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Math 241

Final examination

Instructions. Answer the following problems carefully and completely. Show all your work. Do not use a calculator. You may use both sides of one $8\frac{1}{2} \times 11$ sheet of paper for handwritten notes you wrote yourself. Please turn in your sheet of notes with your exam. There are 100 points possible. Good luck!

Name Solutions

Instructors's name ________________

TA's name and time ________________

1.	(2)	
2.	(14)	
3.	(6)	
4.	(2)	
5.	(3)	
6.	(8)	
7.	(8)	
8.	(5)	
9.	(5)	
10.	(10)	
11.	(11)	
12.	(6)	
13.	(6)	
14.	(14)	
Total	(100)	

Here are some integrals you can use:

$$\int_0^\infty xe^{-x}\sin(cx)\,dx = \frac{2c}{(1+c^2)^2}$$
$$\int_0^\infty xe^{-x}\cos(cx)\,dx = \frac{1-c^2}{(1+c^2)^2}$$

1. Write whether the following statement is true or false. (You do not need to show any work.) The product of an odd function f with an odd function g is an odd function.

FALSE

2. Use a Fourier transform, a sine transform, or a cosine transform to find the displacement $u(x,t)$, for $x > 0$ and $t > 0$, of a semi-infinite string if

$$u(0,t) = 0, \quad u(x,0) = xe^{-x}, \quad \text{and} \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0.$$

You may assume the constant a^2 of the wave equation is equal to 1. Your final answer may contain an integral.

Let $U(\alpha,t) = \int_0^\infty u(x,t)\sin(\alpha x)\,dx$ be the sign transform of $u(x,t)$.

$$\mathcal{F}_s\left\{\frac{\partial^2 u}{\partial x^2}\right\} = -\alpha^2 U(\alpha,t) + \alpha\, \overset{0}{\cancel{u(0,t)}}$$

$$= -\alpha^2 U(\alpha,t)$$

$$\mathcal{F}_s\left\{\frac{\partial^2 u}{\partial t^2}\right\} = \frac{\partial^2 U}{\partial t^2}.$$

$\therefore$ the wave eqn $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$ transforms to

$$-\alpha^2 U(\alpha,t) = \frac{\partial^2 U}{\partial t^2}.$$

$$\Rightarrow \quad U(\alpha,t) = A\cos(\alpha t) + B\sin(\alpha t)$$

where A & B are fcts of α alone.

$$0 = \mathcal{F}_s\{0\} = \mathcal{F}_s\left\{\left.\frac{\partial u}{\partial t}\right|_{t=0}\right\} = \left.\frac{\partial U}{\partial t}\right|_{t=0}.$$

$$\Rightarrow \quad B = 0 \quad \Rightarrow \quad \mathcal{U}(\alpha,t) = A\cos(\alpha t).$$

$$A = \mathcal{U}(\alpha,0) = \mathcal{F}_s\{u(x,0)\} = \mathcal{F}_s\{xe^{-x}\} = \frac{2\alpha}{(1+\alpha^2)^2}.$$

$$\Rightarrow \quad \mathcal{U}(\alpha,t) = \frac{2\alpha}{(1+\alpha^2)^2}\cdot\cos(\alpha t).$$

$$\Rightarrow \quad u(x,t) = \mathcal{F}_s^{-1}\{\mathcal{U}(\alpha,t)\} = \frac{2}{\pi}\int_0^\infty \frac{2\alpha}{(1+\alpha^2)^2}\cos(\alpha t)\sin(\alpha x)\,d\alpha.$$

3. Find *any two* independent solutions $u(x, y)$ to the following PDE:

$$\frac{\partial^2 u}{\partial x \partial y} = u$$

Neither of your solutions can be the zero function.

e^{x+y} is one solution.

$e^{2x+y/2}$ is another solution.

$\frac{e^{x+y}}{e^{2x+y/2}} = e^{-x+y/2}$ is not a constant fct..

$\Rightarrow$ they are independent.

4. Find a and b real numbers such that

$$\frac{10-5i}{6+2i} = a + ib.$$

$$\frac{(10-5i)(6-2i)}{36+4} = \frac{60-30i-20i-10}{40}$$

$$= \frac{5}{4} - \frac{5}{4}i$$

$\Rightarrow \quad a = 5/4, \ b = -5/4.$

5. Let

$$
\begin{aligned}
z_1 &= 2\cos(\pi/8) + 2i\sin(\pi/8) \\
z_2 &= 4\cos(3\pi/8) + 4i\sin(3\pi/8)
\end{aligned}
$$

Find a and b real numbers such that

$$\frac{z_1}{z_2} = a + ib.$$

$$z_1 = 2e^{\pi i/8} \qquad z_2 = 4e^{3\pi i/8}$$

$$\frac{z_1}{z_2} = \frac{2e^{\pi i/8}}{4e^{3\pi i/8}} = \frac{1}{2}e^{-\frac{2\pi i}{8}} = \frac{1}{2}e^{-\frac{\pi i}{4}} = \frac{1}{2}\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)$$

$$a = \frac{1}{2\sqrt{2}} \qquad b = \frac{-1}{2\sqrt{2}}$$

6. Show the complex function $f(z) = \overline{z}$ is not analytic at $z = 0$.

I will show the limit $\lim_{z \to 0} \frac{f(z) - f(0)}{z} = \lim_{z \to 0} \frac{\overline{z}}{z}$ does not exist.

First let $z = x$ for x real.

$$\lim_{z \to 0} \frac{\overline{z}}{z} = \lim_{x \to 0} \frac{\overline{x}}{x} = \lim_{x \to 0} \frac{x}{x} = 1.$$

Now let $z = iy$ for y real.

$$\lim_{z \to 0} \frac{\overline{z}}{z} = \lim_{y \to 0} \frac{\overline{iy}}{iy} = \lim_{y \to 0} \frac{-iy}{iy} = -1.$$

This shows the limit does not exist.

7. Find all points z in $\mathbb{C}$ satisfying the equation

$$\sin z = 2.$$

Write the solutions in the form $a + ib$ for a and b real numbers.

$$2 = \sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$$

$$4i\, e^{iz} = (e^{iz})^2 - 1$$

$$(e^{iz})^2 - 4i\, e^{iz} - 1 = 0.$$

$$e^{iz} = \frac{4i + (-16+4)^{1/2}}{2} = 2i \pm \tfrac{1}{2} i\sqrt{12}$$

$$= 2i \pm i\sqrt{3} = i(2 \pm \sqrt{3}).$$

Let $z = x + iy$. Then

$$e^{iz} = e^{-y} e^{ix} = i(2 \pm \sqrt{3})$$

For $i(2+\sqrt{3})$:

$2+\sqrt{3} > 0 \Rightarrow \arg(i(2+\sqrt{3})) = \pi/2 \Rightarrow x = \frac{\pi}{2} + 2\pi n.$

$|e^{iz}| = |e^{-y}| = e^{-y} = |i(2+\sqrt{3})| = 2 + \sqrt{3}. \Rightarrow y = -\log(2+\sqrt{3})$

So one set of sol'ns is $\frac{\pi}{2} + 2\pi n - i\log(2+\sqrt{3})$ for n any integer.

For $i(2-\sqrt{3})$:

Similarly $2-\sqrt{3} > 0 \Rightarrow$ the other set of sol'ns is $\frac{\pi}{2} + 2\pi n - i\log(2-\sqrt{3})$ for n any integer.

8. Compute the contour integral

$$\oint_C \frac{z}{z^2 - \pi^2}\, dz$$

where C is the circle $|z| = 3$.

The fct. $\frac{z}{z^2 - \pi^2}$ is not analytic at $\pm\pi$.

Neither of these points lie inside C.

$\therefore$ by Cauchy's thm

$$\oint_C \frac{z}{z^2 - \pi^2}\, dz = 0.$$

9. Determine the pole(s) of $5 - 6/z^2$. Find the order(s) of the pole(s). Compute the residue(s) at the pole(s).

The fct. $f(z) = 5 - 6/z^2$ is already written in the form of a Laurent series centered at 0, where it has a pole of order 2 with residue 0.

$f(z)$ has no other poles.

10. Determine the pole(s) of

$$\frac{1}{1-e^z}.$$

Find the order(s) of the pole(s). Compute the residue(s) at the pole(s).

Let $f(z) = \frac{1}{1-e^z}$. $f(z)$ is $2\pi i$-periodic, i.e. $f(z+2\pi i) = f(z)$.

$f(z)$ is not analytic at $z=0$. Let's examine this singularity first.

Consider $g(z) = \frac{e^z - 1}{z} = 1 + \frac{z}{2!} + \frac{z^2}{3!} + \frac{z^3}{4!} + \dots$

By the ratio test this power series converges on all of $\mathbb{C}$.

$\Rightarrow$ $g(z)$ is analytic everywhere. $g(0) = 1$.

$$\Rightarrow \frac{d}{dz}\left(\frac{z}{1-e^z}\right) = \frac{d}{dz}\left(\frac{-1}{g(z)}\right) = -1\cdot(-1)\cdot(g(z))^{-2}\cdot g'(z)$$

$$\Rightarrow \frac{d}{dz}\bigg|_{z=0}\left(\frac{z}{1-e^z}\right) = g'(0) = \frac{1}{2}.$$

$\Rightarrow$ $\frac{z}{1-e^z}$ is analytic at 0. $\Rightarrow$ $f(z)$ has a simple pole at 0. The residue at 0 is

$$\lim_{z\to 0}\left(\frac{z}{1-e^z}\right) = -\frac{1}{\lim_{z\to 0} g(z)} = -1.$$

By the $2\pi i$-periodicity of $f(z)$, the other poles are at $2\pi i n$ for n any integer, and these poles are all simple with residue -1.

11. Compute the integral

$$\int_0^\pi \frac{1}{5+4\cos\theta}\,d\theta$$

$\cos\theta$ is even so

$$\int_0^\pi \frac{d\theta}{5+4\cos\theta} = \frac{1}{2}\int_{-\pi}^{\pi}\frac{d\theta}{5+4\cos\theta} = \frac{1}{2}\int_{-\pi}^{\pi}\frac{d\theta}{5+2e^{i\theta}+2e^{-i\theta}} = \frac{1}{2}\int \frac{1}{ie^{i\theta}}\,\frac{ie^{i\theta}\,d\theta}{5+2e^{i\theta}+2e^{-i\theta}}$$

Let $C(\theta)=e^{i\theta}$ for $-\pi\le\theta\le\pi$.

$$= \frac{1}{2i}\oint_C \frac{1}{z}\,\frac{dz}{5+2z+2/z} = \frac{1}{2i}\oint_C\frac{dz}{2z^2+5z+2} = \frac{1}{2i}\oint_C \frac{dz}{2(z+2)(z+\frac{1}{2})}$$

$$\frac{-5\pm\sqrt{25-16}}{4} = \left\{\frac{-8}{4}, \frac{-2}{4}\right\} = \left\{-2,-\frac{1}{2}\right\}$$

$$=\frac{1}{4i}\cdot 2\pi i\cdot \mathrm{Res}\left(\frac{1}{(z+2)(z+\frac{1}{2})}, -\frac{1}{2}\right)$$

↑ only the simple pole $-\frac{1}{2}$ lies inside C

$$=\frac{\pi}{2}\cdot\frac{1}{(-\frac{1}{2}+2)} = \frac{\pi}{3}$$

12. Let C be the curve in the complex plane parametrized by $C(t) = \cos(t) + i\sin(t)$, for $0 \le t \le \pi$. (Note the π!) Compute the value of the contour integral

$$\int_C \frac{dz}{z^2}$$

$C(t) = \cos t + i \sin t = e^{it} \quad 0 \le t \le \pi$

$C'(t) = ie^{it}$

$$\int_C \frac{dz}{z^2} = \int_0^{\pi} \frac{1}{e^{2it}} \cdot ie^{it}\, dt = i\int_0^{\pi} e^{-it}\, dt$$

$$= \frac{i}{-i} \cdot \left[e^{-it}\right]_0^{\pi} = -1 \cdot (-1-1) = 2.$$

13. Consider the function

$$f(x) = \begin{cases} 0 & \text{for } 0 \le x \le 1 \\ 1 & \text{for } 1 < x \le 2 \end{cases}$$

defined on the interval $[0, 2]$. Let

$$\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{2}\right)$$

be a sine series for $f(x)$. Using the same values for B_n, for all x in the real line define a function

$$g(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{2}\right).$$

Find $g(-5/2)$ and $g(-5)$.

f has a jump discontinuity at 1

$g(x)$ is a 4π-periodic fct..

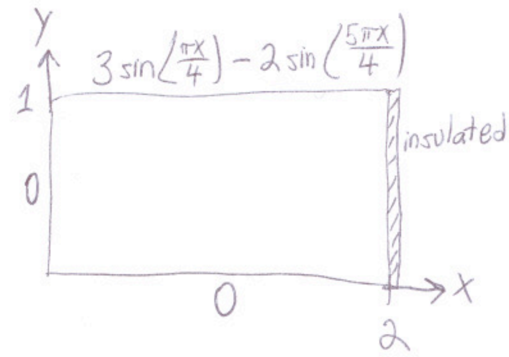
$$\Rightarrow \quad g(-5/2) = g\left(-\tfrac{5}{2} + \tfrac{8}{2}\right) = g\left(\tfrac{3}{2}\right) = f\left(\tfrac{3}{2}\right) = 1.$$

$$g(-5) = g(-1) \underset{\substack{\uparrow \\ g \text{ is odd}}}{=} -g(1) = -f(1) = -\tfrac{1}{2}(0+1) = -\tfrac{1}{2}.$$

14. Solve the Laplace equation $u_{xx} + u_{yy} = 0$ for a function $u(x, y)$ with $0 \leq x \leq 2$, $0 \leq y \leq 1$ and boundary conditions:

$$u(0, y) = 0, \quad \frac{\partial u}{\partial x}(2, y) = 0, \quad u(x, 0) = 0,$$

$$u(x, 1) = 3\sin\left(\frac{\pi x}{4}\right) - 2\sin\left(\frac{5\pi x}{4}\right).$$



$u(x,y) = F(x) \cdot G(y)$

$F''(x)\, G(y) + F(x) \cdot G''(y) = 0$

$\Rightarrow \quad \dfrac{F''(x)}{F(x)} = -\dfrac{G''(y)}{G(y)} = -\lambda.$

$F''(x) + \lambda F(x) = 0 \qquad F(0) = F'(2) = 0$

$\lambda > 0$: Let $\alpha = \sqrt{+\lambda}$.

$F(x) = A\cos(\alpha x) + B\sin(\alpha x)$

$F(0) = 0 \Rightarrow A = 0$

$F'(x) = \alpha B\cos(\alpha \cdot 2) = 0 \quad \Rightarrow \quad 2\alpha = \frac{\pi}{2} + \pi n$

$\Rightarrow \quad \alpha = \frac{\pi}{4} + \frac{\pi}{2} n$ for $n = 0, 1, 2, 3, \ldots$

$\lambda = 0$: $F(x) = Ax + B \qquad F(0) = 0 \Rightarrow B = 0$

$F'(2) = A = 0.$

So 0 is not an eigenvalue.

$\lambda<0$: $F(x) = A\cosh(\alpha x) + B\sinh(\alpha x)$ for $\alpha=\sqrt{\lambda}$.

$F(0)=0 \Rightarrow A=0$.

$F'(2) = B\alpha\cosh(2\alpha) = 0 \Rightarrow B=0$.

$\Rightarrow$ no eigenvalues <0.

So the eigenvalues are $\lambda_n = \left(\frac{\pi}{4}+\frac{\pi n}{2}\right)^2$ for $n=0,1,2,3,\ldots$

with eigen fct $F_n(x) = B_n \sin(\sqrt{+\lambda_n}\cdot x)$.

$\Rightarrow G_n''(y) = \lambda_n G_n(y)$ & $G_n(0)=0$

$\Rightarrow G_n(y) = C_n \sinh(\sqrt{\lambda_n}\cdot y)$.

$\Rightarrow u_n(x,y) = A_n \sinh\left(\left(\frac{\pi}{4}+\frac{\pi n}{2}\right)y\right)\cdot\sin\left(\left(\frac{\pi}{4}+\frac{\pi n}{2}\right)x\right)$ for $n=0,1,2,\ldots$

& $u(x,y) = \sum_{n=0}^{\infty} u_n(x,y)$.

$$u(x,1) = 3\sin\left(\frac{\pi x}{4}\right) - 2\sin\left(\frac{5\pi x}{4}\right) = \sum_{n=0}^{\infty} A_n \sinh\left(\left(\frac{\pi}{4}+\frac{\pi n}{2}\right)\right)\sin\left(\left(\frac{\pi}{4}+\frac{\pi n}{2}\right)x\right)$$

$\Rightarrow$ only the $n=0$ and $n=2$ coeff. are nonzero

$A_0 = \frac{3}{\sinh(\pi/4)}$, $A_2 = \frac{-2}{\sinh(\frac{5\pi}{4})}$

$$\text{\&}\quad u(x,y) = \frac{3}{\sinh(\frac{\pi}{4})}\cdot\sinh\left(\frac{\pi y}{4}\right)\sin\left(\frac{\pi x}{4}\right) - \frac{2}{\sinh(\frac{5\pi}{4})}\cdot\sinh\left(\frac{5\pi y}{4}\right)\sin\left(\frac{5\pi x}{4}\right).$$