Math 241 AP Exam
Spring 2013

University of Pennsylvania

Name:

Please *turn off and put away all electronic devices*. You may use both sides of a $8.5'' \times 11''$ sheet of paper for handwritten notes while you take this exam. No calculators, no course notes, no books, no help from your neighbors. **Show all work**. Please **clearly mark** your final answer. Remember to put your name at the top of this page. Good luck.

My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this examination.

Your signature

Question Number	Points Possible	Your Score
1	10	
2	10	
3	10	
4	10	
5	10	

Question Number	Points Possible	Your Score
6	10	
7	10	
8	10	
9	10	
10	10	
Total	100	

A Partial Table of Integrals

$$\int_0^x u\cos nu\ du = \frac{\cos nx + nx\sin nx - 1}{n^2} \qquad \text{for any real } n \neq 0$$

$$\int_0^x u\sin nu\ du = \frac{\sin nx - nx\cos nx}{n^2} \qquad \text{for any real } n \neq 0$$

$$\int_0^x e^{mu}\cos nu\ du = \frac{e^{mx}(m\cos nx + n\sin nx) - m}{m^2+n^2} \qquad \text{for any real } n, m$$

$$\int_0^x e^{mu}\sin nu\ du = \frac{e^{mx}(-n\cos nx + m\sin nx) + n}{m^2+n^2} \qquad \text{for any real } n, m$$

$$\int_0^x \sin nu\cos mu\ du = \frac{m\sin nx\sin mx + n\cos nx\cos mx - n}{m^2-n^2} \qquad \text{for any real numbers } m \neq n$$

$$\int_0^x \cos nu\cos mu\ du = \frac{m\cos nx\sin mx - n\sin nx\cos mx}{m^2-n^2} \qquad \text{for any real numbers } m \neq n$$

$$\int_0^x \sin nu\sin mu\ du = \frac{n\cos nx\sin mx - m\sin nx\cos mx}{m^2-n^2} \qquad \text{for any real numbers } m \neq n$$

Formulas Involving Bessel Functions

- Bessel's equation: $r^2R'' + rR' + (\alpha^2r^2 - n^2)R = 0$ – The only solutions of this which are bounded at $r = 0$ are $R(r) = cJ_n(\alpha r)$.

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!}\left(\frac{x}{2}\right)^{n+2k}.$$

  $J_0(0) = 1$, $J_n(0) = 0$ if $n > 0$. z_{nm} is the mth positive zero of $J_n(x)$.

- Orthogonality relations:

$$\text{If } m \neq k \text{ then } \int_0^1 xJ_n(z_{nm}x)J_n(z_{nk}x)\,dx = 0 \quad \text{and} \quad \int_0^1 x(J_n(z_{nm}x))^2\,dx = \frac{1}{2}J_{n+1}(z_{nm})^2.$$

- Recursion and differentiation formulas:

$$\frac{d}{dx}(x^nJ_n(x)) = x^nJ_{n-1}(x) \quad \text{or} \quad \int x^nJ_{n-1}(x)\,dx = x^nJ_n(x) + C \quad \text{for } n \geq 1 \tag{1}$$

$$\frac{d}{dx}(x^{-n}J_n(x)) = -x^{-n}J_{n+1}(x) \quad \text{for} \quad n \geq 0 \tag{2}$$

$$J_n'(x) + \frac{n}{x}J_n(x) = J_{n-1}(x) \tag{3}$$

$$J_n'(x) - \frac{n}{x}J_n(x) = -J_{n+1}(x) \tag{4}$$

$$2J_n'(x) = J_{n-1}(x) - J_{n+1}(x) \tag{5}$$

$$\frac{2n}{x}J_n(x) = J_{n-1}(x) + J_{n+1}(x) \tag{6}$$

- Modified Bessel's equation: $r^2R'' + rR' - (\alpha^2r^2 + n^2)R = 0$ – The only solutions of this which are bounded at $r = 0$ are $R(r) = cI_n(\alpha r)$.

$$I_n(x) = i^{-n}J_n(ix) = \sum_{k=0}^{\infty} \frac{1}{k!(k+n)!}\left(\frac{x}{2}\right)^{n+2k}.$$

Formulas Involving Associated Legendre and Spherical Bessel Functions

- Associated Legendre Functions: $\frac{d}{d\phi}\left(\sin\phi\frac{dg}{d\phi}\right) + \left(\mu - \frac{m^2}{\sin\phi}\right)g = 0$. Using the substitution $x = \cos\phi$, this equation becomes $\frac{d}{dx}\left((1-x^2)\frac{dg}{dx}\right) + \left(\mu - \frac{m^2}{1-x^2}\right)g = 0$. This equation has bounded solutions only when $\mu = n(n+1)$ and $0 \le m \le n$. The solution $P_n^m(x)$ is called an associated Legendre function of the first kind.

- Associated Legendre Function Identities:

$$P_n^0(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n \text{ and } P_n^m(x) = (-1)^m(1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x) \text{ when } 1 \le m \le n$$

- Orthogonality of Associated Legendre Functions: If n and k are both greater than or equal to m,

$$\text{If } n \ne k \text{ then } \int_{-1}^{1} P_n^m(x)P_k^m(x)dx = 0 \text{ and } \int_{-1}^{1}\left(P_n^m(x)\right)^2 dx = \frac{2(n+m)!}{(2n+1)(n-m)!}.$$

- Spherical Bessel Functions: $(\rho^2 f')' + (\alpha^2\rho^2 - n(n+1))f = 0$. If we define the spherical Bessel function $j_n(\rho) = \rho^{-\frac{1}{2}}J_{n+\frac{1}{2}}(\rho)$, then only solution of this ODE bounded at $\rho = 0$ is $j_n(\alpha\rho)$.

- Spherical Bessel Function Identity:

$$j_n(x) = x^2\left(-\frac{1}{x}\frac{d}{dx}\right)^n\left(\frac{\sin x}{x}\right).$$

- Spherical Bessel Function Orthogonality: Let z_{nm} be the m-th positive zero of j_n.

$$\text{If } m \ne k \text{ then } \int_0^1 x^2 j_n(z_{nm}x)j_n(z_{nk}x)dx = 0 \quad\text{and}\quad \int_0^1 x^2(j_n(z_{nm}x))^2dx = \frac{1}{2}(j_{n+1}(z_{nm}))^2.$$

One-Dimensional Fourier Transform

$$\mathcal{F}[u](\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x)e^{i\omega x}dx, \qquad \mathcal{F}^{-1}[U](x) = \int_{-\infty}^{\infty} U(\omega)e^{-i\omega x}d\omega$$

Table of Fourier Transform Pairs

Fourier Transform Pairs ($\alpha > 0$)		Fourier Transform Pairs ($\beta > 0$)	
$u(x) = \mathcal{F}^{-1}[U]$	$U(\omega) = \mathcal{F}[u]$	$u(x) = \mathcal{F}^{-1}[U]$	$U(\omega) = \mathcal{F}[u]$
$e^{-\alpha x^2}$	$\frac{1}{\sqrt{4\pi\alpha}}e^{-\frac{\omega^2}{4\alpha}}$	$\sqrt{\frac{\pi}{\beta}}e^{-\frac{x^2}{4\beta}}$	$e^{-\beta\omega^2}$
$e^{-\alpha\lvert x\rvert}$	$\frac{1}{2\pi}\frac{2\alpha}{x^2+\alpha^2}$	$\frac{2\beta}{x^2+\beta^2}$	$e^{-\beta\lvert\omega\rvert}$
$u(x) = \begin{cases} 0 & \lvert x\rvert > \alpha \\ 1 & \lvert x\rvert < \alpha \end{cases}$	$\frac{1}{\pi}\frac{\sin\alpha\omega}{\omega}$	$2\frac{\sin\beta x}{x}$	$U(\omega) = \begin{cases} 0 & \lvert\omega\rvert > \beta \\ 1 & \lvert\omega\rvert < \beta \end{cases}$
$\delta(x - x_0)$	$\frac{1}{2\pi}e^{i\omega x_0}$	$e^{-i\omega_0 x}$	$\delta(\omega - \omega_0)$
$\frac{\partial u}{\partial t}$	$\frac{\partial U}{\partial t}$	$\frac{\partial^2 u}{\partial t^2}$	$\frac{\partial^2 U}{\partial t^2}$
$\frac{\partial u}{\partial x}$	$-i\omega U$	$\frac{\partial^2 u}{\partial x^2}$	$(-i\omega)^2 U$
xu	$-i\frac{\partial U}{\partial\omega}$	$x^2 u$	$(-i)^2\frac{\partial^2 U}{\partial\omega^2}$
$u(x - x_0)$	$e^{i\omega x_0}U$	$\frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)g(x-s)ds$	FG

1. Beginning with the principle of conservation of energy and Fourier's law of heat conduction, derive the PDE that governs heat flow on a long thin rod assuming that density is a function of x but all other quantities (cross-sectional area, thermal conductivity, specific heat) are constant.

2. Solve the Laplace equation on the half disk $0 \leq \theta \leq \pi$, $0 \leq r \leq 1$, subject to the boundary conditions $\frac{\partial u}{\partial r}(1, \theta) = \theta + 2$ and $u(r, 0) = u(r, \pi) = 0$.

3. (a) Write the function $\frac{1}{2} - x$ as a Fourier sine series on the interval $[0, \frac{1}{2}]$.

   (b) Carefully plot the sum of the Fourier series on the interval $[-2, 2]$.

   (c) Use your answer from (a) to compute the Fourier cosine series of $x(1 - x)$ on the interval $[0, \frac{1}{2}]$.

4. Consider the displacement u of a string with source term $Q = -\alpha \frac{\partial u}{\partial t}$ (where α is a small, positive constant), given by

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - \alpha \frac{\partial u}{\partial t}.$$

(a) What is the physical effect of the source term Q on the string?

(b) Using separation of variables, solve the equation subject to the constraints

$$u(0,t) = 0, \quad u(L,t) = 0, \quad u(x,0) = 0, \quad \frac{\partial u}{\partial t}(x,0) = f(x).$$

(c) In the case of part (b), what are the (circular) frequencies of vibration of the string?

5. Express the eigenvalue problem

$$e^{x^2}\frac{d^2\phi}{dx^2} + x\frac{d\phi}{dx} = -\lambda x^2 \phi \text{ on } [1,2] \text{ with } \phi(1) = \phi(2) = 0$$

in the standard Sturm-Liouville form. Explicitly verify that the Sturm-Liouville problem is regular, give the orthogonality relationship satisfied by the eigenfunctions, an asymptotic formula for the eigenvalues, and an approximate expression for the eigenfunctions themselves.

6. Write a series for the solution u of the modified heat equation

$$\frac{\partial u}{\partial t} + tu = k\nabla^2 u$$

on the unit ball in three dimensions (and all $t \geq 0$) subject to homogeneous Dirichlet boundary conditions. You need not relate the coefficients of the series to the initial data.

7. Let $u(x,t)$ be the solution of the initial/boundary value problem for the heat equation:

$$u_t = u_{xx} \quad \text{for } 0 < x < 1,\ t > 0$$

$$u(x,0) = 0, \quad \frac{\partial u}{\partial x}(0,t) = 1, \quad u(1,t) = 1.$$

(a) What is $\lim\limits_{t\to\infty} u(x,t)$?

(b) Find $u(x,t)$.

(c) Use the equilibrium solution and one term from the Fourier series of $u(x,t)$ to estimate how long from $t = 0$ it will take for $u(\frac{1}{2},t)$ to attain half its equilibrium value (in other words, estimate the value of T such that

$$u\left(\tfrac{1}{2},t\right) < \frac{1}{2}\lim_{t\to\infty} u\left(\tfrac{1}{2},t\right) \quad \text{for } t < T$$

and

$$u\left(\tfrac{1}{2},t\right) > \frac{1}{2}\lim_{t\to\infty} u\left(\tfrac{1}{2},t\right) \quad \text{for } t > T.$$

(Bonus points for explaining why this is a really good estimate.)

8. Consider the following wave equation on the interval $[0, L]$:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},$$

$$u(x,0) = \frac{\partial u}{\partial t}(x,0) = 0, \quad \frac{\partial u}{\partial x}(0,t) = 0, \; u(L,t) = \sin \alpha t$$

For what values of α is resonance possible? How does the situation change if the boundary condition at $x = 0$ is changed to $u(x, 0) = 0$? Note that you do not need to write down the solution of this PDE to answer this question.

9. Let

$$f(x) = \begin{cases} 1 & \text{for } 0 \le x \le 2 \\ 0 & \text{for } |x-1| > 1 \end{cases}.$$

(a) Compute the Fourier transform of f.

(b) Solve initial-value problem for the wave equation

$$u_{tt} = \frac{1}{9} u_{xx}, \qquad u(x,0) = f(x), \quad u_t(x,0) = 0$$

with $f(x)$ as given at the beginning of this problem.

(c) Draw a reasonably accurate graph of $u(x,3)$.

10. Solve the initial-value problem

$$u_t = 2u_{xx} + u_x \,, \qquad u(x,0) = e^{-x^2}$$

for $u(x,t)$, $t > 0$ and $-\infty < x < \infty$. Fully simplify your answer.