Exam II: Complex Analysis. (150 points.) *MATH 241, December 16, 2005.*

NAME: ______________________ **PROFESSOR:** ______________________

Answer **all** the questions. Each problem counts as 15 points.

1. Find all values of the cube root of $-4\sqrt{2}+i4\sqrt{2}$. Draw a picture of the roots in the complex plane.

2. Find all values of $\arcsin 2$ (or $\sin^{-1} 2$). Draw a picture of these values in the complex plane.

3. Let $f(z) = u(x, y) + iv(x, y)$ be a function of a complex variable $z = x + iy$. For which of the functions $u(x, y)$ below does there exist a (real-valued) function $v(x, y)$ so that $f(z)$ is analytic? If it exists, find $v(x, y)$, and calculate the complex derivative $f'(z)$.

   a) $u(x, y) = x^2 + 2y$

   b) $u(x, y) = x^2 + 2x - y^2$

   c) $u(x, y) = e^{-x}(x \sin y - y \cos y)$

4. Evaluate the integral

$$\int_c z\bar{z}\,dz,$$

where c is the segment of the parabola $y = x^2$ between the points $(0,0)$ and $(1,1)$ (i.e., from $z = 0$ to $z = 1 + i$). The curve c is oriented from left to right.

5. Evaluate the integral

$$\int_c \frac{1}{z^4}\,dz.$$

The path of integration c is the semicircle $|z| = 1$ in the lower half plane, which starts at $z = 1$ and ends at $z = -1$.

6. For each of the functions f below, give the first four terms of the Laurent series centered at the indicated singularity z_0.

a) $f(z) = z^2 \sin \frac{1}{z}$ at $z_0 = 0$.

b) $f(z) = \frac{1}{z^3+z^2}$ at $z_0 = 0$.

c) $f(z) = \frac{\cos z}{z-\pi/2}$ at $z_0 = \pi/2$.

Then answer the following questions in each case: What is the residue of f at z_0? Is the singularity at z_0 removable, essential, or a pole? If it is a pole, what is its order?

7. Evaluate the contour integral

$$\oint_c \frac{\cos z/4}{z^2 - \pi z}\, dz$$

for each of the following (positively oriented) circles c:

a) $|z| = 4$, b) $|z-4| < 2$, c) $|z-2| = 1$.

8. Evaluate the contour integral

$$\oint_c \frac{z^2}{\cos z}\, dz.$$

The contour c is composed of four straight line segments c_1, c_2, c_3, c_4 (in this order), where c_1 connects $(2, -2)$ to $(2, 2)$, c_2 connects $(2, 2)$ to $(-2, -2)$, c_3 connects $(-2, -2)$ to $(-2, 2)$, and c_4 connects $(-2, 2)$ to $(2, -2)$, closing the loop.

9. Evaluate the real integral

$$\int_{-\infty}^{+\infty} \frac{x^2+1}{(x^2+2)^2}\, dx.$$

10. Evaluate the real integral

$$\int_0^{2\pi} \frac{1}{a + \cos^2\theta}\, d\theta,$$

where $a = -\frac{1}{2} + \frac{1}{2}\sqrt{5}$ (incidently: a is the value of the golden ratio).

Bonus (5 extra points): Find an expression for the value of the integral for general $a > 0$.