Preliminary Examination, Part I

Monday, August 26, 2019 9:30-12:00

This part of the examination consists of six problems. You should work all of the problems. Show all of your work. Try to keep computations well-organized and proofs clear and complete — and justify your assertions.

If a problem has multiple parts, earlier parts may be useful for later parts. Moreover, if you skip some part, you may still use the result in a later part.

Be sure to write your name both on the exam and on any extra sheets you may submit.

All problems have equal weight of 10 points.

1. a) Show that there is no real polynomial $p(x)$ so that $\cos x = p(x)$ for all real x.
   b) Show that $\cos x$ is not a rational function, that is, there are no polynomials $p(x)$ and $q(x)$ so that $\cos x = \frac{p(x)}{q(x)}$ for all real x.

2. Classify finite groups of order 45 (up to isomorphism).

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with the property that $\lim_{t\to\infty} f(t) = 0$. Show that
$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(t)\, dt = 0.$$

4. Let $\mathcal{P}_n$ be the linear space of polynomials $p(x) \in \mathbb{R}[x]$ of degree at most n and let $L : \mathcal{P}_n \to \mathcal{P}_n$ be the linear map defined by $Lu := u'' + bu' + cu$, where b and c are constants. Assume $c \neq 0$.
   a) Find all $p \in \mathcal{P}_n$ that satisfy $Lp = 0$.
   b) Show that for every polynomial $q(x) \in \mathcal{P}_n$ there is one (and only one) solution $p(x) \in \mathcal{P}_n$ of $Lp = q$. In other words, for $c \neq 0$, the map $L : \mathcal{P}_n \to \mathcal{P}_n$ is invertible. [Note: You are not being asked to find a formula for p.]

5. a) Let f be a continuous function on the interval $\{x \,|\, 1 \leq x \leq 3\}$. Compute
$$\lim_{n\to\infty} \int_1^3 f(x) e^{-nx}\, dx.$$
   [Justify your assertions.]
   b) Give an example of a sequence of continuous real-valued functions $f_n(x) \geq 0$ with the property $f_n(x) \to 0$ for all $x \in [0, 1]$ but
$$\int_0^1 f_n(x)\, dx \geq 1 \quad \text{for } \textit{all } n = 1, 2, \ldots$$
   If you prefer, a clear sketch of a graph will be adequate.

6. a) Let M be a complete metric space. Suppose $K \subset M$ is a compact subset and P is a point in M with $P \notin K$. Show there is a point $Q \in K$ that is closest to P, that is,

$$d(P, Q) = \inf_{x \in K} d(P, x).$$

b) Consider the metric space ℓ_2 of real sequences $\{x = (x_1, x_2, \ldots) \,|\, x_j \in \mathbb{R}\}$ with norm $|x|^2 = \sum_j x_j^2 < \infty$, inner product $\langle x,\, y\rangle = x_1y_1 + x_2y_2 + \cdots$, and with metric given by $d(x, y) := |x - y|$.

Let $Q \subset \ell_2$ be the (standard) set of unit orthonormal vectors $\{e_j,\, j = 1, 2, 3, \ldots\}$, where $e_1 = (1, 0, 0, 0, \ldots)$, $e_2 = (0, 1, 0, 0, \ldots)$, $e_3 = (0, 0, 1, 0, \ldots)$, $\ldots$, $e_k = (0, \ldots, 0, 1, 0, \ldots)$ with 1 in the k^{th} slot.

Is the set Q closed in ℓ_2?, Is it bounded? Is it compact? Justify your assertions.

Preliminary Examination, Part II

Monday, August 26, 2019 1:30-4:00

This part of the examination consists of six problems. You should work all of the problems. Show all of your work. Try to keep computations well-organized and proofs clear and complete — and justify your assertions.

If a problem has multiple parts, earlier parts may be useful for later parts. Moreover, if you skip some part, you may still use the result in a later part.

Please write your name on both the exam and any extra sheets you may submit.

All problems have equal weight of 10 points.

7. Let $\Omega \subset \mathbb{R}^3$ be a connected bounded open set with smooth boundary $\partial\Omega$. Suppose $\mathbf{F}(x)$ is an infinitely differentiable vector field defined for $x \in \mathbb{R}^3$, and $u(x)$ is an infinitely differentiable real-valued function defined for $x \in \mathbb{R}^3$.

   NOTATION: ∇u is the gradient of u and $\nabla \cdot \mathbf{F}$ is the divergence of $\mathbf{F}$.

   a) Verify the formula for the derivative of the product
   $$\nabla \cdot \big(u(x)\mathbf{F}(x)\big) = \nabla u \cdot \mathbf{F} + u\, \nabla \cdot \mathbf{F}. \tag{1}$$

   b) Use Part a) to obtain the generalization of *integration by parts*:
   $$\iiint_\Omega u\, \nabla \cdot \mathbf{F}\, dV = \iint_{\partial\Omega} u\, \mathbf{F} \cdot \mathbf{n}\, dA - \iiint_\Omega \nabla u \cdot \mathbf{F}\, dV, \tag{2}$$
   where dV is the element of volume on Ω, dA the element of area on $\partial\Omega$, and $\mathbf{n}$ a unit outer normal vector field on $\partial\Omega$. [Hint: Use the divergence theorem].

   c) In the special case of $\mathbf{F} = \nabla u$, the equation (2) is the identity
   $$\iiint_\Omega u\, \nabla \cdot \nabla u\, dV = \iint_{\partial\Omega} u\, \nabla u \cdot \mathbf{n}\, dA - \iiint_\Omega |\nabla u|^2\, dV. \tag{3}$$
   Use this to show that if $\nabla \cdot \nabla u = 0$ in Ω and $u = 0$ on $\partial\Omega$, then $u = 0$ in all of Ω. [Remark: $\nabla \cdot \nabla u = u_{x_1 x_1} + u_{x_2 x_2} + u_{x_3 x_3}$, the Laplacian, is often written as Δu.]

8. Let $\vec{x}, \vec{y} \in \mathbb{R}^n$ with the usual inner product which we write as $\langle x, y\rangle$ (the notation $\vec{x} \cdot \vec{y}$ is also often used). Also, we write the norm as $|\vec{x}| = \sqrt{\langle \vec{x}, \vec{x}\rangle}$.

   Let A be a real symmetric $n \times n$ matrix with eigenvalues $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$. Show that
   $$\langle \vec{x}, A\vec{x}\rangle \ge \lambda_1 |\vec{x}|^2 \qquad \text{for all} \quad \vec{x}.$$

9. Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function with the properties $f(0) = 3$, $f(1) = 1$, and $f(3) = 5$. Find an explicit positive real number A such that there exists a real number c with $0 < c < 3$ such that $f''(c) \geq A$.

10. Let R denote the ring $\dfrac{\mathbb{Z}[x]}{(2x^2 + 2x + 1)}$. Prove that R is an integral domain.

11. Find an integer N so that $\quad 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{N} > 100.$

12. Let A be an $n \times n$ real or complex matrix.

    a) Show that $\ker A^j \subset \ker A^{j+1}$. If $\ker A^k = \ker A^{k+1}$ for some k, show that $\ker A^j = \ker A^k$ for all $j \geq k$.

    b) Say A is a nilpotent 5×5 matrix. Is it true that $A^5 = 0$? Proof or counterexample.