UNIVERSITY *of* PENNSYLVANIA
MATHEMATICS DEPARTMENT
Mathematics 114–Final Examination
Spring 2005

Your Name:______________________

Professor/Section (check one):

Crotty 001❑ 601❑ Matthews 002❑ Stovall 003❑ 004❑

Your TA: _______________________________ **Rectation #:** __________________________

True/False: Please write the *word* true or false (and any other information required) in the space provided

1		6	
2		7	
3		8	
4		9	
5		10	

Multiple Choice: Write the letter corresponding to your answer in the space provided

1		8	
2		9	
3		10	
4		11	
5		12	
6		13	
7		14	

True/False questions

1. If C is a circle in the xy–plane and $f(x, y)$ is not a constant when it is constrained to C, then there must be at least one point on C where ∇f is perpendicular to C.

For questions 2-5, you are to determine if (0, 0) is a critical point of the given function. If (0, 0) IS a critical point, write TRUE on your answer sheet and indicate the type of critical point (min, max or saddle) as well.

2. $f(x,y)=\cos x\cos y$

3. $f(x,y)=\sin x\cos y$

4. $f(x,y)=5x^4+y^2$

5. $f(x,y)=-6x^2+3y^4$

6. If f is any two variable function, $\int_R f(x,y)dxdy=2\int_S f(x,y)dxdy$ where R is the rectangle $0\le x\le 2$, $0\le y\le 1$ and S is the square $0\le x\le 1,\ 0\le y\le 1$.

7. The equations $x=\cos(t^3),\ y=\sin(t^3)$ parameterize a circle.

8. When solving a first order ordinary differential equation, you find that $y_1 = 2x - 4$ is a solution corresponding to the initial value y(2) = 0. Suppose you find a second solution, y_2, corresponding to the initial point $y(3) = 5$. Is it possible for the solution y_2 to cross the line $y_1 = 2x - 4$? (Answer TRUE if the second solution CAN cross the line; answer FALSE otherwise).

9. The solution of $\frac{dy}{dx}=x+1$ passing through (0, 1) has the same shape as the solution through (0, 0) except it is shifted up by 1 unit.

10. The differential equation $(y\cos x+2xe^y)+(\sin x+x^2e^y+2)\frac{dy}{dx}=0$ is exact.

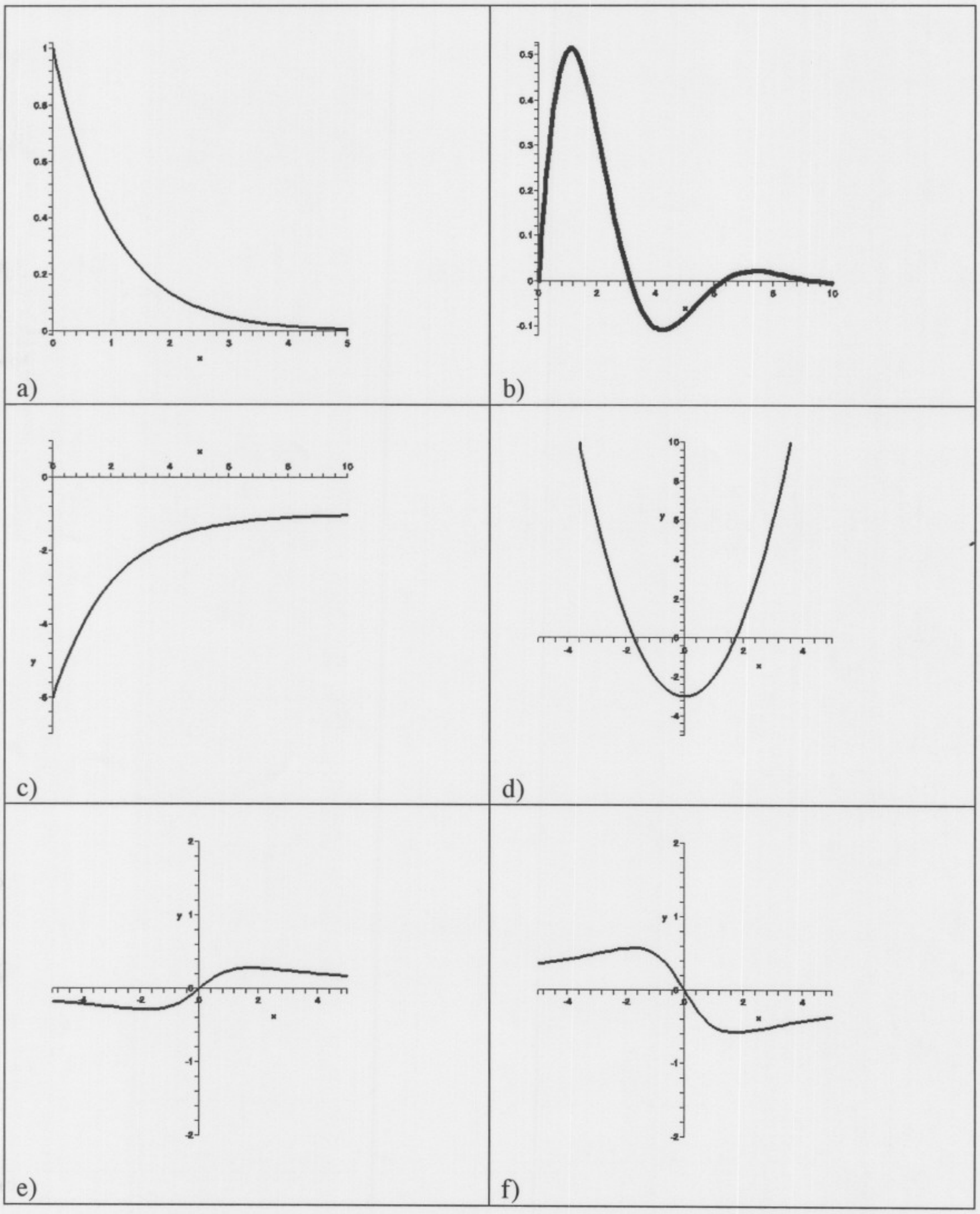
Multiple Guess:

1. The temperature throughout the first quadrant is given by $T(x,y)=\ln(xy)-x-y$. Determine the location of the hottest point in the first quadrant.

a) (0, 0) b) (1, 1) c) (∞, ∞) d) (2, 0) e) (0, 2) f) no hottest spot exists

Answer: ___

2. Suppose $y(x)$ is a solution to the differential equation $\frac{dy}{dx} = f(y)$ and further suppose that $f(y) > 0$ for all y. Which of the following *could* be a graph of $y(x)$?



Answer: ___

3. Consider the differential equation $\frac{dy}{dx} = y(1 - x - y)$. This equation has a constant solution (y = const.) if:

a) $y = 0$ b) $y = x$ c) $x + y = 1$ d) $y = 1$ e) there are no constant solutions

Answer: ___

4. Consider the differential equation $\frac{dy}{dx} = y(1 - x - y)$. The solutions (other than constant solutions, if any) of this equation will have critical points on

a) the line $y = x$ b) the curve y2 $= -xy$ c) at $-\infty$ d) on the line $x + y = 1$

e) no solution has a critical point

Answer: ___

5. Consider the differential equation $y''+10y'+25y=x$. The homogeneous solution will include a function of the form:

a) e^{-x} b) $\sin 5x$ c) $5x$ d) $5\,e^{x}$ e) e^{5x} f) xe^{-5x}

Answer: ___

6. Evaluate $\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{0}\frac{4\sqrt{x^2+y^2}}{1+x^2+y^2}\,dxdy$.

a) $2\pi\ln 2$ b) π c) 0 d) $4\pi-\pi^2$ e) $\frac{4-\pi}{4}$ f) $\frac{\pi-1}{2}$

Answer: ___

7. Solve $y'' - 2y' - 2y = 0$.

a) $y = c_1 e^{2x} + c_2 e^{-2x}$

b) $y = c_1 e^{1+\sqrt{3}x} + c_2 e^{1-\sqrt{3}x}$

c) $y = c_1 e^{\sqrt{3}x} + c_2 e^{x}$

d) $y = c_1 e^{2x} + c_2 e^{-\sqrt{3}x}$

e) $y = (c_1 x + c_2) e^{1-\sqrt{3}x}$

f) $y = (c_1 x + c_2) e^{x}$

Answer:____

8. One of the solutions to $y'' + \lambda^2 y = 0$ is

a) $\sin\lambda x$ b) $e^{\lambda x}$ c) λx d) x^{λ} e) λ^{x} f) λ

Answer: ____

9. At what value of t does the curve $x = t^2 - t,\ y = t^2 + t$ have a horizontal tangent?

a) 2 b) 1 c) 1/2 d) 0 e) –1/2 f) –1 g) –2

Answer: ___

10. If R is the region inside the circle $x^2 + y^2 = 1$, then $\iint_R \sqrt{x^2 + y^2}\ dA$ is equal to

a) $\pi/6$ b) $\pi/4$ c) $\pi/3$ d) $\pi/2$ e) $2\pi/3$ f) $3\pi/4$

Answer: ___

11. The general solution of the differential equation $(4x - 5y)\,dx + (5x - y)\,dy = 0$ is:

a) $2x^2 - y^2 = C$ b) $2x^2 - y^2 + 10xy = C$ c) $(y - 2x)^7 = C(y + 2x)^3$
d) $4x^2y - 5xy - y^2 = C$ e) $(y - 2x)^3 = C(y + 2x)^7$ f) $(2y - x)5 = C(2y + x)3$

Answer: ___

12. Find the unit tangent vector to the curve $r(t) = e^{2t}\cos t\,\mathbf{i} + e^{2t}\sin t\,\mathbf{j} + e^{2t}\mathbf{k}$ at the point where $t = \pi/2$.

a) <2/3, –2/3, 1/3> b) <2/3, –1/3, 2/3> c) <-1/3, 2/3, 2/3>
d) <-1/3, -2/3, -2/3> e) <3/$\sqrt{14}$, 2/$\sqrt{14}$, -1/$\sqrt{14}$> f) <-3/$\sqrt{14}$, -2/$\sqrt{14}$, -1/$\sqrt{14}$>

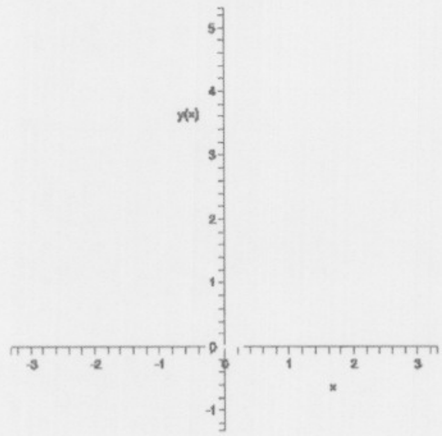
Answer: ___

14. Shown are graphs of the solutions of three differential equations solved using the initial conditions given.

(i) $y' - \frac{1}{x} y = 2x - 1, \; y(1) = 0$ -- 2

(ii) $y'' + 2y' + y = x, \; y(0) = 0, \; y'(0) = -1$ -- 3

(iii) $\frac{dy}{dx} - 2xy = \sin x\,, \; y(0) = 0$ -- 1.



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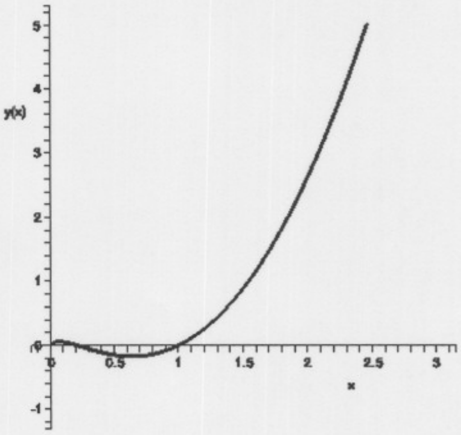

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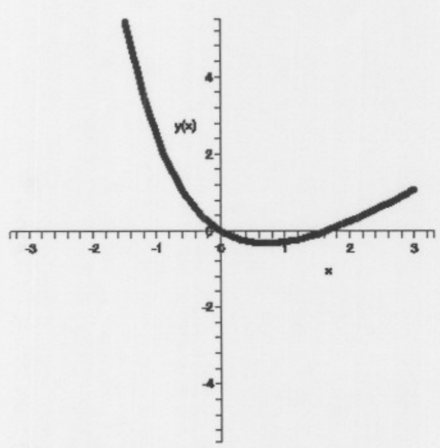
3.



The solution graphs shown match the differential equations (i), (ii) and (iii) in the order:

(a) 1, 2, 3 (b) 1, 3, 2 (c) 2, 1, 3 (d) 2, 3, 1 (e) 3, 1, 2 (f) 3, 2, 1

Answer: ___

Free Response Questions:
Problems 1, 2 and 3 are *REQUIRED* of all students. You may do either 4. Or 5. For a total of 4 problems. If you do both 4 and 5, the better of the two scores will count.

1. Suppose the function $P(t)$ satisfies the following differential equation:

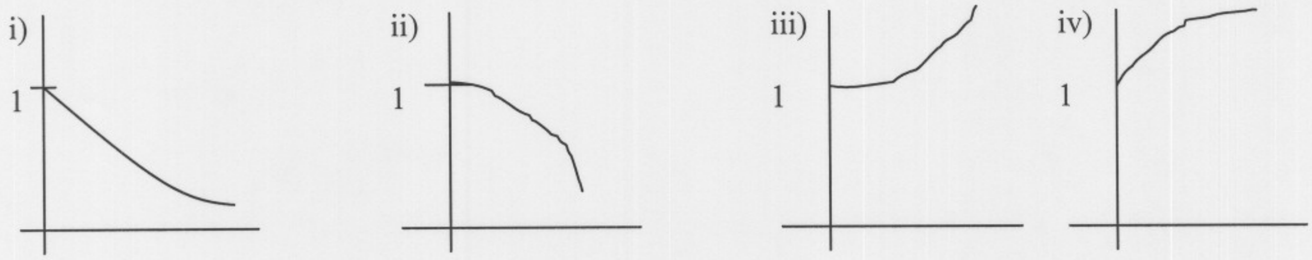
$$P'(t) = P(t)(4 - P(t))$$

with the initial condition $P(0)=1$. Even without knowing a formula for $P(t)$, we can deduce many of its properties. For instance,

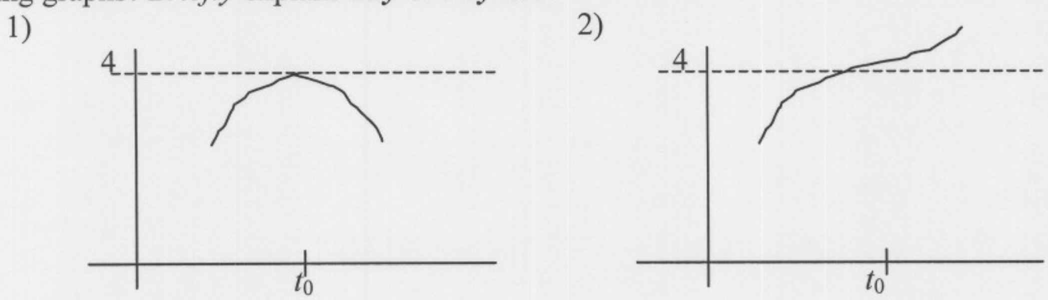
$$P'(0) = P(0)(4 - P(0)) = 1(4-1) = 3$$

a) Find $P''(t)$ in terms of $P(t)$. Find $P''(0)$

b) For small t, which of the following is a possible graph for $P(t)$? Explain *briefly.*



c) Since $P(0)=1$, the function starts out less than 4. If $P(t)$ reaches 4, that is, there is a time, t_0, such that $P(t_0) = 4$, then $P'(t_0) = P(t_0)(4 - P(t_0)) = 4(4-4) = 0$. Would $P(t)$ near $t = t_0$ look like either of the following graphs? *Briefly* explain **why** or ***why not***.



d) Sketch a graph of $P(t)$ showing a complete graph of $P(t)$ for $t > 0$. *Explain* any critical points, points of inflection and the behavior of P as $t \to \infty$.

2. Solve $y'' - y' + 2y = 0$ *subject to* $y(0) = 1,\ y'(0) = 1$.

3. A zoo is designing a giant birdcage consisting of a cylinder of radius r feet and height h feet with a hemispherical top but no bottom (the cage will rest on the ground) (see diagram). The material for the hemisphere costs \$20/ft^2 and the material for the cylindrical part costs \$10/ft^2. The zoo has a budget of \$8000 for materials. Find the values of r and h giving the birds the maximum volume inside their cage assuming that the zoo spends all \$8000 in the budget.

4. Solve $y'' - 8y' + 16y = x^2$ *subject to* $y(0) = 1,\ y'(0) = 0$. You may use the method of your choice to solve for the non-homogeneous part (i.e., the particular solution).

5. Solve the homogeneous differential equation $y' = \dfrac{x+y}{x-y}$.