (where $z_i \notin \mathcal{O}$ for $1 \leq i \leq n$ but it is not necessarily the case that $z_i \in \mathcal{C}$ for each i) if $\eta_{\boldsymbol{\phi}}(\chi) = \eta_{\boldsymbol{\phi}}(\boldsymbol{\psi})$, for all $\boldsymbol{\phi} \in \mathcal{B}_{\mathcal{O}}$, i.e. the value of $\eta_{\boldsymbol{\phi}}(\chi)$ is given by (5) for all $\boldsymbol{\phi} \in \mathcal{B}_{\mathcal{O}}$. Consider then the set $\mathcal{Q}$ of values of $\boldsymbol{z} = (z_1, z_2, \ldots, z_n)$ for which $\boldsymbol{\psi}(\boldsymbol{z}) = \prod_{i=1}^{n} V(\psi_i, z_i)\,\Omega$ is a member of $\widetilde{\mathcal{V}}_{\mathcal{C}}^{\mathcal{O}}$. Then $\mathcal{D}' \subset \mathcal{Q} \subset \mathcal{D}$, where $\mathcal{D}' = \{\boldsymbol{z} : z_i, z_j \in \mathcal{C}, z_i \neq z_j, 1 \leq i < j \leq n\}$ and $\mathcal{D} = \{\boldsymbol{z} : z_i, z_j \in \mathcal{O}^c, z_i \neq z_j, 1 \leq i < j \leq n\}$, where $\mathcal{O}^c$ is the complement of $\mathcal{O}$ in $\mathbb{P}$. We shall show that $\mathcal{Q} = \mathcal{D}$.

If z_b is in $\mathcal{D}^o$, the interior of $\mathcal{D}$, but not in $\mathcal{Q}$, choose a point $z_a \in \mathcal{D}' \subset \mathcal{Q}$ and join it to z_b by a path C inside $\mathcal{D}^o$, $\{\boldsymbol{z}(t) : 0 \leq t \leq 1\}$ with $\boldsymbol{z}(0) = \boldsymbol{z}^a$ and $\boldsymbol{z}(1) = \boldsymbol{z}^b$. (There is such a point $\boldsymbol{z}^a$ because $\mathcal{C}^o \neq \emptyset$; the path C exists because the interior of $\mathcal{O}^c$ is connected, from which it follows that $\mathcal{D}^o$ is.) Let t_c be the supremum of the values of t_0 for which $\{\boldsymbol{z}(t) : 0 \leq t \leq t_0\} \subset \mathcal{Q}^o$, the interior of $\mathcal{Q}$, and let $\boldsymbol{z}^c = \boldsymbol{z}(t_c)$. Then $\boldsymbol{z}^c = (z_1^c, z_2^c, \ldots, z_n^c)$ is inside the open set $\mathcal{D}^o$ and so we can find a neighbourhood of the form $N_1 = \{\boldsymbol{z} : |\boldsymbol{z} - \boldsymbol{z}^c| < 4\delta\}$ which is contained inside $\mathcal{D}^o$. Let $\boldsymbol{z}^d, \boldsymbol{z}^e$ be points each distant less than δ from $\boldsymbol{z}^c$, with $\boldsymbol{z}^d$ outside $\mathcal{Q}$ and $\boldsymbol{z}^e$ inside $\mathcal{Q}^o$. (There must be a point $\boldsymbol{z}^d$ outside $\mathcal{Q}$ in every neighbourhood of $\boldsymbol{z}^c$.) Then the set $N_2 = \{\boldsymbol{z}^e + (\boldsymbol{z}^d - \boldsymbol{z}^e)\omega : |\omega| < 1\}$ is inside N_1 but contains points outside $\mathcal{Q}$. We shall show that $N_2 \subset \mathcal{Q}$ establishing a contradiction to the assumption that there is a point z_b in $\mathcal{D}^o$ but not in $\mathcal{Q}$, so that we must have $\mathcal{D}^o \subset \mathcal{Q} \subset \mathcal{D}$.

The circle $\{\boldsymbol{z}^e + (\boldsymbol{z}^d - \boldsymbol{z}^e)\omega : |\omega| < \epsilon\}$ is inside $\mathcal{Q}$ for some ϵ in the range $0 < \epsilon < 1$. Now, we can form the integral

$$\chi = \int_S \boldsymbol{\psi}(\boldsymbol{z})\mu(\boldsymbol{z})d^r\boldsymbol{z}$$

of $\boldsymbol{\psi}(\boldsymbol{z})$ over any compact r-dimensional sub-manifold $S \subset \mathcal{Q}$, with continuous weight function $\mu(\boldsymbol{z})$, to obtain an element $\chi \in \widetilde{\mathcal{V}}_{\mathcal{C}}^{\mathcal{O}}$, because the approximating sums to the integral will have the necessary uniform convergence property. So the Taylor coefficients

$$\boldsymbol{\psi}_N = \int_{|\omega|=\epsilon} \boldsymbol{\psi}(\boldsymbol{z}^e + (\boldsymbol{z}^d - \boldsymbol{z}^e)\omega)\omega^{-N-1}d\omega \in \widetilde{\mathcal{V}}_{\mathcal{C}}^{\mathcal{O}}$$

and, since the $\sum_{N=0}^{\infty} \eta_{\boldsymbol{\phi}}(\boldsymbol{\psi}_N)\omega^N$ converges to $\eta_{\boldsymbol{\phi}}(\boldsymbol{\psi}(\boldsymbol{z}^e + (\boldsymbol{z}^d - \boldsymbol{z}^e)\omega))$ for $|\omega| < 1$ and all $\boldsymbol{\phi} \in \mathcal{B}_{\mathcal{O}}$, we deduce that $N_2 \subset \mathcal{Q}$, hence proving that $\mathcal{D}^o \subset \mathcal{Q} \subset \mathcal{D}$. Finally, if $\boldsymbol{z}_j$ is a sequence of points in $\mathcal{Q}$ convergent to $\boldsymbol{z}_0 \in \mathcal{D}$, it is straightforward to see that $\boldsymbol{\psi}(\boldsymbol{z}_i)$ will converge to $\boldsymbol{\psi}(\boldsymbol{z}_0)$, so that $\mathcal{Q}$ is closed in $\mathcal{D}$, and so must equal $\mathcal{D}$. □

Appendix D. Möbius Transformation of Vertices

By virtue of the Uniqueness Theorem we can establish the transformation properties

$$e^{\lambda L_{-1}} V(\Psi, z) e^{-\lambda L_{-1}} = V(\Psi, z+\lambda), \tag{D.1}$$

$$e^{\lambda L_0} V(\Psi, z) e^{-\lambda L_0} = e^{\lambda h} V(\Psi, e^{\lambda} z), \tag{D.2}$$

$$e^{\lambda L_1} V(\Psi, z) e^{-\lambda L_1} = (1-\lambda z)^{-2h} V\left(\exp(\lambda(1-\lambda z)L_1)\Psi, z/(1-\lambda z)\right), \tag{D.3}$$

where we have used the relation

$$\begin{pmatrix} 1 & 0 \\ -\lambda & 1 \end{pmatrix} \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{z}{1-\lambda z} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\lambda(1-\lambda z) & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{1-\lambda z} & 0 \\ 0 & 1-\lambda z \end{pmatrix}. \tag{D.4}$$

From these it follows that

$$\langle V(\Phi, z)\rangle = 0, \tag{D.5}$$

$$\langle V(\Phi, z) V(\Psi, \zeta)\rangle = \frac{\varphi(\Phi, \Psi)}{(z-\zeta)^{h_\Phi + h_\Psi}}, \tag{D.6}$$

where $L_0\Phi = h_\Phi \Phi$, $L_0\Psi = h_\Psi \Psi$ and $h_\Phi \neq 0$. The bilinear form $\varphi(\Phi, \Psi)$, defined as being the constant of proportionality in (D.6), has the symmetry property

$$\varphi(\Phi, \Psi) = (-1)^{h_\Phi + h_\Psi} \varphi(\Psi, \Phi). \tag{D.7}$$

If, in addition, $L_1\Phi = L_1\Psi = 0$, it follows from (D.3) applied to (D.6) that $\varphi(\Psi, \Phi) = 0$ unless $h_\Phi = h_\Psi$.

It follows from (D.3) that, if $\Psi \in \mathcal{H}_1$,

$$\begin{aligned} \langle V(\Psi, z)\rangle = &(1-\lambda z)^{-2} \langle V(\Psi, z/(1-\lambda z))\rangle \\ &+ \frac{\lambda}{(1-\lambda z)} \langle V(L_1\Psi, z/(1-\lambda z))\rangle, \end{aligned} \tag{D.8}$$

and from (D.5) that both the left-hand side and the first term on the right-hand side of (D.8) vanish, implying that the second term on the right-hand side also vanishes. But $L_1\Psi \in \mathcal{H}_0$ and, if we assume cluster decomposition, so that the vacuum is unique, $L_1\Psi = \kappa\Omega$ for some $\kappa \in \mathbb{C}$. We deduce that $\kappa = \langle V(L_1\Psi, \zeta)\rangle = 0$, so that Ψ is quasi-primary, *i.e.* $\mathcal{H}_1 = \mathcal{H}_1^Q$.

We can show inductively that $\mathcal{H}_h$ is the direct sum of spaces $L_{-1}^n \mathcal{H}_{h-n}^Q$, where $0 \leq n < h$, i.e. $\mathcal{H}$ is composed of quasi-primary states and their descendants under the action of L_{-1}. Given $\Psi \in \mathcal{H}_h$, we can find $\Phi \in L_{-1}H_{h-1}$ such that $L_1(\Psi - \Phi) = 0$; then Ψ is the sum of the quasi-primary state $\Psi - \Phi$ and Φ which, by an inductive hypothesis, is the sum of descendants of quasi-primary states. To find Φ, note

$$L_1(\Psi + \sum_{n=1}^{h} a_n L_{-1}^n L_1^n \Psi) = \sum_{n=1}^{h} (a_{n-1} + a_n(2nh + n(n+1)) L_{-1}^{n-1} L_1^n \Psi,$$

where $a_0 = 1$. So choosing $a_n = -a_{n-1}/(2nh + n(n+1))$, $1 \leq n \leq h$, we have $L_1(\Psi - \Phi) = 0$ for $\Phi = -\sum_{n=1}^{h} a_n L_{-1}^n L_1^n \Psi \in L_{-1}H_{h-1}$ establishing the result.

Appendix E: Proof that the Extended Amplitudes $\hat{\mathcal{A}}$ Satisfy the Axioms

An alternative description of the extended amplitudes can be given as follows: we define the amplitudes involving vectors in $\hat{V}$ recursively (the recursion being on the number of times L appears in an amplitude) by

$$\langle L(w)\rangle = 0, \tag{E.1}$$

$$\begin{aligned}\langle L(w)\prod_{i=1}^{n} V(\psi_i, z_i)\rangle &= \sum_{l=1}^{n} \frac{c_{\psi_l}}{(w-z_l)^4}\langle \prod_{i\neq l} V(\psi_i, z_i)\rangle \\ &+ \sum_{l=1}^{n} \frac{h_l}{(w-z_l)^2}\langle \prod_{i=1}^{n} V(\psi_i, z_i)\rangle \\ &+ \sum_{l=1}^{n} \frac{1}{(w-z_l)}\frac{d}{dz_l}\langle V(\psi_1, z_1)\cdots V(\psi_l, z_l)\cdots V(\psi_n, z_n)\rangle\end{aligned} \tag{E.2}$$

and

$$\begin{aligned}\langle L(w)\prod_{j=1}^{m} L(w_j)\prod_{i=1}^{n} V(\psi_i, z_i)\rangle &= \sum_{k=1}^{m} \frac{2}{(w-w_k)^2}\langle \prod_{j=1}^{m} L(w_j)\prod_{i=1}^{n} V(\psi_i, z_i)\rangle \\ &+ \sum_{l=1}^{n} \frac{h_l}{(w-z_l)^2}\langle \prod_{j=1}^{m} L(w_j)\prod_{i=1}^{n} V(\psi_i, z_i)\rangle \\ &+ \sum_{k=1}^{m} \frac{1}{(w-w_k)}\frac{d}{dw_k}\langle \prod_{j=1}^{m} L(w_j)\prod_{i=1}^{n} V(\psi_i, z_i)\rangle \\ &+ \sum_{l=1}^{n} \frac{1}{(w-z_l)}\frac{d}{dz_l}\langle \prod_{j=1}^{m} L(w_j)\prod_{i=1}^{n} V(\psi_i, z_i)\rangle \\ &+ \sum_{k=1}^{m} \frac{c/2}{(w-w_k)^4}\langle \prod_{j\neq k} L(w_j)\prod_{i=1}^{n} V(\psi_i, z_i)\rangle \\ &+ \sum_{l=1}^{n} \frac{c_{\psi_l}/2}{(w-z_l)^4}\langle \prod_{j=1}^{m} L(w_j)\prod_{i\neq l} V(\psi_i, z_i)\rangle.\end{aligned} \tag{E.3}$$

Here h_i is the grade of the vector $\psi_i \in V$, c is an arbitrary (real) number, and c_ψ is zero unless ψ is of grade two. It is not difficult to see that the functions defined by (E.1)–(E.3) agree with those defined in the main part of the text: for a given set of fields, the difference between the two amplitudes does not have any poles in w_j, and therefore is constant as a function of w_j; this constant is easily determined to be zero.

The diagrammatical description of the amplitudes immediately implies that the amplitudes are local. We shall now use the formulae (E.1)–(E.3) to prove that they are also Möbius covariant. The Möbius group is generated by translations, scalings and the inversion $z \mapsto 1/z$. It is immediate from the above formulae that the amplitudes (with the grade of L being 2) are covariant under translations and scalings, and we therefore only have to check the covariance under the inversion $z \mapsto 1/z$. First, we calculate (setting for the moment $c_\psi = 0$ for all ψ of grade two)

$$\langle L(1/w) \prod_{i=1}^{n} V(\psi_i, 1/z_i)\rangle = \sum_{l=1}^{n} \frac{h_l}{(1/w - 1/z_l)^2} \langle \prod_{i=1}^{n} V(\psi_i, 1/z_i)\rangle$$
$$+ \sum_{l=1}^{n} \frac{1}{(1/w - 1/z_l)} \;\; \frac{d}{d\tilde{z}_l} \langle V(\psi_1, 1/z_1) \cdots V(\psi_l, \tilde{z}_l) \cdots V(\psi_n, 1/z_n)\rangle \Big|_{\tilde{z}_l = 1/z_l} . \tag{E.4}$$

Using the Möbius covariance of the original amplitudes, we find

$$\frac{d}{d\tilde{z}_l} \langle V(\psi_1, 1/z_1) \cdots V(\psi_l, \tilde{z}_l) \cdots V(\psi_n, 1/z_n)\rangle \Big|_{\tilde{z}_l = 1/z_l} = -z_l^2 \frac{d}{dz_l} \langle \prod_{i=1}^{n} V(\psi_i, 1/z_i)\rangle$$
$$= -z_l^2 \prod_{i \neq l} \left(\frac{-1}{z_i^2}\right)^{-h_i} \frac{d}{dz_l} \left[\left(\frac{-1}{z_l^2}\right)^{-h_l} \langle \prod_{i=1}^{n} V(\psi_i, z_i)\rangle \right]$$
$$= -z_l^2 \prod_{i=1}^{n} \left(\frac{-1}{z_i^2}\right)^{-h_i} \frac{d}{dz_l} \langle V(\psi_1, z_1) \cdots V(\psi_l, z_l) \cdots V(\psi_n, z_n)\rangle$$
$$- z_l^2 \prod_{i=1}^{n} \left(\frac{-1}{z_i^2}\right)^{-h_i} (-h_l) \frac{2}{z_l^3} \left(-\frac{1}{z_l^2}\right)^{-1} \langle \prod_{i=1}^{n} V(\psi_i, z_i)\rangle$$
$$= \prod_{i=1}^{n} \left(\frac{-1}{z_i^2}\right)^{-h_i} \left[-z_l^2 \frac{d}{dz_l} \langle V(\psi_1, z_1) \cdots V(\psi_l, z_l) \cdots V(\psi_n, z_n)\rangle \right.$$
$$\left. -2\, h_l\, z_l \, \langle \prod_{i=1}^{n} V(\psi_i, z_i)\rangle \right].$$

Inserting this formula in the above expression, we get

$$\langle L(1/w) \prod_{i=1}^{n} V(\psi_i, 1/z_i)\rangle$$
$$= \left(\frac{-1}{w^2}\right)^{-2} \prod_{i=1}^{n} \left(\frac{-1}{z_i^2}\right)^{-h_i} \left\{ \sum_{l=1}^{n} h_l \left[\frac{z_l^2}{w^2 (w - z_l)^2} + \frac{2 z_l^2}{w^3\,(w - z_l)} \right] \langle \prod_{i=1}^{n} V(\psi_i, z_i)\rangle \right.$$
$$\left. + \sum_{l=1}^{n} \frac{z_l^3}{w^3 (w - z_l)} \; \frac{d}{dz_l} \langle V(\psi_1, z_1) \cdots V(\psi_l, z_l) \cdots V(\psi_n, z_n)\rangle \right\}$$
$$= \left(\frac{-1}{w^2}\right)^{-2} \prod_{i=1}^{n} \left(\frac{-1}{z_i^2}\right)^{-h_i} \left\{ \sum_{l=1}^{n} h_l \frac{3 z_l^2 w - 2\, z_l^3}{w^3 (w - z_l)^2} \langle \prod_{i=1}^{n} V(\psi_i, z_i)\rangle \right.$$
$$\left. + \sum_{l=1}^{n} \frac{z_l^3}{w^3 (w - z_l)} \, \frac{d}{dz_l} \langle V(\psi_1, z_1) \cdots V(\psi_l, z_l) \cdots V(\psi_n, z_n)\rangle \right\} .$$

It remains to show that the expression in brackets actually agrees with (E.4). To prove this, we observe, that because of Möbius invariance of the amplitudes we have

$$\sum_{l=1}^{n} \frac{d}{dz_l} \langle V(\psi_1, z_1) \cdots V(\psi_l, z_l) \cdots V(\psi_n, z_n) \rangle = 0,$$

$$\sum_{l=1}^{n} h_l \langle \prod_{i=1}^{n} V(\psi_i, z_i) \rangle + \sum_{l=1}^{n} z_l \frac{d}{dz_l} \langle V(\psi_1, z_1) \cdots V(\psi_l, z_l) \cdots V(\psi_n, z_n) \rangle = 0$$

and

$$\sum_{l=1}^{n} 2\, h_l\, z_l \langle \prod_{i=1}^{n} V(\psi_i, z_i) \rangle + \sum_{l=1}^{n} z_l^2 \frac{d}{dz_l} \langle V(\psi_1, z_1) \cdots V(\psi_l, z_l) \cdots V(\psi_n, z_n) \rangle = 0.$$

The claim then follows from the observation that

$$\begin{aligned}
&\sum_{l=1}^{n} h_l \left[\frac{1}{(w - z_l)^2} - \frac{3 z_l^2 w - 2 z_l^3}{w^3 (w - z_l)^2} \right] \langle \prod_{i=1}^{n} V(\psi_i, z_i) \rangle \\
&+ \sum_{l=1}^{n} \left[\frac{1}{(w - z_l)} - \frac{z_l^3}{w^3 (w - z_l)} \right] \\
&\cdot \frac{d}{dz_l} \langle V(\psi_1, z_1) \cdots V(\psi_l, z_l) \cdots V(\psi_n, z_n) \rangle \\
= &\frac{1}{w^3} \left\{ w^2 \sum_{l=1}^{n} \frac{d}{dz_l} \langle \prod_{i}^{n} V(\psi_i, z_i) \rangle \right. \\
&+ w \left[\sum_{l=1}^{n} h_l \langle \prod_{i=1}^{n} V(\psi_i, z_i) \rangle + \sum_{l=1}^{n} z_l \frac{d}{dz_l} \langle \prod_{i}^{n} V(\psi_i, z_i) \rangle \right] \\
&\left. + \sum_{l=1}^{n} 2\, h_l\, z_l \langle \prod_{i=1}^{n} V(\psi_i, z_i) \rangle + \sum_{l=1}^{n} z_l^2 \frac{d}{dz_l} \langle \prod_{i=1}^{n} V(\psi_i, z_i) \rangle \right\} \\
= &\, 0.
\end{aligned}$$

We have thus shown that the functions of the form (E.2) (for $c_\psi = 0$) have the correct transformation property under Möbius transformations. This implies, as L is quasiprimary, that the functions of the form (E.3) have the right transformation property for $c = 0$. However, the sum involving the c-terms has also (on its own) the right transformation property, and thus the above functions have. This completes the proof.

Finally, we want to show that the amplitudes $\hat{\mathcal{A}}$ have the cluster property provided the amplitudes $\mathcal{A}$ do. We want to prove the cluster property by induction on the number N_L of L-fields in the extended amplitudes. If $N_L = 0$, then the result follows from the assumption about the original amplitudes. Let us therefore assume that the result has been proven for $N_L = N$, and consider the amplitudes with $N_L = N + 1$. For a given amplitude, we subdivide the fields into two groups, and we consider the limit, where the parameters z_i of one group are scaled to zero, whereas the parameters ζ_j of the other group are kept fixed. Because of the Möbius covariance, we may assume that the group whose parameters z_i are scaled to zero contain at least one L-field, $L(z_1)$, say, and we can use (E.2) (or (E.3)) to rewrite the amplitudes involving $L(z_1)$ in terms of amplitudes which do not involve $L(z_1)$ and which have $N_L \leq N$. It then follows from (E.2) (or (E.3)) together with the induction hypothesis that the terms involving $(z_1 - \zeta_j)^{-l}$ (where $l = 1, 2$ or $l = 4$) are not of leading order in the limit where the z_i are scaled to zero, whereas all terms with $(z_1 - z_i)^{-l}$ are. This implies, again by the induction hypothesis, that the amplitudes satisfy the cluster property for $N_L = N + 1$, and the result follows by induction.

13. Appendix F: Möbius Transformation of Zhu's Modes

We want to prove formula (110) in this Appendix. We have to show that

$$\begin{aligned} V^{(N)}_{u_1,u_2}(\psi) &= U(\gamma)V^{(N)}(U(\gamma)^{-1}\psi)U(\gamma)^{-1} \\ &= U(\gamma)\oint_0 V\left[(\zeta+1)^{L_0}U(\gamma)^{-1}\psi,\zeta\right]\frac{d\zeta}{\zeta^{N+1}}U(\gamma)^{-1} \\ &= \oint_0 U(\gamma)V\left[(\zeta+1)^{L_0}U(\gamma)^{-1}\psi,\zeta\right]U(\gamma)^{-1}\frac{d\zeta}{\zeta^{N+1}}. \end{aligned}$$

We therefore have to find an expression for the transformed vertex operator. By the uniqueness theorem, it is sufficient to evaluate the expression on the vacuum; then we find

$$U(\gamma)V\left[(\zeta+1)^{L_0}U(\gamma)^{-1}\psi,\zeta\right]U(\gamma)^{-1}\Omega = U(\gamma)e^{\zeta L_{-1}}(\zeta+1)^{L_0}U(\gamma)^{-1}\psi.$$

To calculate the product of the Möbius transformations, we write them in terms of 2×2 matrices, determine their product and rewrite the resulting matrix in terms of the generators $L_0, L_{\pm 1}$. After a slightly lengthy calculation we then find

$$\begin{aligned} &U(\gamma)e^{\zeta L_{-1}}(\zeta+1)^{L_0}U(\gamma)^{-1}\psi = \\ &V\left[\left(\frac{\zeta+1}{(1-\frac{u_2\zeta}{(u_1-u_2)})^2}\right)^{L_0}\exp\left(\frac{\zeta}{u_2\zeta+(u_2-u_1)}L_1\right)\psi,\ \frac{u_1u_2\zeta}{u_2\zeta+(u_2-u_1)}\right]\Omega. \end{aligned}$$

In the integral for $V^{(N)}_{u_1,u_2}$ we then change variables to

$$w = \frac{u_1u_2\zeta}{u_2\zeta+(u_2-u_1)} = \gamma(\zeta);$$

in terms of w the relevant expressions become

$$1+\zeta = \frac{u_1(w-u_2)}{u_2(w-u_1)} \quad 1-\frac{u_2\zeta}{(u_1-u_2)} = \frac{u_1}{(u_1-w)} \quad d\zeta = \frac{u_1(u_2-u_1)}{u_2}\frac{dw}{(w-u_1)^2}.$$

Putting everything together, we then obtain formula (110).

14. Appendix G: Rings and Algebras

In this Appendix we review various concepts in algebra; the treatment follows closely the book [24].

We restrict attention to rings, R which have a unit element, $1 \in R$. An algebra, A, over a field F, is a ring which is also a vector space over F in such a way that the structures are compatible [$\lambda(xy) = (\lambda x)y, \lambda \in F, x, y \in R$]. The dimension of A is its dimension as a vector space. We shall in general consider complex algebras, i.e. algebras over $\mathbb{C}$. Since $1 \in A$ we have $F \subset A$.

A (left) module for a ring R is an additive group M with a map $R \times M \to M$, compatible with the structure of R [i.e. $(rs)m = r(sm), (r+s)m = rm + sm, \quad r, s \in R, m \in M$]. A module M for an algebra A, viewed as a ring, is necessarily a vector space over F (because $F \subset A$) and provides a representation of A as an algebra in terms of endomorphisms of the vector space M.

R provides a module for itself, the adjoint module. A submodule N of a module M for R is an additive subgroup of M such that $rN \subset N$ for all $r \in R$. A simple or irreducible module is one which has no proper submodules. A (left) ideal J of R is a submodule of the adjoint module, *i.e.* an additive subgroup $J \subset R$ such that $rj \in I$ for all $r \in R$, $j \in J$. The direct sum $M_1 \oplus M_2$ of the R modules M_1, M_2 is the additive group $M_1 \oplus M_2$ with $r(m_1, m_2) = (rm_1, rm_2), r \in R, m_1 \in M_1, m_2 \in M_2$. The direct sum of a (possibly infinite) set $M_i, i \in I$, of R modules consists of elements $(m_i, i \in I)$, with all but finitely many $m_i = 0$. The module M is decomposable if it can be written as the direct sum of two non-zero modules and completely reducible if it can be written as the direct sum of a (possibly infinite) sum of irreducible modules.

A representation of an algebra A is irreducible if it is irreducible as a module of the ring A.

An ideal J is maximal in R if $K \supset J$ is another ideal in R, then $K = R$. If M is an irreducible module for the ring R, then $M \cong R/J$ for some maximal ideal $J \subset R$. [Take $m \in M, m \neq 0$ and consider $Rm \subset M$. This is a submodule, so $Rm = M$. The kernel of the map $r \mapsto rm$ is an ideal, $J \subset R$. So $M \cong R/J$. If $J \subset K \subset R$ and K is an ideal, then K/J defines a submodule of M, so that $K/J = M$ and $K = R$, i.e. J is maximal.] Thus an irreducible representation of a finite-dimensional algebra A is necessarily finite-dimensional.

The coadjoint representation A' of an algebra A is defined on the dual vector space to A consisting of linear maps $\rho : A \to F$ with $(r\rho)(s) = \rho(sr)$. If M is an n-dimensional irreducible representation of A and $d_{ij}(r)$ the corresponding representation matrices, the n elements $d_{ij}(r)$, $1 \leq j \leq n$, i fixed, define an n-dimensional invariant subspace A' corresponding to a representation equivalent to M. So the sum of the dimensions of the inequivalent representations of A does not exceed $\dim A$. This shows that each irreducible representation of a finite-dimensional algebra is finite-dimensional and there are only finitely many equivalence classes of such representations.

This is not such a strong statement as it seems because A may have indecomposable representations. In fact A may have an infinite number of inequivalent representations of a given dimension even if $\dim A < \infty$. [E.g. consider the three dimensional complex algebra, consisting of $\lambda + \mu x + \nu y$, $\lambda, \mu, \nu \in \mathbb{C}$, subject to $x^2 = y^2 = xy = yx = 0$, which has the faithful three dimensional representation $\begin{pmatrix} \lambda & 0 & \mu \\ 0 & \lambda & \nu \\ 0 & 0 & \lambda \end{pmatrix}$ and the inequivalent two-dimensional representations $\begin{pmatrix} \lambda & \mu + \xi \nu \\ 0 & \lambda \end{pmatrix}$, for each $\xi \in \mathbb{C}$.] The situation is more under control if the algebra is semi-simple.

The ring R is semi-simple if the adjoint representation is completely reducible. If R is semi-simple, 1 is the sum of a finite number of elements of R, one in each of a number of the summands in the expression of R as a sum of irreducible modules, $1 = \sum_{i=1}^n e_i$ and, since any $r = \sum_{i=1}^n r e_i$, it follows that there is a finite number, n, of summands $R_i = R e_i$ and $R = \bigoplus_{i=1}^n R_i$. If R is semi-simple, every R module is completely reducible (though not necessarily into a finite number of irreducible summands). [Any module M is the quotient of the free module $\mathcal{R} = \bigoplus_{m \in M} R_m$, where R_m is a copy of the adjoint module R, by the ideal consisting of those $(r_m)_{m \in M}$ such that $\sum_{m \in M} r_m m = 0$. The result follows since $\mathcal{R}$ is completely reducible if R is and the quotient of a completely reducible module is itself completely reducible.]

An R module M is finitely-generated if $M = \{\sum_{i=1}^n r_i m_i : r_i \in R\}$ for a finite number, n, of fixed elements $m_i \in M$, $1 \le i \le n$. If R is semi-simple, any finitely generated R module is completely reducible into a finite number of summands. [This follows because M is the quotient of $\bigoplus_{i=1}^n R_i$, where $R_i \cong R$, by the ideal $\{(r_i) : \sum_{i=1}^n r_i m_i = 0\}$.]

If M and N are R modules, an R-homomorphism $f : M \to N$ is a map satisfying $rf = fr$. If M and N are simple modules, Schur's Lemma implies that the set of R-homomorphisms $\mathrm{Hom}_R(M, N) = 0$ if M and N are not equivalent. If $M = N$, $\mathrm{Hom}_R(M, M) \equiv \mathrm{End}_R(M)$ is a division ring, that is every ring in which every non-zero element has an inverse. In the case of an algebra, if $\dim M < \infty$, $\mathrm{End}_A(M) = F$, the underlying field.

If the R-module M is completely decomposable into a finite number of irreducible submodules, we can write $M = \sum_{i=1}^N M_i^{n_i}$, where each M_i is irreducible and M_i and M_j are inequivalent if $i \ne j$. Since $\mathrm{Hom}_R(M_i, M_j) = 0$ if $i \ne j$,

$$\mathrm{End}_R(M) = \prod_{i=1}^N \mathrm{End}_R(M_i^{n_i}) = \prod_{i=1}^n \mathcal{M}_{n_i}(D_i),$$

where the division algebra $D_i = \mathrm{End}_R(M_i)$ and $\mathcal{M}_n(D)$ is the ring of $n \times n$ matrices with entries in the division algebra D.

If R is a semi-simple ring, we can write $R = \oplus_{i=1}^N R_i^{n_i}$, where the R_i are irreducible as R modules and inequivalent for $i \ne j$. So $\mathrm{End}_R R = \prod_{i=1}^N \mathcal{M}_{n_i}(D_i)$, where $D_i = \mathrm{End}_R(R_i)$. But $\mathrm{End}_R R = R^o$, the reverse ring to R defined on the set R by taking the product of r and s to be sr rather than rs. Since, evidently, $(R^o)^o = R$,

$$R = \prod_{i=1}^N \mathcal{M}_{n_i}(D_i^o),$$

i.e. every semi-simple ring is isomorphic to the direct product of a finite number of finite-dimensional matrix rings over division algebras [Wedderburn's Structure Theorem].

In the case of a semi-simple algebra, each $D_i = F$, the underlying field, so

$$A = \prod_{i=1}^N \mathcal{M}_{n_i}(F),$$

where $\mathcal{M}_n(F)$ is the algebra of $n \times n$ matrices with entries in the field F. In particular, any semi-simple algebra is finite-dimensional.

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