LAST NAME:____________________ **NAME:**____________________ PENN ID # :____________________

Please bubble-in the required information.

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Instructions: *You have 120 minutes to complete this exam.* **Do not detach this sheet from the body of the test.** *This is a multiple-choice test. There is no penalty for guessing. No partial credit will be awarded. Answers with no supporting work will be given no points. If you change an answer, please either erase or cross out the answer you do not want considered; questions with more than one answer will be marked wrong. Bubble-in your choice both in the body of the test and in the grid below.*

Questions 1—7	Questions 8–14	Questions 15-20
1. (A) (B) (C) (D) (E) (F)	8. (A) (B) (C) (D) (E) (F)	15. (A) (B) (C) (D) (E) (F)
2. (A) (B) (C) (D) (E) (F)	9. (A) (B) (C) (D) (E) (F)	16. (A) (B) (C) (D) (E) (F)
3. (A) (B) (C) (D) (E) (F)	10. (A) (B) (C) (D) (E) (F)	17. (A) (B) (C) (D) (E) (F)
4. (A) (B) (C) (D) (E) (F)	11. (A) (B) (C) (D) (E) (F)	18. (A) (B) (C) (D) (E) (F)
5. (A) (B) (C) (D) (E) (F)	12. (A) (B) (C) (D) (E) (F)	19. (A) (B) (C) (D) (E) (F)
6. (A) (B) (C) (D) (E) (F)	13. (A) (B) (C) (D) (E) (F)	20. (A) (B) (C) (D) (E) (F)
7. (A) (B) (C) (D) (E) (F)	14. (A) (B) (C) (D) (E) (F)	

1. Consider the region in the first quadrant bounded by $y = x^3$ and $x = y^3$. Rotate this region about the line $y = -1$. What is the volume of the resulting solid?

   (A) $\frac{16}{35}\pi$ (B) $\frac{51}{35}\pi$ (C) $\frac{23}{14}\pi$ (D) $\frac{86}{35}\pi$ (E) $\frac{31}{10}\pi$ (F) $\frac{166}{35}\pi$

2. Find the average value of the function

$$f(\theta) = \sec(\theta)\tan(\theta)$$

   on the interval $[0, \pi/4]$.

   (A) 0 (B) $\sqrt{2} - 1$ (C) $\frac{2}{\pi}$ (D) $\frac{1}{2}$ (E) $\frac{4\sqrt{2} - 4}{\pi}$ (F) diverges

3. The base of a solid is a circular disk of radius 3. Find the volume of the solid if parallel cross-sections perpendicular to the base are isosceles right triangles with hypotenuse lying along the base. Refer to figure 1

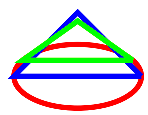

Figure 1: Problem 3.

   (A) 36 (B) 18 (C) 9 (D) 1 (E) $\sqrt{2}$ (F) π

4. Find $\frac{d}{dx}\left(e^{e^{e^x}}\right)$.

   (A) $e^{\ln x}$ (B) $e^{e^{e^x}}$ (C) $e^x e^{e^{e^x}}$ (D) $e^x e^{e^x} e^{e^{e^x}}$ (E) $e^x e^{e^x} e^{e^{e^x}} e^{e^{e^{e^x}}}$ (F) $e^{e^x} e^{e^{e^x}}$

5. Let

$$f(x) = \frac{10^x}{10^x + 1}.$$

   Find the inverse function f^{-1}.

   (A) $f^{-1}(x) = \log_{10}\left(\frac{x}{1-x}\right)$

   (B) $f^{-1}(x) = \log_{10}\left(\frac{x}{1+x}\right)$

   (C) $f^{-1}(x) = \log_{10}\left(\frac{x-1}{x+1}\right)$

   (D) $f^{-1}(x) = \log_{10}\left(\frac{x+1}{x-1}\right)$

   (E) $f^{-1}(x) = \log_{10}\left(\frac{x+1}{1-x}\right)$

   (F) $f^{-1}(x) = \log_{10}\left(\frac{1-x}{x+1}\right)$

6. Find the limit

$$\lim_{x\to\infty} \arcsin\left(\frac{1 + \sqrt{3}\,x^3}{1 + 2x^3}\right).$$

   (A) 0 (B) $\frac{\pi}{6}$ (C) $\frac{\pi}{4}$ (D) $\frac{\pi}{3}$ (E) $\frac{\pi}{2}$ (F) 1

7. $\displaystyle\int_2^3 \frac{x^3 - x^2 + 1}{x^2 - x}\,\mathrm{d}x =$

(A) $\frac{1}{2} + \ln 2$ (B) $\frac{5}{2} + \ln 3$ (C) $\frac{5}{2} - \ln 3$ (D) $\ln 3 - \ln 4$ (E) $\frac{5}{2} + \ln 3 - 2\ln 2$ (F) $\frac{5}{2} - \ln 3 + 2\ln 2$

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8. $\displaystyle\int \frac{e^{3x}}{1 + e^{6x}}\mathrm{d}x =$

(A) $3\arctan e^{3x} + C$

(B) $\frac{1}{3}\arctan e^{3x} + C$

(C) $\arctan e^{3x} + C$

(D) $\arctan e^{x/3} + C$

(E) $3\arctan e^{x/3} + C$

(F) $\frac{1}{3}\arctan e^{x^3} + C$

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9. $\displaystyle\int_0^3 \frac{\mathrm{d}x}{(x-1)^3} =$

(A) $\frac{3}{8}$ (B) $\frac{1}{2}$ (C) $\frac{9}{4}$ (D) 0 (E) $\frac{8}{3}$ (F) diverges

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10. Find the surface area generated by revolving the curve

$$y = \sqrt{1 - x^2}, \qquad 0 \le x \le \frac{1}{2}$$

about the x-axis. Refer to figure 2

(A) 2π (B) π (C) $\frac{4}{3}\pi$ (D) $\frac{3}{4}\pi$ (E) 1 (F) $\frac{4}{3}$

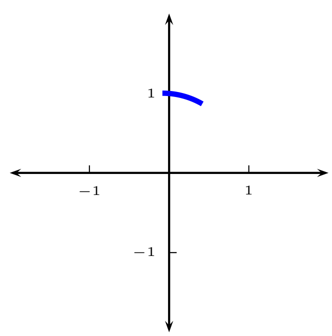

Figure 2: Problem 10.

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11. Compute the arc length of the curve $y = 1 + \frac{2}{3}(x-1)^{3/2}$ for $x \in [1, 4]$.

(A) $\frac{14}{3}$ (B) $\frac{2}{3}\ln 3$ (C) $\frac{4}{3}\pi$ (D) 4.75 (E) 1 (F) $\frac{1}{2}$

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12. Let $\mathscr{C}$ be the curve defined by:

$$x = t - t^2, \qquad y = t^2 + 4, \qquad t \in \mathbb{R}.$$

What is the slope of the line tangent to $\mathscr{C}$ at $(0, 5)$?

(A) -2 (B) 2 (C) $\frac{1}{2}$ (D) $-\frac{1}{2}$ (E) -1 (F) 1

13. Find the length of the curve with parametric equations

$$x = \sin t + \cos t \qquad y = \sin t - \cos t$$

for $0 \le t \le 2\pi$.

(A) $\sqrt{2}$ (B) $2\sqrt{2}$ (C) $2\sqrt{2}\,\pi$ (D) 3π (E) 9 (F) $\sqrt{2} + \ln(1+\sqrt{2})$

14. Four polar plots appear below.

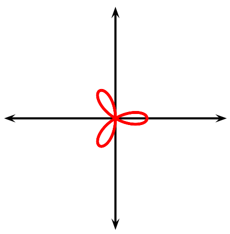
Figure 3: I

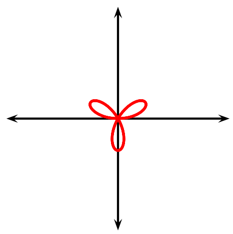
Figure 4: II

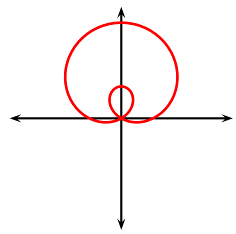
Figure 5: III

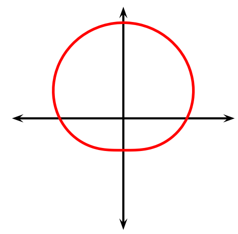
Figure 6: IV

Also, four polar equations are given below

$$\alpha : r = \sin 3\theta; \qquad \beta : r = \cos 3\theta; \qquad \gamma : r = 1 + 2\sin\theta; \qquad \delta : r = 2 + \sin\theta$$

Which choice gives the correct match with the polar equation?

(A) (I, β), (II, α), (III, δ), (IV, γ)

(B) (I, β), (II, α), (III, γ), (IV, δ)

(C) (I, β), (II, γ), (III, α), (IV, δ)

(D) (I, δ), (II, γ), (III, β), (IV, α)

(E) (I, α), (II, β), (III, γ), (IV, δ)

(F) (I, γ), (II, α), (III, β), (IV, δ)

15. Find the area of the region between the two curves with polar equations

$$r = \sqrt{2} \qquad \text{and} \qquad r = \frac{1}{\sin\theta}, \qquad \frac{\pi}{4} \le \theta \le \frac{3}{4}\pi.$$

(A) $\frac{\pi}{2}$ (B) $\frac{\pi - 1}{2}$ (C) $\frac{\pi - 2}{2}$ (D) $\frac{1}{2}$ (E) $\ln 2$ (F) $\ln\frac{\pi}{2}$

16. Find the limit of the sequence

$$\lim_{n\to\infty} n \tan\frac{1}{n}.$$

(A) 2 (B) 1 (C) -1 (D) $\frac{\pi}{2}$ (E) 0 (F) divergent

17. A sequence is defined by

$$a_1 = 2; \quad a_2 = 2^{1+1/2}; \quad a_3 = 2^{1+1/2+1/4}; \quad \ldots \quad a_n = 2^{1+1/2+1/2^2+\cdots+1/2^{n-1}}.$$

Determine $\lim_{n\to\infty} a_n$.

(A) 0 (B) 1 (C) $\sqrt{2}$ (D) 2 (E) 4 (F) $+\infty$

18. Find the interval of convergence of $\sum_{n=2}^{\infty} \frac{1}{\ln n}(x-1)^n$.

(A) $0 < x < 2$ (B) $-2 \le x < 2$ (C) $0 \le x \le 2$ (D) $0 \le x < 1$ (E) $0 \le x < 2$ (F) $1 - \frac{1}{e} \le x < 1 + \frac{1}{e}$

19. Consider the following series:

$$I : \sum_{n=1}^{\infty} \frac{1}{n^{5/3}}; \quad II : \sum_{n=1}^{\infty} 5^{n/3}; \quad III : \sum_{n=1}^{\infty} \frac{1}{n^{3/5}}; \quad IV : \sum_{n=1}^{\infty} \frac{1}{5^{n/3}}.$$

Which of the following statements is true?

(A) I, II, III, IV all converge

(B) I and II converge, but III and IV diverge

(C) I and III converge, but II and IV diverge

(D) I and IV converge, but II and III diverge

(E) I converges, but II, III, and IV diverge

(F) all diverge

20. Let $\sum_{n=0}^{\infty} a_n x^n = (1-x)^{1/2} + (1+x)^{1/2}$ be the Maclaurin expansion of $(1-x)^{1/2} + (1+x)^{1/2}$. Find

$$(a_0)^2 + (a_1)^2 + (a_2)^2.$$

(A) $\frac{65}{16}$ (B) $\frac{63}{16}$ (C) $\frac{7}{4}$ (D) $\frac{9}{4}$ (E) $\frac{81}{16}$ (F) $\frac{15}{4}$

1. (B) In the first quadrant, the curves meet at $(0,0)$, and $(1,1)$. See figure 7. The volume can be computed by both shells or washers. Hence we obtain

$$\pi\int_0^1 \left((x^{1/3}+1)^2-(x^3+1)^2\right)\mathrm{d}x = 2\pi\int_0^1 (1+y)(y^{1/3}-y^3)\mathrm{d}y = \frac{51}{35}\pi.$$

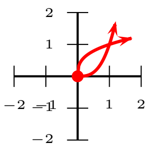

Figure 7: Problem 1.

2. (E) The average value sought is plainly

$$\frac{4}{\pi}\int_0^{\pi/4}\sec\theta\tan\theta\mathrm{d}\theta = \frac{4}{\pi}\sec\theta\Big|_0^{\pi/4} = \frac{4}{\pi}(\sec\frac{\pi}{4}-\sec 0) = \frac{4}{\pi}\left(\sqrt{2}-1\right) = \frac{4\sqrt{2}-4}{\pi}.$$

3. (A) Let $A(x)$ be the area of a typical triangle. The area of an isosceles right triangle is $\dfrac{a^2}{2}=\dfrac{h^2}{4}$, where a is the length of a cathetus and h is the length of the hypotenuse. Now, if one is at distance $|x|$ from the centre of the circle, $h=2\sqrt{9-x^2}$, and so $A(x)=9-x^2$. The volume sought is

$$\int_{-3}^{3}A(x)\mathrm{d}x = \int_{-3}^{3}(9-x^2)\mathrm{d}x = 36$$

cubic units.

4. (D) Let $f(x)=e^x$. By the Chain Rule

$$(f\circ f\circ f)'(x) = f'(x)f'(f(x))f'(f(f(x))) = e^x e^{e^x} e^{e^{e^x}}.$$

5. (A) Putting $y=\dfrac{10^x}{10^x+1}$ one gathers

$$10^x y + y = 10^x \implies 10^x(1-y)=y \implies 10^x = \frac{y}{1-y} \implies x=\log_{10}\left(\frac{y}{1-y}\right),$$

whence $f^{-1}(x)=\log_{10}\left(\dfrac{x}{1-x}\right)$.

6. (D) As $x\to\infty$,

$$\frac{1+\sqrt{3}x^3}{1+2x^3}\sim\frac{\sqrt{3}}{2} \implies \arcsin\left(\frac{1+\sqrt{3}x^3}{1+2x^3}\right)\to\arcsin\frac{\sqrt{3}}{2}=\frac{\pi}{3}.$$

7. (F) We have

$$\frac{x^3-x^2+1}{x^2-x} = \frac{x^3-x^2}{x^2-x}+\frac{1}{x^2-x} = \frac{x^2(x-1)}{x(x-1)}+\frac{1}{x(x-1)} = x-\frac{1}{x}+\frac{1}{x-1},$$

from where it follows that

$$\int_2^3\frac{x^3-x^2+1}{x^2-x}\mathrm{d}x = \int_2^3\left(x-\frac{1}{x}+\frac{1}{x-1}\right)\mathrm{d}x = \left(\frac{x^2}{2}-\ln|x|+\ln|x-1|\right)\Big|_2^3 = \frac{5}{2}-\ln 3+2\ln 2.$$

8. (B) Put $u = e^{3x}$. Then $\mathrm{d}u = 3e^{3x}\mathrm{d}x$. Thus

$$\int \frac{e^{3x}}{1+e^{6x}}\mathrm{d}x = \frac{1}{3}\int \frac{\mathrm{d}u}{1+u^2} = \frac{1}{3}\arctan u + C = \frac{1}{3}\arctan e^{3x} + C.$$

9. (F) Diverges, by inspection.

10. (B) The surface area sought is $\frac{1}{4}$ of the surface area of a unit sphere, therefore $\frac{4\pi}{4} = \pi$.

11. (A) One has

$$\mathrm{d}y = (x-1)^{1/2}\mathrm{d}x \implies (\mathrm{d}y)^2 = (x-1)(\mathrm{d}x)^2.$$

The surface area sought

$$\int_{x=1}^{x=4} \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2} = \int_1^4 \sqrt{1+(x-1)}\mathrm{d}x = \int_1^4 \sqrt{x}\mathrm{d}x = \frac{14}{3}.$$

12. (A) $\{t : t - t^2 = 0\} \cap \{t : t^2 + 4 = 5\} = \{1\}$. The slope of the tangent is $\frac{y'(1)}{x'(1)} = \frac{2(1)}{1-2(1)} = -2$.

13. (C) One has

$$\mathrm{d}x = (\cos t - \sin t)\mathrm{d}t, \qquad \mathrm{d}y = (\cos t + \sin t)\mathrm{d}t \implies \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2} = \sqrt{2}\,\mathrm{d}t,$$

whence the arc length sought is

$$\int_0^{2\pi} \sqrt{2}\,\mathrm{d}t = 2\sqrt{2}\,\pi.$$

14. (B)

15. (C) $r = \sqrt{2}$ is a circle of radius $\sqrt{2}$ centred at the origin and $r = \frac{1}{\sin\theta} \implies y = 1$. This is asking for the area of the circular segment in figure 8. This is

$$\frac{1}{2}\int_{\pi/4}^{3\pi/4}\left(2 - \frac{1}{\sin^2\theta}\right)\mathrm{d}\theta = \theta + \frac{1}{2}\cot\theta\Big|_{\pi/4}^{3\pi/4} = \frac{\pi}{2} - 1.$$

Aliter: One can observe that the area of the circular segment is the area of a circular sector forming $\frac{1}{4}$ of the area of the circle minus the area of an isosceles right triangle with catheti of length $\sqrt{2}$, hence the desired area is

$$\frac{\pi(\sqrt{2})^2}{4} - \frac{1}{2}(\sqrt{2})^2 = \frac{\pi}{2} - 1.$$

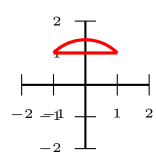

Figure 8: Problem 15.

16. (B) $n\tan\frac{1}{n} \sim n \cdot \frac{1}{n} = 1.$

17. (E) $a_n = 2^{1+1/2+1/2^2+\cdots+1/2^{n-1}} \to 2^{\frac{1}{1-\frac{1}{2}}} = 4.$

18. (E) If $a_n = \dfrac{|x-1|^n}{\ln n}$ then $a_n^{1/n} = \dfrac{|x-1|}{(\ln n)^{1/n}} \to |x-1|$. The root test gives absolute convergence for

$$|x-1| < 1 \implies -1 < x-1 < 1 \implies 0 < x < 2.$$

At $x = 0$ the series becomes $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ which converges conditionally by Leibniz's test. At $x = 2$ the series becomes $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$ which diverges. The interval of convergence is thus $0 \le x < 2$.

19. (D) I is a p-series with $p > 1$ and so converges. II is a geometric series with ratio $5^{1/3} > 1$ and so diverges. III is a p-series with $p < 1$ and so diverges. IV is a geometric series with ratio $\dfrac{1}{5^{1/3}} < 1$ and so converges.

20. (A) By the binomial series

$$(1-x)^{1/2} = 1 + \binom{1/2}{1}(-x) + \binom{1/2}{2}(-x)^2 + \cdots,$$

$$(1+x)^{1/2} = 1 + \binom{1/2}{1}x + \binom{1/2}{2}x^2 + \cdots,$$

whence

$$(1-x)^{1/2} + (1+x)^{1/2} = 2 + 2\binom{1/2}{2}x^2 + \cdots = 2 + 2\left(\frac{(\frac{1}{2})(\frac{1}{2}-1)}{2\cdot 1}\right)x^2 = 2 - \frac{1}{4}x^2 + \cdots,$$

giving

$$a_0 = 2, a_2 = \frac{1}{4} \implies a_0^2 + a_1^2 + a_2^2 = 4 + \frac{1}{16} = \frac{65}{16}.$$

(Note: The function f with $f(x) = (1-x)^{1/2} + (1+x)^{1/2}$ is even, and so $a_1 = 0$.)