Math 241 Final Exam
Spring 2014

Name:

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Please clear your desk and put away all notes, books, electronic devices, etc. No outside material is allowed during the exam. Make sure to clearly indicate your responses; box your final answers if necessary.

My signature below certifies that I will comply with the University of Pennsylvania's Code of Academic Integrity in completing this examination.

Your signature

Question Number	Points Possible	Your Score
1	6	
2	6	
3	6	
4	6	
5	9	
6	9	
7	9	
8	9	
Total	60	

A Partial Table of Integrals

$$\int_0^x u\cos nu\,du = \frac{\cos nx + nx\sin nx - 1}{n^2} \qquad \text{for any real } n \neq 0$$

$$\int_0^x u\sin nu\,du = \frac{\sin nx - nx\cos nx}{n^2} \qquad \text{for any real } n \neq 0$$

$$\int_0^x e^{mu}\cos nu\,du = \frac{e^{mx}(m\cos nx + n\sin nx) - m}{m^2+n^2} \qquad \text{for any real } n, m$$

$$\int_0^x e^{mu}\sin nu\,du = \frac{e^{mx}(-n\cos nx + m\sin nx) + n}{m^2+n^2} \qquad \text{for any real } n, m$$

$$\int_0^x \sin nu\cos mu\,du = \frac{m\sin nx\sin mx + n\cos nx\cos mx - n}{m^2-n^2} \qquad \text{for any real numbers } m \neq n$$

$$\int_0^x \cos nu\cos mu\,du = \frac{m\cos nx\sin mx - n\sin nx\cos mx}{m^2-n^2} \qquad \text{for any real numbers } m \neq n$$

$$\int_0^x \sin nu\sin mu\,du = \frac{n\cos nx\sin mx - m\sin nx\cos mx}{m^2-n^2} \qquad \text{for any real numbers } m \neq n$$

Facts About Bessel Functions

- Bessel's equation: $r^2 f''(r) + rf'(r) + (\alpha^2 r^2 - m^2)f(r) = 0$ for each integer $m \geq 0$. The only solutions which are bounded at $r = 0$ are $f(r) = cJ_m(\sqrt{\alpha}r)$ for a constant c.
- Orthogonality relation: Writing z_{mn} as the nth zero of $J_m(z)$,

$$\int_0^1 rJ_n(z_{mn}r)J_n(z_{mk}r)\,dr = 0, \qquad n \neq k$$

for each $m = 0, 1, \ldots$

Fourier Transform

$$\mathcal{F}[u](\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x)e^{i\omega x}dx, \qquad \mathcal{F}^{-1}[U](x) = \int_{-\infty}^{\infty} U(\omega)e^{-i\omega x}d\omega$$

Table of Fourier Transform Pairs ($\alpha, \beta > 0$)

$u(x) = \mathcal{F}^{-1}[U]$	$U(\omega) = \mathcal{F}[u]$	$u(x) = \mathcal{F}^{-1}[U]$	$U(\omega) = \mathcal{F}[u]$
$e^{-\alpha x^2}$	$\frac{1}{\sqrt{4\pi\alpha}}e^{-\frac{\omega^2}{4\alpha}}$	$\sqrt{\frac{\pi}{\beta}}e^{-\frac{x^2}{4\beta}}$	$e^{-\beta\omega^2}$
$e^{-\alpha\lvert x\rvert}$	$\frac{1}{2\pi}\frac{2\alpha}{x^2+\alpha^2}$	$\frac{2\beta}{x^2+\beta^2}$	$e^{-\beta\lvert\omega\rvert}$
$u(x) = \begin{cases} 0 & \lvert x\rvert > \alpha \\ 1 & \lvert x\rvert < \alpha \end{cases}$	$\frac{1}{\pi}\frac{\sin\alpha\omega}{\omega}$	$2\frac{\sin\beta x}{x}$	$U(\omega) = \begin{cases} 0 & \lvert\omega\rvert > \beta \\ 1 & \lvert\omega\rvert < \beta \end{cases}$

Laplacian in Polar Coordinates

$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$$

1. Consider the wave equation for a vibrating string

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L$$

subject to the boundary conditions

$$u(0,t) = u(L,t) = 0.$$

Which of the following statements are correct? Note that these statements are graded +1 for each correct answer, −1 for each incorrect answer, 0 for no answer.

(a) u represents the horizontal displacement of the string. Y N

(b) Newton's second law was used to derive this equation. Y N

(c) c has units of speed. Y N

(d) The model is valid for large displacements in the string. Y N

(e) The endpoints of the string are fixed. Y N

(f) Increasing the length of the string, while keeping the density and tension fixed, will increase the frequency at which the string vibrates. Y N

2. (a) Write the forward time, centered spatial finite difference scheme for the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.$$

Here $0 < x < L$ and $0 < t < T$. Use the notation $u_j^{(m)}$ for the approximation of the values $u(j\Delta x, m\Delta t)$ for the true solution for $j = 0, 1, \ldots, N$, and $m = 0, 1, \ldots, M$.

(b) When is the approximation method from part (a) numerically stable?

3. Set

$$f(x) = \begin{cases} x - 1, & -1 \le x \le 0 \\ x + 1, & \phantom{-1 \le{}} 0 < x \le 1 \end{cases} .$$

(a) Compute the coefficients in the Fourier series of $f(x)$

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\pi x) + b_n \sin(n\pi x)).$$

(b) Plot the Fourier series of $f(x)$ on the interval $[-3, 3]$.

4. (a) Find the harmonic function $u(r, \theta)$ on the disk $r^2 \leq 1$ that satisfies the boundary condition

$$u(1, \theta) = 1 + \sin(2\theta).$$

(b) Explain why this solution is always less than or equal to 2.

5. Solve the heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}, \qquad -\infty < x < \infty,\ t > 0$$

subject to the initial condition

$$u(x,0) = e^x.$$

6. Find $u(x,t)$ that satisfies the nonhomogeneous heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + e^{-\pi^2 t}\sin(3\pi x) \qquad \text{for } 0 < x < 1,\ t > 0$$

with boundary conditions $u(0,t) = u(1,t) = 0$ and initial condition $u(x,0) = 2$.

7. Consider a vibrating circular membrane of radius 1 that has no displacement on the boundary. The associated PDE is the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u.$$

(a) After separating time, we are lead to the eigenvalue problem

$$\nabla^2 \phi + \lambda \phi = 0, \quad \phi(1, \theta) = 0.$$

Separate variables $\phi(r, \theta) = f(r) g(\theta)$ and derive ODE for $f(r)$ and $g(\theta)$.

(b) Find the eigenvalues λ_{mn} and the corresponding eigenfunctions ϕ_{mn}. You may assume all eigenvalues are positive.

(c) Find a general solution $u(r,t)$ when the membrane is initially circularly symmetric

$$u(r,0) = \alpha(r), \quad \frac{\partial u}{\partial t}(r,0) = 0.$$

8. (**Note:** Respond if HYND is your instructor) Consider the energy

$$E(t) = \frac{1}{2}\int_0^1 \left(\frac{\partial u}{\partial t}(x,t)\right)^2 + \left(\frac{\partial u}{\partial x}(x,t)\right)^2 \, dx.$$

associated with a solution $u(x,t)$ of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \qquad 0 < x < 1,\ t > 0.$$

Show that if the boundary conditions

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t) \quad \text{and} \quad \frac{\partial u}{\partial t}(0,t) = \frac{\partial u}{\partial t}(1,t)$$

are satisfied for all $t > 0$, then

$$E(t) = E(0), \quad t \geq 0.$$

8. (**Note:** Respond if WEBER is your instructor) Consider the second order differential equation on the domain $[1, 2]$:

$$x^2 \frac{d^2 f}{dx^2} + 4x \frac{df}{dx} + \left(\lambda - x^2\right) f \;=\; 0,\; f(1) = 0,\; f(2) = 0. \tag{0.1}$$

This is almost, but not quite, a Bessel-type differential equation.

a) Put the equation into Sturm-Liouville form. What are $p(x)$, $q(x)$, and $\sigma(x)$?

b) According to the Sturm-Liouville theory, the eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ come as a discrete list, and to each eigenvalue λ_n corresponds an eigenfunction $\varphi_n(x)$. The eigenfunctions satisfy certain orthogonality relations. For the differential equation (0.1), write out this relation in terms of the appropriate integral or integrals.

c) If $\{\lambda, \varphi(x)\}$ constitute an eigenvalue-eigenfunction pair for this Sturm-Liouville equation, show that necessarily $\lambda > 0$. Give a specific reason why $\lambda = 0$ is not actually an eigenvalue. (Hint: Remember the Rayleigh quotient.)