MATHEMATICS 241 FALL 2010 FINAL EXAM

(Bruzzo, Donagi)

December 17, 2010, 9 am

Problems 1 through 10 are “multiple choice”. No partial credit will be given. Problems 11 through 16 are “partial credit”. Full credit will be obtained with the correct answer with some motivation. Wrong answers with evidence of some work done will obtain partial credit.

Each multiple choice problem will count 3 points, and each partial credit problem up to 5.

To answer the multiple choice problems, circle the ENTIRE statement you deem correct in the problem concerned.

No books, tables, notes, calculators, computers, phones or electronic equipment allowed. One $11 \times 8\frac{1}{2}$ inch sheet handwritten both sides allowed.

YOUR NAME (print please):

YOUR PENN ID NUMBER:

YOUR SIGNATURE:

YOUR INSTRUCTOR’S NAME (CIRCLE): Bruzzo Donagi

DON’T WRITE BELOW THIS LINE. THIS TABLE IS FOR GRADING PURPOSES ONLY

1		2	
3		4	
5		6	
7		8	
9		10	
11		12	
13		14	
15		16	
		Total	

MULTIPLE CHOICE

1. Consider the complex number $z = e^{w \ln i}$, where w is an unspecified complex number. If we want the value of z to lie on the unit circle in the complex plane for all values of $\ln i$, then w needs to

a) lie in the right half plane

b) lie in the upper half plane

c) have vanishing real part

d) lie in the lower half plane

e) lie in the left half plane

f) be real

2. A solution to the wave equation $c^2 u_{xx} = u_{tt}$ on the interval $0 \le x \le L$ corresponds to the initial conditions

$$u(x,0) = \sum_{k=1}^{\infty} b_k \sin \frac{k\pi x}{L}$$

$$u_t(x,0) = 0.$$

Which of the following functions is the solution $u(x,t)$?

a) $\displaystyle\sum_{k=1}^{\infty} b_k \cos \frac{k\pi x}{L} \sin \frac{k\pi ct}{L}$

b) $\displaystyle\sum_{k=1}^{\infty} b_k \sin \frac{k\pi x}{L} \cos \frac{k\pi ct}{L}$

c) $\displaystyle\sum_{k=1}^{\infty} b_k \cos \frac{k\pi x}{L} \cos \frac{k\pi ct}{L}$

d) $\displaystyle\sum_{k=1}^{\infty} b_k \sin \frac{k\pi x}{L} \sin \frac{k\pi ct}{L}$

e) $\displaystyle\sum_{k=1}^{\infty} b_k \sin \frac{k\pi x}{L} \cos kct$

f) $\displaystyle\sum_{k=1}^{\infty} \sin \frac{k\pi x}{L} \cos \frac{k\pi ct}{L}$

3. The value of the integral

$$\int_C \mathrm{Re}(z)\, dz$$

where C is the unit circle $|z| = 1$ traversed counterclockwise, is

a) $4\pi i$

b) $3\pi i$

c) $2\pi i$

d) πi

e) 0

4. Consider the power series

$$\sum_{n=0}^{\infty} \frac{1}{n^2} \left(\frac{3+4i}{3-4i} \right)^n (z-3)^n.$$

Its center z_0 is

a) $z_0 = 0$

b) $z_0 = 4$

c) $z_0 = 3$

d) $z_0 = 3 + 4i$,

e) $z_0 = 3 - 4i$,

f) $z_0 = \frac{3+4i}{3-4i}$

and the radius of convergence R is

a') $R = 0$

b') $R = 3$

c') $R = 1$

d') $R = 2\sqrt{2}$

e') $R = 5$

f') $R = \infty$

5. The value of the integral

$$\int_0^{2\pi} \frac{dt}{5 - 3\cos t}$$

is

a) 0

b) π

c) 1

d) $\pi/2$

e) $-1/4$

6. The value of the integral

$$\int_{-\infty}^{+\infty} \frac{dx}{x^2 - 6x + 25}$$

is

a) $1/4$

b) $\pi/8$

c) $1/8$

d) $\pi/4$

e) $-1/3$

7. A solution of a heat conduction problem has the form

$$u(x,t) = \sum_{n=1}^{\infty} A_n \, e^{-kn^2\pi^2 t/L^2} \sin \frac{n\pi x}{L}$$

and satisfies $u(x,0) = 2$.

Which of the following is true

a) $u_x(L,t) = u(L,t) = 0$

b) $u(0,t) = u(L,t) = 0$

c) $u_x(0,t) = u(0,t) = 0$

d) $u_x(0,t) = u_x(L,t) = 0$

e) $u_x(0,t) = u_x(L,t)$

8. The residue of the function $f(z) = z^2 e^{3/z}$ at the origin is

a) $9/2$

b) $2\pi i$

c) $1/3$

d) $3\pi i$

e) 0

9. For which value of the constant k is the function $x^2 + ky^2$ the real part of an analytic function?

a) all k

b) no k

c) more than one value of k but not all

d) $k = -2$

e) $k = -1$

f) $k = 0$

g) $k = 1$

h) $k = 2$

10. A thin rod of length π starts out at time $t = 0$ at a temperature $f(x) = \sin x$. If the ends are kept always at zero temperature, and units are chosen so that the heat equation is written $u_t = u_{xx}$, the temperature at one-quarter of the way from either end at $t = 1$ is

a) $\sqrt{2}/e$

b) e

c) $\frac{1}{2e}$

d) $\frac{\sqrt{2}}{2e}$

e) $\frac{e}{2\sqrt{2}}$

PARTIAL CREDIT

11. Let

$$f(x) = \begin{cases} 0 & \text{if} \quad x \notin [-1,1] \\ 2 & \text{if} \quad x \in [-1,1] \end{cases}$$

$$g(x) = \begin{cases} 0 & \text{if} \quad x < -1 \\ (x+1)^2 & \text{if} \quad x \in [-1,1] \\ 2+2x & \text{if} \quad x > 1 \end{cases}$$

a) Compute the Fourier transform of f.

b) Check that for $x \neq -1$ and $x \neq 1$, one has $g''(x) = f(x)$.

c) Compute the Fourier transform of g.

12. Solve the Sturm-Liouville problem

$$\begin{cases} y'' + \lambda y = 0 \\ y'(0) = 0 \\ y'(\pi) = 0 \end{cases}$$

on the interval $[0, \pi]$. Be sure to find all eigenvalues and eigenfunctions.

13. Compute the integral of $f(z) = \dfrac{e^z}{(z-1)^3}$ around a small circle centered at $z = 1$ traversed counterclockwise.

14. Let

$$f(w) = \oint_C \frac{2z - 20}{z - w}\, dz$$

where C is the contour $|z| = 10$ oriented counterclockwise.

(a) Compute $f(w)$ when $|w| > 10$.

(b) Compute $f(w)$ when $|w| < 10$.

15. Compute the integral

$$\int_{-\infty}^{+\infty} \frac{\sin x}{x(4x^2+9)}\, dx$$

16. Let C the star-shaped contour that joins, in this order, the points

$$-2+i$$

$$1-2i$$

$$1+2i$$

$$-2-i$$

$$3$$

and back to

$$-2+i$$

in the complex plane by line segments, traversed counterclockwise. Compute

$$\oint_C \frac{z-5}{z(z-2)}\,dz$$