Penn
UNIVERSITY *of* PENNSYLVANIA

MATH 240 FINAL EXAM
FALL 2014 MAKE-UP

1. Find the inverse of the matrix A and clearly indicate your answer.

$$A := \begin{bmatrix} 1 & 0 & 2 & 0 \\ 5 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 6 & 0 & 4 & 1 \end{bmatrix}.$$

Circle the option below which equals the nullity (that is, the dimension of the null space) of A.

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4 (f) none of these

Answer: (a) The nullity of any invertible matrix must be zero. $A^{-1} = \begin{bmatrix} 1 & 0 & -2 & 0 \\ -5 & 1 & 7 & 0 \\ 0 & 0 & 1 & 0 \\ -6 & 0 & 8 & 1 \end{bmatrix}.$

2. For which value of k will the span of the vectors below be two-dimensional?

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ k \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ k \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -1 \\ k \end{bmatrix}, \begin{bmatrix} 0 \\ k-1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

(a) -2 (b) -1 (c) 0 (d) 1 (e) 2 (f) none of these

Answer: (b) $k = -1$

3. Consider the 3×3 matrix

$$A := \begin{bmatrix} 0 & 0 & a-1 \\ 3 & a & -5 \\ 4 & 0 & -4 \end{bmatrix}.$$

What value(s) of a make this matrix *non-invertible*?

(a) $a = 0, 1$ (b) $a = 0, 2$ (c) $a = 1, 2$ (d) $a = \pm\sqrt{2}$ (e) $a = 2$ (f) none of these

Answer: (a) $a = 0, 1$.

4. Let $T : V \to V$ be a linear map of a finite dimensional vector space V to itself. The trace of T is defined as the trace of the matrix $\mathfrak{M}^{\mathfrak{B}}_{\mathfrak{B}}(T)$ of T for any basis $\mathfrak{B}$ of V. It is a true fact that this trace is independent of the choice of the basis $\mathfrak{B}$. Let V be the vector space of polynomials of degree at most three, and let T be given by

$$T(P) := x^2 P''' - x^2 P'' - P' + P.$$

Find the trace of T.

(a) -4 (b) -3 (c) -2 (d) -1 (e) 0 (f) none of these

Answer: (a) -4.

5. In the matrix A from problem 3, let $a = 0$ so that

$$A = \begin{bmatrix} 0 & 0 & -1 \\ 3 & 0 & -5 \\ 4 & 0 & -4 \end{bmatrix}.$$

Which of the following ordered lists of vectors is a cycle of generalized eigenvectors corresponding to eigenvalue $\lambda = -2$?

(a) $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix}$ (c) $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

(d) $\begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ (e) $\begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ (f) none of these

Answer: (d).

6. Considering again the matrix

$$A = \begin{bmatrix} 0 & 0 & -1 \\ 3 & 0 & -5 \\ 4 & 0 & -4 \end{bmatrix},$$

Find the vector-valued function $X(t)$ satisfying

$$\frac{d}{dt}X(t) = AX(t) \text{ and } X(0) = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}.$$

(a) $X(t) = \begin{bmatrix} 4(t+1)e^{-2t} \\ (14t+4)e^{-2t} \\ 4(2t+1)e^{-2t} \end{bmatrix}$ (b) $X(t) = \begin{bmatrix} 4(t+1) \\ (14t+11) - 7e^{2t} \\ 4(2t+1) \end{bmatrix}$ (c) $X(t) = \begin{bmatrix} 4(2t+1) \\ (14t+11) - 7e^{2t} \\ 4(t+1) \end{bmatrix}$

(d) $X(t) = \begin{bmatrix} 4(t+1)e^{-2t} \\ (14t+11)e^{-2t} - 7 \\ 4(2t+1)e^{-2t} \end{bmatrix}$ (e) $X(t) = \begin{bmatrix} 4(t+1)e^{2t} \\ (14t+11)e^{2t} - 7e^{-t} \\ 4(2t+1)e^{2t} \end{bmatrix}$ (f) none of these

Answer: (d).

7. Solve the ODE

$$y'' + y' - 6y = 0$$

subject to the initial conditions $y(0) = 0$, $y'(0) = 5$ and clearly indicate your result. Which option below equals $y'''(0)$?

(a) 7 (b) 14 (c) 21 (d) 28 (e) 35 (f) none of these

Answer: (e) $y'''(0) = 35$. The solution of the IVP is $y(t) = e^{2t} - e^{-3t}$.

8. Solve the ODE

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = (4t+2)e^{-t}$$

subject to the initial conditions $y(0) = -1$, $y'(0) = 1$ and clearly indicate your result. Which option below equals $y(1)$?

(a) -2 (b) -1 (c) 0 (d) 1 (e) 2 (f) none of these

Answer: (c) $y(1) = 0$. The solution to the IVP is $y(t) = (t^2 - 1)e^{-t}$.

9. A weight of 1 g is hanging on the end of a spring suspended inside a fluid-filled container from the top. The spring constant of this spring is $k = 4\,\text{dyn}\,\text{cm}^{-1} (= 4\,\text{g}\,\text{s}^{-2})$, and the fluid exerts 4 dyn of resistive force for every $1\,\text{cm}\,\text{s}^{-1}$ of velocity. The spring is displaced 5 cm from its equilibrium position in the upwards/positive direction and given an initial velocity of $11\,\text{cm}\,\text{s}^{-1}$ moving towards the equilibrium position. Find the formula for the displacement from equilibrium as a function of time and clearly indicate the result. At what time (if ever) will the weight first reach the equilibrium position?

(a) 2 s (b) 3 s (c) 5 s (d) 7 s (e) 11 s (f) none of these

Answer: (c) 5 s. The height in centimeters as a function of seconds after the start is $x(t) = (5-t)e^{-2t}$.

10. Consider the differential equation

$$y'' + (4x+2)y' + (4x^2+4x+2)y = 0.$$

The function $y_1 = e^{-x^2}$ solves this ODE. Use reduction of order to find the general solution. After recording the general soltuion, solve the IVP $y(-1) = 1$, $y'(-1) = 0$ and circle the option below which agrees with your answer.

(a) $y = e^{-x^2}$ (b) $y = e^{-(x-1)^2}$ (c) $y = e^{-(x+1)^2}$
(d) $y = e^{x^2}$ (e) $y = e^{(x-1)^2}$ (f) $y = e^{(x+1)^2}$

Answer: (c) $y = e^{-(x+1)^2}$. The general solution is $y = C_1e^{-x^2} + C_2e^{-(x+1)^2}$.

11. Let A equal the matrix

$$\begin{bmatrix} 0 & 2 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Compute e^{tA} for all t. Which of the following expressions equals the entry of e^{tA} in row 1 and column 2?
(a) e^t (b) $\cos t$ (c) $\sin t$ (d) $e^t - \cos t + \sin t$ (e) $e^t - \cos t - \sin t$ (f) none of these

Answer: (d). The matrix exponential is $e^{tA} = \begin{bmatrix} \cos t & e^t - \cos t + \sin t & -\sin t \\ 0 & e^t & 0 \\ \sin t & e^t - \cos t - \sin t & \cos t \end{bmatrix}.$

12. The following linear autonomous system has two equilibrium points. Describe the types of these points.

$$\frac{dx}{dt} = (y^2 - 1)$$
$$\frac{dy}{dt} = -y + x$$

(a) stable node, center (b) saddle point, center (c) saddle point, stable spiral
(d) unstable node, center (e) sadle point, unstable spiral (f) degenerate node, center

Answer: (c) This system has a stable spiral at $(-1,-1)$ and a saddle point at $(1,1)$.