Final examination, Math 241: Calculus IV
April 29, 2002, 11:00AM–1:00PM

No books, papers, calculators or electronic device may be used, other than a hand-written note sheet at most $5'' \times 7''$ in size. *No questions will be answered or clarifications offered, during this exam.*

This examination consists of ten multiple-choice questions and two short-answer question. The multiple-choice questions are worth eight (8) points each, with no partial credit. The correct and most appropriate answer to a multiple-choice question will be, in each case, just one of the five choices (A), (B), (C), (D) and (E). Answer all multiple-choice questions on the answer sheet, which is page 13 of this exam. Only the answers on the answer sheet will be considered for grading.

The short-answer questions are worth ten (10) points each. You must **show all your work** and **fill in your answers** at the underlined space; credit for these questions are based mostly on your short answers. Partial credit will be given only when a substantial part of a problem has been worked out. Merely displaying some formulas is not a sufficient ground for receiving partial credit.

For this posted version, the answers appear at the end of the exam.

1.. Let S be the set of all complex numbers w such that

$$e^w (= \exp(w)) = 1 + i \quad \text{and} \quad |e^{w^2}| (= \exp(w^2)) > 1\,.$$

How many elements does the set S have? (In other words, how many complex numbers w are there satisfying both $e^w (:= \exp(w)) = 1 + i$ and $|e^{w^2}| > 1$?)

A. 0 B. 1 C. 2 D. 3 E. None of the above.

2. Let $f(x)$ be a periodic function on the real line with period 2π such that

$$f(x) = \begin{cases} 0, & \text{if} \quad -\pi < x < 0 \\ \pi - x & \text{if} \quad 0 < x < \pi \end{cases}$$

Write the Fourier series expansion of $f(x)$ as

$$f(x) \sim \sum_{n=0}^{\infty} a_n \cos(nx) + \sum_{n=0}^{\infty} b_n \sin(nx)\,.$$

Then the infinite series

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots$$

A. converges to 0. B. converges to 2π. C. converges to $\frac{\pi^2}{6}$.

D. converges to $\frac{\pi}{2}$. E. diverges.

3. Consider the following five complex-valued functions in the complex variable $z = x + i\,y$:

$$f(z) = \tfrac{x}{x^2+y^2} - i\tfrac{y}{x^2+y^2} \qquad g(z) = e^z$$

$$h(z) = x - iy^2 \qquad j(z) = \frac{z^2}{\cos\left(\frac{\pi z}{2}\right)}$$

$$k(z) = z^{100}$$

Which ones of the above functions are holomorphic (or, complex analytic) at $z = 5$?

A. $f(z)$, $g(z)$ and $k(z)$ only.

B. $f(z)$, $g(z)$ and $h(z)$ only.

C. $f(z)$, $g(z)$, $h(z)$ and $j(z)$ only.

D. $g(z)$ and $k(z)$ only.

E. $f(z)$, $g(z)$, $h(z)$ and $k(z)$ only.

4. Denote by I the complex line integral

$$I = \int_C z^2 \, dz \,,$$

where C is the segment of the curve $y^2 = 4 - x$ going **from** $(0, -2)$ **to** $(0, 2)$.
What is the absolute value $|I|$ of this line integral?

A. $|I| = \frac{16}{3}$ B. $|I| = 0$ C. $|I| = \frac{8}{3}$ D. $|I| = 2\pi$ E. None of the above.

5. Suppose that $u(x, t)$ is a function defined for $0 \le x \le 1$, $t \ge 0$ such that

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} &= 0 \quad \text{for all } 0 \le x \le 1, \text{ all t } \ge 0 \\ \frac{\partial u}{\partial x}(0, t) &= 0 \quad \text{for all } t \ge 0 \\ \frac{\partial u}{\partial x}(1, t) &= 0 \quad \text{for all } t \ge 0 \\ u(x, 0) &= \cos(\pi\, x) \quad \text{for all } 0 \le x \le 1 \\ \frac{\partial u}{\partial t}(x, 0) &= \cos(3\pi x) \quad \text{for all } 0 \le x \le 1 \end{aligned}$$

What is the value of $u(\frac{1}{3}, \frac{1}{2})$?

A. $\frac{1}{2}$ B. $-\frac{1}{3\pi}$ C. 0 D. $\frac{\sqrt{2}}{4}$ E. None of the above

6. Which of the following complex line integral is *not* equal to 0?

A. $\oint_{|z|=1} \frac{dz}{z^2+9}$ B. $\oint_{|z|=1} \bar{z}^{-1} \, dz$ C. $\oint_{|z|=1} \frac{dz}{4\,z^2-1}$
D. $\oint_{|z|=1} \frac{e^z-1}{z^3} \, dz$ E. None of the above

7. Suppose that $f(t)$ is a function defined for $t \ge 0$ such that its Laplace transform $\mathcal{L}\{f(t)\}(s)$ is equal to $\frac{e^{-3s}}{(s-1)^3}$. What is the value of $f(5)$?

A. $5\,e^3$ B. $2\,e^2$ C. $\frac{25}{2}\,e^2$ D. $\frac{25}{2}\,e^5$ E. None of the above

8. Consider the Sturm-Liouville equation for a function $y(x)$ defined for $0 \leq x \leq \pi$:

$$y'' + \lambda y = 0$$

with boundary conditions

$$y'(0) = 0, \qquad y'(\pi) = 0\,.$$

Let $\lambda_0 < \lambda_1 < \lambda_2 < \cdots$ be the set of of all eigenvalues of the above equation, and let $y_n(x)$ be the eigenfunction for the eigenvalue λ_n such that $y(0) = 1$, $n \geq 1$. Which one of the following statements is true?

A. $\int_0^\pi y_n^2(x)\,dx = 0$ for all $n \geq 1$.

B. $\int_0^\pi x\,y_m(x)\,y_n(x)\,dx = 0$ for all $m \neq n$.

C. $\lim_{n\to\infty} y_n(x) = 0$.

D. $\sum_{n=0}^{\infty} y_n^2(x)$ converges for all $x \in [0, \pi]$.

E. None of the above

9. Let $u(x,t)$ be a function defined for $0 \leq x \leq \pi$, $t \geq 0$ such that

$$\frac{\partial u}{\partial t} = 4\,\frac{\partial^2 u}{\partial x^2} + 4\,\frac{\partial u}{\partial x}$$

and

$$u(0,t) = u(\pi,t) = 0 \qquad \text{for all } t \geq 0\,,$$

$$u(x,0) = e^{-\frac{x}{2}}\,\sin(x)\,.$$

What is the value of $u(\frac{\pi}{2}, 1)$?

A. $e^{-2-\frac{\pi}{4}}$ B. $e^{-1-\frac{\pi}{4}}$ C. $e^{-1} + e^{-\frac{\pi}{4}}$ D. $e^{-\frac{\pi}{4}}$ E. None of the above

10. Suppose that $u(x, y, z)$ is a function on $\mathbb{R}^3$ which satisfies the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

and

$$u(x, y, z) = 5z^3 \qquad \text{if} \quad x^2 + y^2 + z^2 = 1\,.$$

What is the value of $u(1, 1, 1)$?

(Notice that the values of $u(x, y, z)$ at the boundary sphere $\{x^2 + y^2 + z^2 = 1\}$ depends only on the angle ϕ with the positive z-axis, so that the solution $u(x, y, z)$ is symmetric under arbitrary rotation about the z-axis. The formula for the Laplacian in spherical coordinates is

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi \frac{\partial u}{\partial \phi}\right) + \frac{1}{r^2 \sin^2\phi}\frac{\partial^2 u}{\partial \theta^2}$$

Some Legendre polynomials are: $P_0(w) = 1$, $P_1(w) = w$, $P_2(w) = \frac{3w^2-1}{2}$, $P_3(w) = \frac{5w^3-3w}{2}$.)

A. 5 B. 0 C. -1 D. 3 E. None of the above

11. The complex valued function $f(z) = \frac{e^z}{\cos z}$ has a Laurent series expansion centered at $z = 0$, for $|z| < 1/100$:

$$f(z) = \frac{e^z}{\cos z} = \sum_{n=-\infty}^{\infty} a_n\, z^n \qquad |z| < 1/100$$

Find the coefficients a_0, a_1, a_2, a_3.

Your Answer:

$a_0 =$ ______________

$a_1 =$ ______________

$a_2 =$ ______________

$a_3 =$ ______________

12. Suppose that $u(x,y)$ is a function defined for $x^2+y^2 \le 1$ such that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

and

$$u(x,y) = 4x^2 + x \quad \text{if} \quad x^2+y^2 = 1\,.$$

Find a closed formula for $u(x,y)$, and use it to evaluate $u(\frac{1}{2},\frac{1}{2})$. (Recall that in polar coordinates, we have

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{1}{r^2}\left[\left(r\frac{\partial}{\partial r}\right)^2 + \frac{\partial^2}{\partial \theta^2}\right].$$

You might want to use the method of separation of variables.)

Your Answer:

$u(x,y) =$ ____

$u(\frac{1}{2},\frac{1}{2}) =$ ____________________

ANSWERS

1. A. (Since the real part of $\log(\sqrt{2})$ is less than $\frac{\pi}{4}$.)
2. D
3. A.
4. A.
5. E. $u(x,t) = \cos(\pi x)\,\cos(\pi t) + \frac{1}{3\pi}\,\cos(3\pi x)\,\sin(3\pi t)$; $u(\frac{1}{3},\frac{1}{2}) = \frac{1}{3\pi}$
6. D.
7. B. $f(5) = 8\,e^2$; $f(t) = \frac{1}{2}\,u(t-3)\,(t-3)^2\,e^{t-3}$
8. E.
9. E. Separation of variables gives the general solution: $u(x,t) = \sum_n e^{-(4n^2+1)t}\,e^{-\frac{x}{2}}\,\sin(nx)$. Using the initial conditions we get $u(x,t) = e^{-5t}\,e^{-\frac{x}{2}}\,\sin(x)$. Thus $u(\frac{\pi}{2},1) = e^{-5-\frac{\pi}{4}}$.
10. C. $u(x,y,z) = 2\,r^3\,P_3(\cos\phi) + 3\,r\,P_1(\cos\phi) = -3\,x^2 z - 3\,y^2 z + 2\,z^3 + 3z$, $u(1,1,1) = -1$
11. $f(z) = 1 + z + z^2 + \frac{2z^3}{3} + \cdots$.
12. C. $u(\frac{1}{2},\frac{1}{2}) = \frac{5}{2}$; $u(x,y) = 2x^2 - 2y^2 + x + 2$