Math 241 Final Exam
Spring 2013

Name:

Instructor (circle one): Epstein Hynd Wong

Please *turn off and put away all electronic devices.* You may use both sides of a $3'' \times 5''$ card for handwritten notes while you take this exam. No calculators, no course notes, no books, no help from your neighbors. **Show all work**. Please **clearly mark** your final answer. Remember to put your name at the top of this page. Good luck.

My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this examination.

Your signature

Question Number	Points Possible	Your Score
1	10	
2	10	
3	10	
4	10	

Question Number	Points Possible	Your Score
5	15	
6	10	
7	10	
8	15	
Total	90	

$$\int_0^x u\cos nu\; du = \frac{\cos nx + nx\sin nx - 1}{n^2} \qquad \text{for any real } n \neq 0$$

$$\int_0^x u\sin nu\; du = \frac{\sin nx - nx\cos nx}{n^2} \qquad \text{for any real } n \neq 0$$

$$\int_0^x e^{mu}\cos nu\; du = \frac{e^{mx}(m\cos nx + n\sin nx) - m}{m^2+n^2} \qquad \text{for any real } n, m$$

$$\int_0^x e^{mu}\sin nu\; du = \frac{e^{mx}(-n\cos nx + m\sin nx) + n}{m^2+n^2} \qquad \text{for any real } n, m$$

$$\int_0^x \sin nu\cos mu\; du = \frac{m\sin nx\sin mx + n\cos nx\cos mx - n}{m^2-n^2} \qquad \text{for any real numbers } m \neq n$$

$$\int_0^x \cos nu\cos mu\; du = \frac{m\cos nx\sin mx - n\sin nx\cos mx}{m^2-n^2} \qquad \text{for any real numbers } m \neq n$$

$$\int_0^x \sin nu\sin mu\; du = \frac{n\cos nx\sin mx - m\sin nx\cos mx}{m^2-n^2} \qquad \text{for any real numbers } m \neq n$$

Formulas Involving Bessel Functions

- Bessel's equation: $r^2R'' + rR' + (\alpha^2r^2 - n^2)R = 0$ – The only solutions of this which are bounded at $r=0$ are $R(r) = cJ_n(\alpha r)$.

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!}\left(\frac{x}{2}\right)^{n+2k}.$$

  $J_0(0) = 1$, $J_n(0) = 0$ if $n > 0$. z_{nm} is the mth positive zero of $J_n(x)$.

- Orthogonality relations:

  If $m \neq k$ then $\int_0^1 xJ_n(z_{nm}x)J_n(z_{nk}x)\,dx = 0$ and $\int_0^1 x(J_n(z_{nm}x))^2\,dx = \frac{1}{2}J_{n+1}(z_{nm})^2$.

- Recursion and differentiation formulas:

$$\frac{d}{dx}(x^nJ_n(x)) = x^nJ_{n-1}(x) \quad \text{or} \quad \int x^nJ_{n-1}(x)\,dx = x^nJ_n(x) + C \quad \text{for } n \geq 1 \tag{1}$$

$$\frac{d}{dx}(x^{-n}J_n(x)) = -x^{-n}J_{n+1}(x) \quad \text{for} \quad n \geq 0 \tag{2}$$

$$J_n'(x) + \frac{n}{x}J_n(x) = J_{n-1}(x) \tag{3}$$

$$J_n'(x) - \frac{n}{x}J_n(x) = -J_{n+1}(x) \tag{4}$$

$$2J_n'(x) = J_{n-1}(x) - J_{n+1}(x) \tag{5}$$

$$\frac{2n}{x}J_n(x) = J_{n-1}(x) + J_{n+1}(x) \tag{6}$$

- Modified Bessel's equation: $r^2R'' + rR' - (\alpha^2r^2 + n^2)R = 0$ – The only solutions of this which are bounded at $r=0$ are $R(r) = cI_n(\alpha r)$.

$$I_n(x) = i^{-n}J_n(ix) = \sum_{k=0}^{\infty} \frac{1}{k!(k+n)!}\left(\frac{x}{2}\right)^{n+2k}.$$

Formulas Involving Associated Legendre and Spherical Bessel Functions

- Associated Legendre Functions: $\frac{d}{d\phi}\left(\sin\phi\frac{dg}{d\phi}\right)+\left(\mu-\frac{m^2}{\sin\phi}\right)g=0$. Using the substitution $x=\cos\phi$, this equation becomes $\frac{d}{dx}\left((1-x^2)\frac{dg}{dx}\right)+\left(\mu-\frac{m^2}{1-x^2}\right)g=0$. This equation has bounded solutions only when $\mu=n(n+1)$ and $0\le m\le n$. The solution $P_n^m(x)$ is called an associated Legendre function of the first kind.

- Associated Legendre Function Identities:

$$P_n^0(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n \text{ and } P_n^m(x)=(-1)^m(1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x) \text{ when } 1\le m\le n$$

- Orthogonality of Associated Legendre Functions: If n and k are both greater than or equal to m,

$$\text{If } n\neq k \text{ then } \int_{-1}^{1}P_n^m(x)P_k^m(x)dx=0 \text{ and } \int_{-1}^{1}\left(P_n^m(x)\right)^2dx=\frac{2(n+m)!}{(2n+1)(n-m)!}.$$

- Spherical Bessel Functions: $(\rho^2 f')'+(\alpha^2\rho^2-n(n+1))f=0$. If we define the spherical Bessel function $j_n(\rho)=\rho^{-\frac{1}{2}}J_{n+\frac{1}{2}}(\rho)$, then only solution of this ODE bounded at $\rho=0$ is $j_n(\alpha\rho)$.

- Spherical Bessel Function Identity:

$$j_n(x)=x^2\left(-\frac{1}{x}\frac{d}{dx}\right)^n\left(\frac{\sin x}{x}\right).$$

- Spherical Bessel Function Orthogonality: Let z_{nm} be the m-th positive zero of j_n.

$$\text{If } m\neq k \text{ then } \int_0^1 x^2 j_n(z_{nm}x)j_n(z_{nk}x)dx=0 \text{ and } \int_0^1 x^2(j_n(z_{nm}x))^2dx=\frac{1}{2}(j_{n+1}(z_{nm}))^2.$$

One-Dimensional Fourier Transform

$$\mathcal{F}[u](\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}u(x)e^{i\omega x}dx,\qquad \mathcal{F}^{-1}[U](x)=\int_{-\infty}^{\infty}U(\omega)e^{-i\omega x}d\omega$$

Table of Fourier Transform Pairs

Fourier Transform Pairs ($\alpha>0$)		Fourier Transform Pairs ($\beta>0$)	
$u(x)=\mathcal{F}^{-1}[U]$	$U(\omega)=\mathcal{F}[u]$	$u(x)=\mathcal{F}^{-1}[U]$	$U(\omega)=\mathcal{F}[u]$
$e^{-\alpha x^2}$	$\frac{1}{\sqrt{4\pi\alpha}}e^{-\frac{\omega^2}{4\alpha}}$	$\sqrt{\frac{\pi}{\beta}}e^{-\frac{x^2}{4\beta}}$	$e^{-\beta\omega^2}$
$e^{-\alpha\lvert x\rvert}$	$\frac{1}{2\pi}\frac{2\alpha}{x^2+\alpha^2}$	$\frac{2\beta}{x^2+\beta^2}$	$e^{-\beta\lvert\omega\rvert}$
$u(x)=\begin{cases}0 & \lvert x\rvert>\alpha\\ 1 & \lvert x\rvert<\alpha\end{cases}$	$\frac{1}{\pi}\frac{\sin\alpha\omega}{\omega}$	$2\frac{\sin\beta x}{x}$	$U(\omega)=\begin{cases}0 & \lvert\omega\rvert>\beta\\ 1 & \lvert\omega\rvert<\beta\end{cases}$
$\delta(x-x_0)$	$\frac{1}{2\pi}e^{i\omega x_0}$	$e^{-i\omega_0 x}$	$\delta(\omega-\omega_0)$

1. The following are questions about Bessel functions. [+2.5 points for a correct answer, -2.5 points for an incorrect answer.]

   (a) The modified Bessel functions $\{I_m(x)\}$ remains bounded as x tends to 0. True / False

   (b) The Bessel functions J_0 and Y_0 have infinitely many zeros on the positive real axis. True / False

   (c) If $\{z_5\}$ is a zero of $J_5(x)$, and $\{z_4\}$ is a zero of $J_4(x)$, then True / False

$$\int_0^1 J_5(z_5 r) J_4(z_4 r) r dr = 0. \tag{1}$$

   (d) The functions $\{K_n(r) : n = 0, 1, \dots\}$ are used to solve the boundary value problem: True / False

$$\begin{cases} \nabla^2 u(r,\theta) = 0 \text{ where } 1 < r \\ u(1,\theta) = f(\theta) \\ \lim_{r\to\infty} u(r,\theta) = 0. \end{cases} \tag{2}$$

2. In the study of forced vibrating membranes, one may encounter a differential equation such as

$$\frac{d^2A}{dt^2} + \omega^2 A = \cos \omega t,$$

where $\omega > 0$ is a fixed constant. Which of the following functions is a *particular* solution to this ordinary differential equation

(a) $A(t) = \sin \omega t$

(b) $A(t) = \cos \omega t$

(c) $A(t) = \frac{1}{2\omega} t \sin \omega t$

(d) $A(t) = \frac{1}{2\omega} t \cos \omega t$

Is this an instance of resonance?

3. Decide whether the following statements regarding the Fourier transformation $\mathcal{F}$:

$$\mathcal{F}(f)(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx \tag{3}$$

are true or false. [+2.5 points for a correct answer, -2.5 points for an incorrect answer.]

(a) A Gaussian is a function of the form $\alpha e^{-\beta x^2}$, for constants α and β. The Fourier transform of a Gaussian is always a Gaussian. True / False

(b) If $F(\omega) = \mathcal{F}[f(x)]$, then for any constant β, True / False

$$\mathcal{F}[f(x-\beta)] = e^{-i\omega\beta} F(\omega).$$

(c) If we let

$$f * g(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(y) g(x-y) dy,$$

then True / False

$$\mathcal{F}[f * g] = \mathcal{F}[f]\mathcal{F}[g].$$

(d) If $F(\omega)$ is the Fourier transform of $f(x)$, then $\frac{\partial F}{\partial \omega}$ is the Fourier transform of $\frac{\partial f}{\partial x}$. True / False

4. Suppose that u satisfies the heat equation in a rod $(0 < x < L)$ with the boundary conditions on the left. Match the boundary conditions on the left with the reference temperature u_0 on the right so that $v = u - u_0$ satisfies homogeneous boundary conditions.

i) $\dfrac{\partial u}{\partial x}(0,t) = A(t), u(L,t) = B(t)$ (a) $u_0(x,t) = xA(t) + \frac{x^2}{2L}[B(t) - A(t)]$

ii) $u(0,t) = A(t), u(L,t) = B(t)$ (b) $u_0(x,t) = B(t) + A(t)[x - L]$

iii) $\dfrac{\partial u}{\partial x}(0,t) = A(t), \dfrac{\partial u}{\partial x}(L,t) = B(t)$ (c) $u_0(x,t) = A(t) + \frac{x}{L}[B(t) - A(t)]$.

5. Compute the Fourier transforms of the following functions

a) $f(x) = -3e^{-x^2/4}$

b) $f(x) = 2xe^{-4x^2}$

c) $f(x) = \begin{cases} x \text{ for } x > 0 \\ 0 \text{ for } x \leq 0 \end{cases}$

6. Solve Laplace's equation on the half space

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad -\infty < x < \infty, \quad y > 0,$$

subject to the boundary conditions $\lim_{x\to\pm\infty} u(x,y) = \lim_{y\to\infty} u(x,y) = 0$ and

$$u(x,0) = \begin{cases} 1, & x > 0 \\ 0, & x \leq 0 \end{cases}.$$

7. The vertical displacement of a circular membrane of radius 1, fixed at the boundary satisfies the PDE

$$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u$$

with

$$u(1, \theta, t) = 0.$$

Let $J_0(z)$ be the 0th Bessel function of the first kind with roots $z_1, z_2, \ldots, z_n, \ldots$. Find a formula for the vertical displacement $u(r, \theta, t)$ of a vibrating circular membrane subject to the initial conditions

$$u(r, \theta, 0) = -5J_0(z_3 r) + 4J_0(z_6 r), \quad \text{and} \quad \frac{\partial u}{\partial t}(r, \theta, 0) = 0.$$

Hint: The solution should be circularly symmetric.

8. Find a solution in the unit disk, $r^2 \leq 1$, to the inhomogeneous Laplace equation

$$\nabla^2 u = 2r^4$$

with

$$u(1, \theta) = \sin(2\theta).$$