

MATH 241 FINAL EXAM SPRING 2013

NAME:

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Please *turn off and put away all electronic devices*. You may use both sides of a 3" × 5" card for handwritten notes while you take this exam. No calculators, no course notes, no books, no help from your neighbors. **Show all work**. Please **clearly mark** your final answer. Remember to put your name at the top of this page. Good luck.

My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this examination.

Your signature

QUESTION NUMBER	POINTS POSSIBLE	YOUR SCORE
1	10	
2	10	
3	10	
4	10	

QUESTION NUMBER	POINTS POSSIBLE	YOUR SCORE
5	15	
6	10	
7	10	
8	15	
TOTAL	90	

$$\int_0^x u \cos nu \, du = \frac{\cos nx + nx \sin nx - 1}{n^2} \quad \text{for any real } n \neq 0$$

$$\int_0^x u \sin nu \, du = \frac{\sin nx - nx \cos nx}{n^2} \quad \text{for any real } n \neq 0$$

$$\int_0^x e^{mu} \cos nu \, du = \frac{e^{mx}(m \cos nx + n \sin nx) - m}{m^2 + n^2} \quad \text{for any real } n, m$$

$$\int_0^x e^{mu} \sin nu \, du = \frac{e^{mx}(-n \cos nx + m \sin nx) + n}{m^2 + n^2} \quad \text{for any real } n, m$$

$$\int_0^x \sin nu \cos mu \, du = \frac{m \sin nx \sin mx + n \cos nx \cos mx - n}{m^2 - n^2} \quad \text{for any real numbers } m \neq n$$

$$\int_0^x \cos nu \cos mu \, du = \frac{m \cos nx \sin mx - n \sin nx \cos mx}{m^2 - n^2} \quad \text{for any real numbers } m \neq n$$

$$\int_0^x \sin nu \sin mu \, du = \frac{n \cos nx \sin mx - m \sin nx \cos mx}{m^2 - n^2} \quad \text{for any real numbers } m \neq n$$

FORMULAS INVOLVING BESSEL FUNCTIONS

- Bessel's equation: $r^2 R'' + rR' + (\alpha^2 r^2 - n^2)R = 0$ – The only solutions of this which are bounded at $r = 0$ are $R(r) = cJ_n(\alpha r)$.

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!} \left(\frac{x}{2}\right)^{n+2k}.$$

$J_0(0) = 1$, $J_n(0) = 0$ if $n > 0$. z_{nm} is the m th positive zero of $J_n(x)$.

- Orthogonality relations:

If $m \neq k$ then $\int_0^1 x J_n(z_{nm}x) J_n(z_{nk}x) \, dx = 0$ and $\int_0^1 x (J_n(z_{nm}x))^2 \, dx = \frac{1}{2} J_{n+1}(z_{nm})^2$.

- Recursion and differentiation formulas:

$$\frac{d}{dx}(x^n J_n(x)) = x^n J_{n-1}(x) \quad \text{or} \quad \int x^n J_{n-1}(x) \, dx = x^n J_n(x) + C \quad \text{for } n \geq 1 \quad (1)$$

$$\frac{d}{dx}(x^{-n} J_n(x)) = -x^{-n} J_{n+1}(x) \quad \text{for } n \geq 0 \quad (2)$$

$$J'_n(x) + \frac{n}{x} J_n(x) = J_{n-1}(x) \quad (3)$$

$$J'_n(x) - \frac{n}{x} J_n(x) = -J_{n+1}(x) \quad (4)$$

$$2J'_n(x) = J_{n-1}(x) - J_{n+1}(x) \quad (5)$$

$$\frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x) \quad (6)$$

- Modified Bessel's equation: $r^2 R'' + rR' - (\alpha^2 r^2 + n^2)R = 0$ – The only solutions of this which are bounded at $r = 0$ are $R(r) = cI_n(\alpha r)$.

$$I_n(x) = i^{-n} J_n(ix) = \sum_{k=0}^{\infty} \frac{1}{k!(k+n)!} \left(\frac{x}{2}\right)^{n+2k}.$$

- Associated Legendre Functions: $\frac{d}{d\phi} \left(\sin \phi \frac{dg}{d\phi} \right) + \left(\mu - \frac{m^2}{\sin^2 \phi} \right) g = 0$. Using the substitution $x = \cos \phi$, this equation becomes $\frac{d}{dx} \left((1-x^2) \frac{dg}{dx} \right) + \left(\mu - \frac{m^2}{1-x^2} \right) g = 0$. This equation has bounded solutions only when $\mu = n(n+1)$ and $0 \leq m \leq n$. The solution $P_n^m(x)$ is called an associated Legendre function of the first kind.

- Associated Legendre Function Identities:

$$P_n^0(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n \text{ and } P_n^m(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_n(x) \text{ when } 1 \leq m \leq n$$

- Orthogonality of Associated Legendre Functions: If n and k are both greater than or equal to m ,

$$\text{If } n \neq k \text{ then } \int_{-1}^1 P_n^m(x) P_k^m(x) dx = 0 \text{ and } \int_{-1}^1 (P_n^m(x))^2 dx = \frac{2(n+m)!}{(2n+1)(n-m)!}$$

- Spherical Bessel Functions: $(\rho^2 f')' + (\alpha^2 \rho^2 - n(n+1))f = 0$. If we define the spherical Bessel function $j_n(\rho) = \rho^{-\frac{1}{2}} J_{n+\frac{1}{2}}(\rho)$, then only solution of this ODE bounded at $\rho = 0$ is $j_n(\alpha\rho)$.

- Spherical Bessel Function Identity:

$$j_n(x) = x^2 \left(-\frac{1}{x} \frac{d}{dx} \right)^n \left(\frac{\sin x}{x} \right)$$

- Spherical Bessel Function Orthogonality: Let z_{nm} be the m -th positive zero of j_n .

$$\text{If } m \neq k \text{ then } \int_0^1 x^2 j_n(z_{nm}x) j_n(z_{nk}x) dx = 0 \text{ and } \int_0^1 x^2 (j_n(z_{nm}x))^2 dx = \frac{1}{2} (j_{n+1}(z_{nm}))^2$$

ONE-DIMENSIONAL FOURIER TRANSFORM

$$\mathcal{F}[u](\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u(x) e^{i\omega x} dx, \quad \mathcal{F}^{-1}[U](x) = \int_{-\infty}^{\infty} U(\omega) e^{-i\omega x} d\omega$$

TABLE OF FOURIER TRANSFORM PAIRS
 FOURIER TRANSFORM PAIRS FOURIER TRANSFORM PAIRS
 $(\alpha > 0)$ $(\beta > 0)$

$u(x) = \mathcal{F}^{-1}[U]$	$U(\omega) = \mathcal{F}[u]$	$u(x) = \mathcal{F}^{-1}[U]$	$U(\omega) = \mathcal{F}[u]$
$e^{-\alpha x^2}$	$\frac{1}{\sqrt{4\pi\alpha}} e^{-\frac{\omega^2}{4\alpha}}$	$\sqrt{\frac{\pi}{\beta}} e^{-\frac{x^2}{4\beta}}$	$e^{-\beta\omega^2}$
$e^{-\alpha x }$	$\frac{1}{2\pi} \frac{2\alpha}{x^2 + \alpha^2}$	$\frac{2\beta}{x^2 + \beta^2}$	$e^{-\beta \omega }$
$u(x) = \begin{cases} 0 & x > \alpha \\ 1 & x < \alpha \end{cases}$	$\frac{1}{\pi} \frac{\sin \alpha\omega}{\omega}$	$2 \frac{\sin \beta x}{x}$	$U(\omega) = \begin{cases} 0 & \omega > \beta \\ 1 & \omega < \beta \end{cases}$
$\delta(x - x_0)$	$\frac{1}{2\pi} e^{i\omega x_0}$	$e^{-i\omega_0 x}$	$\delta(\omega - \omega_0)$

1. The following are questions about Bessel functions. [+2.5 points for a correct answer, -2.5 points for an incorrect answer.]

(a) The modified Bessel functions $\{I_m(x)\}$ remains bounded as x tends to 0. True / False

(b) The Bessel functions J_0 and Y_0 have infinitely many zeros on the positive real axis. True / False

(c) If $\{z_5\}$ is a zero of $J_5(x)$, and $\{z_4\}$ is a zero of $J_4(x)$, then True / False

$$\int_0^1 J_5(z_5 r) J_4(z_4 r) r dr = 0. \quad (1)$$

(d) The functions $\{K_n(r) : n = 0, 1, \dots\}$ are used to solve the boundary value problem: True / False

$$\begin{cases} \nabla^2 u(r, \theta) = 0 \text{ where } 1 < r \\ u(1, \theta) = f(\theta) \\ \lim_{r \rightarrow \infty} u(r, \theta) = 0. \end{cases} \quad (2)$$

2. In the study of forced vibrating membranes, one may encounter a differential equation such as

$$\frac{d^2 A}{dt^2} + \omega^2 A = \cos \omega t,$$

where $\omega > 0$ is a fixed constant. Which of the following functions is a *particular* solution to this ordinary differential equation

(a) $A(t) = \sin \omega t$

(b) $A(t) = \cos \omega t$

(c) $A(t) = \frac{1}{2\omega} t \sin \omega t$

(d) $A(t) = \frac{1}{2\omega} t \cos \omega t$

Is this an instance of resonance?

3. Decide whether the following statements regarding the Fourier transformation \mathcal{F} :

$$\mathcal{F}(f)(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)e^{i\omega x} dx \quad (3)$$

are true or false. [+2.5 points for a correct answer, -2.5 points for an incorrect answer.]

(a) A Gaussian is a function of the form $\alpha e^{-\beta x^2}$, for constants α and β . The Fourier transform of a Gaussian is always a Gaussian. True / False

(b) If $F(\omega) = \mathcal{F}[f(x)]$, then for any constant β , True / False

$$\mathcal{F}[f(x - \beta)] = e^{-i\omega\beta} F(\omega).$$

(c) If we let

$$f * g(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(y)g(x - y)dy,$$

then

True / False

$$\mathcal{F}[f * g] = \mathcal{F}[f]\mathcal{F}[g].$$

(d) If $F(\omega)$ is the Fourier transform of $f(x)$, then $\frac{\partial F}{\partial \omega}$ is the Fourier transform of $\frac{\partial f}{\partial x}$. True / False

4. Suppose that u satisfies the heat equation in a rod ($0 < x < L$) with the boundary conditions on the left. Match the boundary conditions on the left with the reference temperature u_0 on the right so that $v = u - u_0$ satisfies homogeneous boundary conditions.

i) $\frac{\partial u}{\partial x}(0, t) = A(t), u(L, t) = B(t)$ (a) $u_0(x, t) = xA(t) + \frac{x^2}{2L}[B(t) - A(t)]$

ii) $u(0, t) = A(t), u(L, t) = B(t)$ (b) $u_0(x, t) = B(t) + A(t)[x - L]$

iii) $\frac{\partial u}{\partial x}(0, t) = A(t), \frac{\partial u}{\partial x}(L, t) = B(t)$ (c) $u_0(x, t) = A(t) + \frac{x}{L}[B(t) - A(t)]$.

5. Compute the Fourier transforms of the following functions

(a) $f(x) = e^{-5|x-1|}$

(b) $f(x) = \frac{4x}{(1+x^2)^2}$

(c) $f(x) = \begin{cases} x & \text{for } -1 \leq x \leq 1 \\ 0 & \text{for } 1 < |x| \end{cases}$

6. Solve Laplace's equation on the half space

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad -\infty < x < \infty, \quad y > 0,$$

subject to the boundary conditions $\lim_{x \rightarrow \pm\infty} u(x, y) = \lim_{y \rightarrow \infty} u(x, y) = 0$ and

$$u(x, 0) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases}.$$

7. The vertical displacement of a circular membrane of radius 1, fixed at the boundary satisfies the PDE

$$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u \quad (4)$$

with

$$u(1, \theta, t) = 0. \quad (5)$$

Let $J_0(z)$ be the 0th Bessel function of the first kind with roots $z_1, z_2, \dots, z_n, \dots$. Find a formula for the vertical displacement $u(r, \theta, t)$ of a vibrating circular membrane subject to the initial conditions

$$u(r, \theta, 0) = 2J_0(z_2 r) - 3J_0(z_5 r), \quad \text{and} \quad \frac{\partial u}{\partial t}(r, \theta, 0) = 0.$$

Hint: The solution should be circularly symmetric.

8. Find a solution in the unit disk, $r^2 \leq 1$, to the inhomogeneous Laplace equation

$$\nabla^2 u = r^2 \quad (6)$$

with

$$u(1, \theta) = \cos(2\theta). \quad (7)$$