Signature	Printed Name

Preliminary Examination, Part I

Friday, April 27, 2018 — 9:30am-12:00pm

This part of the examination consists of six problems. You should do all of the problems. Show all of your work and justify your assertions. Try to keep computations well-organized and proofs clear and complete. Be sure to write your name both on the exam and on any extra sheets you may submit.

All problems have equal weight of ten points.

Score	
1	
2	
3	
4	
5	
6	
initials	
For faculty use only.	

1. For each of the following series, either prove that it converges or prove that it diverges.
   a) $1 + 1/2 - 1/3 + 1/4 + 1/5 - 1/6 + 1/7 + 1/8 - 1/9 + 1/10 + 1/11 - 1/12 + 1/13 + 1/14 - 1/15 + \cdots$
   b) $1 + 1/2 + 1/3 - 1 + 1/4 + 1/5 + 1/6 - 1/2 + 1/7 + 1/8 + 1/9 - 1/3 + 1/10 + 1/11 + 1/12 - 1/4 + \cdots$

2. Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & a & a & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix},$$

where $a \in \mathbb{R}$.

a) Determine all values of $a \in \mathbb{R}$ for which the matrix A is invertible.

b) For each such a, find the determinant of A.

3. Assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function, and that

$$|f(x) - f(y)| \geq |x - y| \quad \text{for all } x, y \in \mathbb{R}.$$

a) Prove that there is a function $g : \mathbb{R} \to \mathbb{R}$ such that the compositions $f \circ g$ and $g \circ f$ are both equal to the identity map.

b) Prove that the above function g is continuously differentiable.

4. Let G be a group of order 66.

   a) Find an integer n with $1 < n < 66$ such that G must have a normal subgroup N of index equal to n. Justify your assertion.

   b) For this value of n, prove that every $g \in G$ has the property that $g^n \in N$.

5. Let $a, b \in \mathbb{R}$ and consider the differential equation $f''(x) + af'(x) + bf(x) = 0$. For which values of a, b does there exist a non-zero solution $f : \mathbb{R} \to \mathbb{R}$ to this equation such that f is bounded on $[0, \infty)$? For each such a, b, find such a solution.

6. For each of the following, either give an example or prove that no such example exists.

   a) A closed subset $S \subset \mathbb{R}$ that contains $\mathbb{Q}$, such that $S \neq \mathbb{R}$.

   b) An open subset $S \subset \mathbb{R}$ that contains $\mathbb{Q}$, such that $S \neq \mathbb{R}$.

   c) A connected subset $S \subset \mathbb{R}$ that contains $\mathbb{Q}$, such that $S \neq \mathbb{R}$.

Extra page for work

Signature

Printed Name

Preliminary Examination, Part II

Friday, April 27, 2018 1:30pm-4:00pm

This part of the examination consists of six problems. You should do all of the problems. Show all of your work and justify your assertions. Try to keep computations well-organized and proofs clear and complete. Please write your name on both the exam and any extra sheets you may submit.

All problems have equal weight of ten points.

Score	
7	
8	
9	
10	
11	
12	
initials	
For faculty use only.	

7. For each continuous function $f(x, y)$ on the x, y-plane, and each path C from $(0, 1)$ to $(\pi, 1)$, consider the contour integral

$$\int_C y \sin^2(x)\, dx + f(x, y)\, dy.$$

a) Find a choice of the function $f(x, y)$ such that the value of the above integral is independent of the choice of the path C from $(0, 1)$ to $(\pi, 1)$.

b) For your choice of f, evaluate the above integral for any choice of path C as above.

8. Let q be a power of a prime number, and let $\mathbb{F}_q$ be the field of q elements. Let V be an n-dimensional vector space over $\mathbb{F}_q$. For each positive integer k, let S_k be the set of ordered k-tuples $(v_1, \ldots, v_k)$ of linearly independent vectors in V.

   a) Show that the number of elements in S_k is $(q^n - 1)(q^n - q) \cdots (q^n - q^{k-1})$. What does this say if $k > n$?

   b) Using part (a), determine the number of invertible $n \times n$ matrices over $\mathbb{F}_q$.

9. Let $I \subset \mathbb{R}$ be an open interval, and let $f : I \to \mathbb{R}$ be a twice differentiable function. Suppose that $a, b, c \in I$ are distinct, and that the three points

$$(a, f(a)), (b, f(b)), (c, f(c)) \in \mathbb{R}^2$$

lie on a line. Prove that $f''(x) = 0$ for some $x \in I$.

10. a) Consider the ideal $I = (2x^2 + 2x + 1)$ in $\mathbb{Z}[x]$. Determine whether I is a prime ideal, and whether it is maximal.

    b) Is $R = \mathbb{Z}[x]/I$ an integral domain? If your answer is no, find a zero-divisor in R. If your answer is yes, find a complex number α such that the fraction field of R is isomorphic to $\mathbb{Q}[\alpha]$.

11. a) Show that in some open neighborhood of the origin in the (x, y)-plane $\mathbb{R}^2$, there is a differentiable function $z = f(x, y)$ satisfying

$$z^5 - z = x^2 + y^2.$$

b) On a sufficiently small neighborhood of the origin, how many such implicit functions f are there?

c) For each such implicit function f, determine whether the origin is a critical point.

12. Let M be the 4×4 real matrix each of whose entries is equal to 1.

   a) Find the kernel, image, rank, nullity (dimension of the kernel), trace, and determinant of M.

   b) Find the characteristic polynomial of M, the eigenvalues of M, and the dimensions of the corresponding eigenspaces.

   c) Determine whether M is diagonalizable.

Extra page for work