Math 240, FINAL EXAM
December 14, 2006

INSTRUCTIONS:

1. Please complete the information requested below and on the second page of this exam. There are 15 multiple choice problems. No partial credit will be given.
2. You must show all your work on the exam itself. Blind guessing will not be credited. An answer with no supporting work may receive no credit even if correct.
3. For each problem which you want graded, transfer your answer to the answer sheet *on the second page of the exam.* We will not grade problems for which answers have not been marked on the answer sheet. Do NOT detach the answer sheet from the rest of the exam.
4. You are allowed to use one hand-written sheet of paper with formulas. NO CALCULATORS, BOOKS OR OTHER AIDS ARE ALLOWED.

- **Name** (please print):
- **Name of your professor:**

  ◯ **Dr. Kadison** ◯ **Dr. Pantev** ◯ **Dr. Shatz**
- **Signature:**
- **Last** 4 **digits of student ID number:**
- **Recitation day and time:**

NAME (please print):

ANSWERS TO BE GRADED

1. A ◯ B ◯ C ◯ D ◯ E ◯ F ◯
2. A ◯ B ◯ C ◯ D ◯ E ◯ F ◯
3. A ◯ B ◯ C ◯ D ◯ E ◯ F ◯
4. A ◯ B ◯ C ◯ D ◯ E ◯ F ◯
5. A ◯ B ◯ C ◯ D ◯ E ◯ F ◯
6. A ◯ B ◯ C ◯ D ◯ E ◯ F ◯
7. A ◯ B ◯ C ◯ D ◯ E ◯ F ◯
8. A ◯ B ◯ C ◯ D ◯ E ◯ F ◯
9. A ◯ B ◯ C ◯ D ◯ E ◯ F ◯
10. A ◯ B ◯ C ◯ D ◯ E ◯ F ◯
11. A ◯ B ◯ C ◯ D ◯ E ◯ F ◯
12. A ◯ B ◯ C ◯ D ◯ E ◯ F ◯
13. A ◯ B ◯ C ◯ D ◯ E ◯ F ◯
14. A ◯ B ◯ C ◯ D ◯ E ◯ F ◯
15. A ◯ B ◯ C ◯ D ◯ E ◯ F ◯

(1) For which real values of c is the matrix $A = \begin{pmatrix} c & 1 \\ c^2 - 1 & c \end{pmatrix}$ diagonalizable as a real matrix?

(A) $c = \pm 1$ **(B)** $-1 < c < 1$ **(C)** $c \leq -1$
(D) $c \geq 1$ **(E)** $|c| > 1$ **(F)** $c = -1$

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(2) For the matrix A and the vector of unknowns $\overrightarrow{x}$ given by

$$A = \begin{pmatrix} 1 & -1 & -1 & -1 & 2 & 3 \\ 2 & 8 & -1 & 1 & 2 & 0 \\ 5 & 5 & -4 & -2 & 8 & 9 \\ 1 & 9 & 0 & 2 & 0 & -3 \end{pmatrix}, \qquad \overrightarrow{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}$$

we have

(A) $\text{rank}(A) = 1$ and the solutions of the homogeneous linear system $A\overrightarrow{x} = \overrightarrow{0}$ depend on 3 parameters.

(B) $\text{rank}(A) = 2$ and the solutions of the homogeneous linear system $A\overrightarrow{x} = \overrightarrow{0}$ depend on 2 parameters.

(C) $\text{rank}(A) = 3$ and the solutions of the homogeneous linear system $A\overrightarrow{x} = \overrightarrow{0}$ depend on 3 parameters.

(D) $\text{rank}(A) = 2$ and the solutions of the homogeneous linear system $A\overrightarrow{x} = \overrightarrow{0}$ depend on 4 parameters.

(E) $\text{rank}(A) = 1$ and the solutions of the homogeneous linear system $A\overrightarrow{x} = \overrightarrow{0}$ depend on 5 parameters.

(F) $\text{rank}(A) = 3$ and the solutins of the homogeneous linear system $A\overrightarrow{x} = \overrightarrow{0}$ depend on 2 parameters.

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(3) Which of the following statements is ***false***:

(A) If A is an 3×3 matrix with real entries, and the only solution of the system $A\overrightarrow{x} = \overrightarrow{0}$ is the zero vector, then A is invertible.

(B) There exists a 2×2 matrix A so that $A \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \end{pmatrix}$ and $A \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$.

(C) If A and B are diagonalizable 3×3 matrices with the same eigenvectors, then $A + B$ is diagonalizable.

(D) There exists a real 3×3 matrix A which satisfies $A^4 = -I_3$.

(E) An $n \times n$ real matrix can have at most n real eigenvalues.

(F) If $\overrightarrow{v} = \begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix}$, then the matrix $A = \overrightarrow{v}\,\overrightarrow{v}^T$ has three linearly independent eigenvectors.

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(4) Consider the matrix

$$B = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

It is known that 2 is an eigenvalue of B. Then

(A) B is diagonalizable with eigenvalues 1, 1, and 2.

(B) B is diagonalizable with eigenvalues 1, 2, and 2.

(C) B is diagonalizable with eigenvalues -1, 1, and 2.

(D) B is not diagonalizable, and has eigenvalues -1, 1, and 2.

(E) B is not diagonalizable, and has eigenvalues 1, 1, and 2.

(F) B is not diagonalizable, and has a unique real eigenvalue 2.

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(5) A surface Σ is given by the equation $z = x^2 + 2y$ Find the unit normal vector to Σ at the point $(0, 1, 2)$

(A) $\frac{2}{\sqrt{5}}\,\widehat{\mathbf{\imath}} + \frac{1}{\sqrt{5}}\,\widehat{\mathbf{k}}$ **(B)** $\frac{-2}{\sqrt{5}}\,\widehat{\mathbf{\jmath}} + \frac{1}{\sqrt{5}}\,\widehat{\mathbf{k}}$ **(C)** $\frac{2}{\sqrt{5}}\,\widehat{\mathbf{\jmath}} + \frac{1}{\sqrt{5}}\,\widehat{\mathbf{k}}$

(D) $\frac{-1}{\sqrt{5}}\,\widehat{\mathbf{\imath}} + \frac{2}{\sqrt{5}}\,\widehat{\mathbf{k}}$ **(E)** $\frac{1}{\sqrt{5}}\,\widehat{\mathbf{\imath}} + \frac{-2}{\sqrt{5}}\,\widehat{\mathbf{\jmath}}$ **(F)** $\frac{1}{\sqrt{5}}\,\widehat{\mathbf{\jmath}} + \frac{-2}{\sqrt{5}}\,\widehat{\mathbf{k}}$

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(6) Let

$$\overrightarrow{F} = (xz + 2e^{yz})\,\widehat{\mathbf{\imath}} + x^2 z\,\widehat{\mathbf{\jmath}} + \left(\cos^2 y - z\right)\,\widehat{\mathbf{k}},$$

and let S be the sphere of radius 5 centered at the origin. Evaluate $\iint_S \overrightarrow{F} \cdot \overrightarrow{n}\, dS$, where $\overrightarrow{n}$ is the outward unit normal.

(A) 125π **(B)** -25π **(C)** 0

(D) $\dfrac{125\pi}{3}$ **(E)** $-\dfrac{25\pi}{3}$ **(F)** $-\dfrac{500\pi}{3}$

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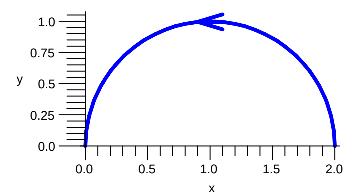
(7) Evaluate the line integral

$$\int_C (e^x \sin y - 5y)\, dx + (e^x \cos y - 5)\, dy,$$

where C is the semicircle $x^2 + y^2 = 2x$, $y \geq 0$, centered at $(1, 0)$ and oriented counterclockwise from $(2, 0)$ to $(0, 0)$.



(A) 0 **(B)** $-e^{\pi}$ **(C)** $e^{\pi} - 10$

(D) $\dfrac{5\pi}{2}$ **(E)** $\dfrac{\pi}{2}$ **(F)** $-\dfrac{3\pi}{2} - 10$

(8) Let C be the curve parameterized by $\overrightarrow{r}(t) = t\,\widehat{\mathbf{\imath}} + \sin(t)\,\widehat{\mathbf{\jmath}} + t^2\cos(t)\,\widehat{\mathbf{k}}$, for $0 \leq t \leq \pi$, and let $f(x, y, z) = z^2 e^{\sin(x)} + y^2\cos(x)$. Compute the line integral $\int_C \overrightarrow{\nabla} f \cdot d\overrightarrow{r}$ of the gradient vector field $\overrightarrow{\nabla} f$ of f.

(A) 0 **(B)** $-\pi$ **(C)** e

(D) π^4 **(E)** $e - \pi$ **(F)** none of the above

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(9) Let $y(x)$ be the general solution of the linear differential equation

$$x^2y'' + 5xy' + 3y = 0,$$

for which $y(1) = 1$ and $y'(1) = -3$. Find $y(1/2)$.

(A) -9 **(B)** 2 **(C)** 0

(D) $1/8$ **(E)** 8 **(F)** $1/16$

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(10) The differential equation $xy' - y = 100$, $x > 0$ has a unique solution for which $y(1) = 50$. Find $y(3)$

(A) 250 **(B)** 300 **(C)** 350
(D) 400 **(E)** 450 **(F)** 500

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(11) Consider the differential equation

$$y''' - y' - y = t.$$

Then the solution $y(t)$ for which $y(0) = 0$, $y'(0) = 1$, $y''(0) = 1$ has Laplace transform

(A) $\dfrac{s^3 + s^2 + 1}{s^2(s^3 - s - 1)}$ **(B)** $\dfrac{s^3 - s - 1}{s(s^3 + s^2 + 1)}$ **(C)** $\dfrac{s^3 - s - 1}{s^2(s^3 + s^2 + 1)}$

(D) $\dfrac{s^3 + s^2 + 1}{s(s^3 - s - 1)}$ **(E)** $\dfrac{s^3 + s + 1}{s(s^3 - s - 1)}$ **(F)** $\dfrac{s^2 + s + 1}{s^2(s^3 - s - 1)}$

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(12) Let $Y(s)$ be the Laplace transform of the function $y(t)$. Find the Laplace transform of the spring-mass equation with damping and sharp impulse function,

$$y'' + 2y' + y = \delta(t-1), \quad y(0) = 1, \; y'(0) = -1$$

(A) $s^2Y(s) - s - 1 + 2sY(s) + Y(s) = \mathcal{U}(t-1)$.

(B) $s^2Y(s) - s - 1 + 2sY(s) + Y(s) = e^{-s}$.

(C) $s^2Y(s) + s - 3 + 2sY(s) + Y(s) = e^{-s}$.

(D) $s^{-2}Y(s) + s^{-1} + 3 + 2s^{-1}Y(s) + Y(s) = s^{-1}$.

(E) $4Y(s) - 3s - s^2 = e^s$.

(F) $4sY(s) - 3s - s^2 = e^s$.

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(13) Consider the differential equation

$$x^2y'' + 2x(x+1)y' - y = 0.$$

It has a Frobenius power series solution $y(x) = x^\alpha \sum_{j=0}^\infty a_j x^j$ for

(A) $\alpha = \pm\frac{1}{2}$ **(B)** $\alpha = \pm\frac{\sqrt{3}}{2}$ **(C)** $\alpha = 0$ and $\alpha = \frac{\sqrt{5}}{2}$

(D) $\alpha = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}$ **(E)** $\alpha = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}$ **(F)** $\alpha = \frac{1}{2}$ and $\alpha = \frac{\sqrt{5}}{2}$

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(14) Find the recursion formula for the coefficients of the power series solution $y(x) = \sum_{n=0}^{+\infty} c_n x^n$ about the ordinary point $x = 0$ of the differential equation,

$$y'' + 2xy' + 2y = 0$$

(A) $c_{n+2} = \dfrac{-c_n}{n+2}$ **(B)** $c_{n+2} = \dfrac{c_{n-1}}{n+1}$ **(C)** $c_{n+2} = \dfrac{-3c_n}{n+1}$

(D) $c_{n+1} = c_n$ **(E)** $c_{n+2} = \dfrac{-3c_n + 1}{n+1}$ **(F)** $c_{n+2} = \dfrac{-2c_n}{n+2}$

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(15) Solve the system of first-order linear differential equations with initial value,

$$\frac{dx}{dt} = -4x + 2y, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$
$$\frac{dy}{dt} = 2x - 4y,$$

(A) $\begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-6t} + \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-2t}$ **(B)** $\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-6t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t}$

(C) $\begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{6t} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t}$ **(D)** $\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-6t}$

(E) $\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t}$ **(F)** $\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t}$
