# MATH 241 Final Exam

Firday, 4 May 2012

Name:

This exam consists of 16 problems. Show *all* your work. You will receive credit for a correct answer only if *your work is shown* and *your work supports your answer.* No credit will be given for correct answers which are not supported by work.

Please clearly indicate your final answer by boxing it. Answers not clearly indicated as final may not receive points.

You will have 120 minutes to complete this exam.

The following table is for grading only. Do not write in it!

| Problem | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | TOTAL |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Points | | | | | | | | | | | | | | | | | |

1. Is the function $\tanh z = \frac{\sinh z}{\cosh z}$ periodic? If so, find its period.

2. TRUE or FALSE. For each part, indicate whether the statement is true or false. You do not need to give a formal argument, but you should indicate your reasoning.

   (a) If $u(x, y)$ is a function which satisfies Laplace's equation in a rectangle $a < x < b, c < y < d$, and $u(a, y) = u(b, y) = u(x, c) = u(x, d) = 0$, then $u(x, y) = 0$ in the rectangle.

   (b) $\mathrm{Ln}\left((1 + i)(1 - i)\right) = \mathrm{Ln}(1 + i) + \mathrm{Ln}(1 - i)$

   (c) $\mathrm{Ln}\left((1 + i)(-1 + i)\right) = \mathrm{Ln}(1 + i) + \mathrm{Ln}(-1 + i)$

   (d) $\exp\left((1 + i) + (-1 + i)\right) = \exp(1 + i) \exp(-1 + i)$

3. Find a solution $u(x, y)$ of the boundary-value problem for Laplace's equation on the rectangle $\{(x, y)|0 < x < \pi, 0 < y < 1\}$:

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \\ \frac{\partial u}{\partial x}(0, y) = \frac{\partial u}{\partial x}(\pi, y) = 0 \\ u(x, 0) = 0 \\ u(x, 1) = \sin^2(3x) \end{cases}$$

4. Find a solution $u(x,t)$ of the initial-value problem for the heat equation on the half-line $x > 0$:

$$\begin{cases} \frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial^2 x} \\ u(0,t) = 0 \\ u(x,0) = \begin{cases} 0 & 0 < x < 5 \\ \pi & 5 \leq x < 10 \\ 0 & x \geq 10 \end{cases} \end{cases}$$

5. In the Fourier series expansion of $f(x) = x - 1$ on $(-1, 1)$,

   (a) Find the coefficient on the term $\sin(3\pi x)$.

   (b) At $x = 1$, to what value does the Fourier series converge?

6. Find the eigenvalues and eigenfunctions of the Sturm-Liouville problem

$$\begin{cases} x^2y'' + xy' + \lambda y = 0 \\ y'(1) = 0 \\ y(e) = 0 \end{cases}$$

7. The steady-state temperature $u(r,\theta)$ in a semicircular plate is determined from the boundary-value problem

$$\begin{cases} \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 & 0 < \theta < \pi, \quad 0 < r < 2 \\ u(2,\theta) = u_0 & 0 < \theta < \pi \\ u(r,0) = u(r,\pi) = 0 & 0 < r < 2 \end{cases}$$

where $u_0$ is a constant. Solve for $u(r,\theta)$.

8. Based on the given information, setup the corresponding boundary-value problem of the steady-state temperature $u$ in a circular cylinder.
The cylinder has radius 1 and height 2.
The temperature $u$ has radial symmetry.
The lateral side is kept at temperature zero.
The top $z = 2$ is kept at temperature $u_0$.
The base $z = 0$ is insulated.

9. Evaluate $\oint_C \frac{1}{(z-1)^2(z-2)}dz$, where $C$ is the circle $|z| = 3$ with counterclockwise orientation.

10. Evaluate $\int_C ze^z dz$, where $C$ is $z(t) = 2t^3 + i(t^4 - 4t^3 + 2),\ -1 \leq t \leq 1$.

11. Find the Laurent series expansion of $f(z) = \dfrac{1}{z^2 - 1}$

(a) which is valid for $z$ with $|z + i| < \sqrt{2}$.

(b) which is valid for $z$ with $|z + i| > \sqrt{2}$.

12. Evaluate the integral

$$\int_0^{2\pi} \frac{\cos 2\theta}{5 + 3\cos\theta}\, d\theta$$

13. Evaluate the integral

$$P.V. \int_{-\infty}^{\infty} \frac{x^2}{1+x^4}\, dx.$$

14. Find the residue of

$$\frac{1}{z \sin z}$$

at $z = 0$.

15. Let $f(x+iy) = x^2y^2 + i\frac{2}{3}xy^3$.

(a) For which $z = x + iy$ does $f$ satisfy the Cauchy-Riemann equations?

(b) For which $z$ is $f$ analytic?

16. Find $\displaystyle\int_C (z-i)^2 \left[\exp\left(\frac{1}{z-i}\right) + \frac{1}{z-1}\right] dz$ where $C$ is the contour indicated below: