# Math 241 MAKE-UP FINAL EXAM
## Fall 2013

Name:

Instructor: **Wong** **Pantev**

Recitation Number and Day/Time:

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Please *turn off and put away all electronic devices.* You may use both sides of a $8.5'' \times 11''$ sheet of paper for handwritten notes while you take this exam. No calculators, no course notes, no books, no help from your neighbors. **Show all work**. Please **clearly mark** your final answer. Remember to put your name at the top of this page. Good luck!

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My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this examination.

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Your signature

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| Question Number | Points Possible | Your Score |
|---|---|---|
| 1 | 10 | |
| 2 | 10 | |
| 3 | 10 | |
| 4 | 10 | |
| 5 | 10 | |
| 6 | 10 | |
| 7 | 10 | |
| 8 | 10 | |

| Total Score | /80 |
|---|---|

# A Partial Table of Integrals

$$\int_0^x u\cos nu\;du = \frac{\cos nx + nx\sin nx - 1}{n^2} \qquad \text{for any real } n \neq 0$$

$$\int_0^x u\sin nu\;du = \frac{\sin nx - nx\cos nx}{n^2} \qquad \text{for any real } n \neq 0$$

$$\int_0^x e^{mu}\cos nu\;du = \frac{e^{mx}(m\cos nx + n\sin nx) - m}{m^2+n^2} \qquad \text{for any real } n, m$$

$$\int_0^x e^{mu}\sin nu\;du = \frac{e^{mx}(-n\cos nx + m\sin nx) + n}{m^2+n^2} \qquad \text{for any real } n, m$$

$$\int_0^x \sin nu\cos mu\;du = \frac{m\sin nx\sin mx + n\cos nx\cos mx - n}{m^2-n^2} \qquad \text{for any real numbers } m \neq n$$

$$\int_0^x \cos nu\cos mu\;du = \frac{m\cos nx\sin mx - n\sin nx\cos mx}{m^2-n^2} \qquad \text{for any real numbers } m \neq n$$

$$\int_0^x \sin nu\sin mu\;du = \frac{n\cos nx\sin mx - m\sin nx\cos mx}{m^2-n^2} \qquad \text{for any real numbers } m \neq n$$

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# Formulas Involving Bessel Functions

- Bessel's equation: $r^2R'' + rR' + (\alpha^2r^2 - n^2)R = 0$ – The only solutions of this which are bounded at $r = 0$ are $R(r) = cJ_n(\alpha r)$.

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!}\left(\frac{x}{2}\right)^{n+2k}.$$

  $J_0(0) = 1,\ J_n(0) = 0$ if $n > 0$. $z_{nm}$ is the $m$th positive zero of $J_n(x)$.

- Orthogonality relations:

$$\text{If } m \neq k \text{ then } \int_0^1 xJ_n(z_{nm}x)J_n(z_{nk}x)\,dx = 0 \quad \text{and} \quad \int_0^1 x(J_n(z_{nm}x))^2\,dx = \frac{1}{2}J_{n+1}(z_{nm})^2.$$

- Recursion and differentiation formulas:

$$\frac{d}{dx}(x^nJ_n(x)) = x^nJ_{n-1}(x) \quad \text{or} \quad \int x^nJ_{n-1}(x)\,dx = x^nJ_n(x) + C \quad \text{for } n \geq 1 \tag{1}$$

$$\frac{d}{dx}(x^{-n}J_n(x)) = -x^{-n}J_{n+1}(x) \quad \text{for} \quad n \geq 0 \tag{2}$$

$$J_n'(x) + \frac{n}{x}J_n(x) = J_{n-1}(x) \tag{3}$$

$$J_n'(x) - \frac{n}{x}J_n(x) = -J_{n+1}(x) \tag{4}$$

$$2J_n'(x) = J_{n-1}(x) - J_{n+1}(x) \tag{5}$$

$$\frac{2n}{x}J_n(x) = J_{n-1}(x) + J_{n+1}(x) \tag{6}$$

- Modified Bessel's equation: $r^2R'' + rR' - (\alpha^2r^2 + n^2)R = 0$ – The only solutions of this which are bounded at $r = 0$ are $R(r) = cI_n(\alpha r)$.

$$I_n(x) = i^{-n}J_n(ix) = \sum_{k=0}^{\infty} \frac{1}{k!(k+n)!}\left(\frac{x}{2}\right)^{n+2k}.$$

# Formulas Involving Associated Legendre and Spherical Bessel Functions

- Associated Legendre Functions: $\frac{d}{d\phi}\left(\sin\phi\frac{dg}{d\phi}\right)+\left(\mu-\frac{m^2}{\sin\phi}\right)g=0$. Using the substitution $x=\cos\phi$, this equation becomes $\frac{d}{dx}\left((1-x^2)\frac{dg}{dx}\right)+\left(\mu-\frac{m^2}{1-x^2}\right)g=0$. This equation has bounded solutions only when $\mu=n(n+1)$ and $0\le m\le n$. The solution $P_n^m(x)$ is called an associated Legendre function of the first kind.

- Associated Legendre Function Identities:

$$P_n^0(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n \text{ and } P_n^m(x)=(-1)^m(1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x) \text{ when } 1\le m\le n$$

- Orthogonality of Associated Legendre Functions: If $n$ and $k$ are both greater than or equal to $m$,

$$\text{If } n\neq k \text{ then } \int_{-1}^{1} P_n^m(x)P_k^m(x)dx=0 \text{ and } \int_{-1}^{1}\left(P_n^m(x)\right)^2 dx=\frac{2(n+m)!}{(2n+1)(n-m)!}.$$

- Spherical Bessel Functions: $(\rho^2 f')'+(\alpha^2\rho^2-n(n+1))f=0$. If we define the spherical Bessel function $j_n(\rho)=\rho^{-\frac{1}{2}}J_{n+\frac{1}{2}}(\rho)$, then only solution of this ODE bounded at $\rho=0$ is $j_n(\alpha\rho)$.

- Spherical Bessel Function Identity:

$$j_n(x)=x^2\left(-\frac{1}{x}\frac{d}{dx}\right)^n\left(\frac{\sin x}{x}\right).$$

- Spherical Bessel Function Orthogonality: Let $z_{nm}$ be the $m$-th positive zero of $j_m$.

$$\text{If } m\neq k \text{ then } \int_0^1 x^2 j_n(z_{nm}x)j_n(z_{nk}x)dx=0 \quad\text{and}\quad \int_0^1 x^2(j_n(z_{nm}x))^2dx=\frac{1}{2}(j_{n+1}(z_{nm}))^2.$$

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# One-Dimensional Fourier Transform

$$\mathcal{F}[u](\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}u(x)e^{i\omega x}dx,\qquad \mathcal{F}^{-1}[U](x)=\int_{-\infty}^{\infty}U(\omega)e^{-i\omega x}d\omega$$

## Table of Fourier Transform Pairs

## Fourier Transform Pairs ($\alpha > 0$)

| $u(x) = \mathcal{F}^{-1}[U]$ | $U(\omega) = \mathcal{F}[u]$ |
|---|---|
| $e^{-\alpha x^2}$ | $\frac{1}{\sqrt{4\pi\alpha}} e^{-\frac{\omega^2}{4\alpha}}$ |
| $e^{-\alpha\|x\|}$ | $\frac{1}{2\pi} \frac{2\alpha}{x^2 + \alpha^2}$ |
| $u(x) = \begin{cases} 0 & \|x\| > \alpha \\ 1 & \|x\| < \alpha \end{cases}$ | $\frac{1}{\pi} \frac{\sin \alpha\omega}{\omega}$ |
| $\delta(x - x_0)$ | $\frac{1}{2\pi} e^{i\omega x_0}$ |
| $\frac{\partial u}{\partial t}$ | $\frac{\partial U}{\partial t}$ |
| $\frac{\partial u}{\partial x}$ | $-i\omega U$ |
| $xu$ | $-i\frac{\partial U}{\partial \omega}$ |
| $u(x - x_0)$ | $e^{i\omega x_0} U$ |

## Fourier Transform Pairs ($\beta > 0$)

| $u(x) = \mathcal{F}^{-1}[U]$ | $U(\omega) = \mathcal{F}[u]$ |
|---|---|
| $\sqrt{\frac{\pi}{\beta}} e^{-\frac{x^2}{4\beta}}$ | $e^{-\beta\omega^2}$ |
| $\frac{2\beta}{x^2 + \beta^2}$ | $e^{-\beta\|\omega\|}$ |
| $2\frac{\sin \beta x}{x}$ | $U(\omega) = \begin{cases} 0 & \|\omega\| > \beta \\ 1 & \|\omega\| < \beta \end{cases}$ |
| $e^{-i\omega_0 x}$ | $\delta(\omega - \omega_0)$ |
| $\frac{\partial^2 u}{\partial t^2}$ | $\frac{\partial^2 U}{\partial t^2}$ |
| $\frac{\partial^2 u}{\partial x^2}$ | $(-i\omega)^2 U$ |
| $x^2 u$ | $(-i)^2 \frac{\partial^2 U}{\partial \omega^2}$ |
| $\frac{1}{2\pi} \int_{-\infty}^{\infty} f(s) g(x - s) ds$ | $FG$ |

**(1)** 10 POINTS Derive the heat equation for a one-dimensional rod assuming that the cross-sectional area $A(x)$ is a non-constant function of $x$, where $0 < x < L$. Assume all other thermal properties are constant, and there is no heat source.

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**(2)** 10 POINTS The function $u(r, \theta)$ describes the steady state temperature distribution in a thin plate $R$ shaped as an anulus with outer radius 2 and inner radius 1. Suppose that heat flux across the boundary of $R$ is given by $u_r(1, \theta) = 2$ for the inner circle, and $u_r(2, \theta) = c\sin^2(3\theta)$ for the outer circle.

**(a)** (4 points) What must the value of the constant $c$? That is: what must the value of $c$ be so that the boundary value problem

$$\left| \begin{array}{l} \nabla^2 u = 0 \\ u_r(1, \theta) = 2 \\ u_r(2, \theta) = c\sin^2(3\theta) \end{array} \right.$$

will have a solution.

**(b)** (6 points) Find the general solution $u(r, \theta)$, you don't need to compute the coefficients.

**(3)** 10 POINTS Denote by $g(x)$ the Fourier sine series of the function $e^x$ on the interval $[0,1]$, that is,

$$e^x \sim \sum_{n=1}^{\infty} b_n \sin n\pi x =: g(x).$$

Decide whether the following statements are true or false. To obtain the full credit, you must justify your reasoning.

(i) (2.5 points) $g$ is an even function. True / False

(ii) (2.5 points) $g$ is periodic. True / False

(iii) (2.5 points) $g$ is bounded. True / False

(iv) (2.5 points) $g(1) = e$. True / False

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**(4)** 10 POINTS Let $u(x,t)$ be the vertical displacement of a vibrating string with $1 < x < 3$. The string has free boundaries, and has constant density $\rho = 4$ and tension with constant magnitude $T = 1$ and thus obeys the wave equation $4u_{tt} = u_{xx}$, with boundary conditions $u_x(1,t) = 0$ and $u_x(3,t) = 0$. The initial position and velocity are given by $u(x,0) = 0$ and $u_t(x,0) = \sqrt{2x}$ for all $1 < x < 3$. Calculate the total energy

$$E(t) = \frac{1}{2}\int_1^3 \left(\rho u_t^2 + T u_x^2\right) dx,$$

of the string.

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**(5)** 10 POINTS Estimate the *large* eigenvalues for the following eigenvalue problem:

$$\begin{cases} \dfrac{d}{dx}\left(e^{2x}\dfrac{d\phi}{dx}\right) + \left(\lambda e^{4x} + e^{3x}\right)\phi = 0 \text{ for } 0 < x < 1 \\ \phi(0) = \phi(1) = 0. \end{cases}$$

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**(6)** 10 POINTS Let $u(\rho, \theta, \phi)$ be a solution of the Laplace equation $\nabla^2 u = 0$ inside a sphere of radius 2 centered at the origin, subject to the boundary condition $u(2, \theta, \phi) = 5P_2^0(\cos\phi)$, where $P_2^0(x)$ is the associated Legendre function of the first kind.

**(a)** (5 points) Compute $u(1, \pi, 0)$.

**(b)** (5 points) Compute $\lim_{\rho \to 0} u(\rho, \pi, 0)$.

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**(7)** 10 POINTS Solve the Poisson equation in a unit disk ($r < 1$, $-\pi \leq \theta \leq \pi$):

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = r^3$$

subject to the boundary condition

$$u(1, \theta) = \sin\theta.$$

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**(8)** 10 POINTS Compute the Fourier transforms of the following functions

**(a)** (3 points) $2e^{-x^2/9}$,

**(b)** (4 points) $e^{-(x+1)^2} * e^{-(x-1)^2}$,

**(c)** (3 points) $\frac{d^3}{dx^3}e^{-x^2}$

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