Preliminary exam–Spring 2017, Part I

1. List all the finite fields (up to isomorphism) of order less than or equal to 10. Show that the ones you list exist, and no others.

2. (a) In the polynomial ring $\mathbf{Z}[x]$, is the ideal generated by x^4-1 and $2x^3-2x$ principal?

(b) Same question in the polynomial ring $\mathbf{Q}[x]$.

(c) Same question in the polynomial rings $\mathbf{Z}[x,y]$ and $\mathbf{Q}[x,y]$.

3. Use the ε-δ definition to prove that the first derivative of $f(x) = x^3$ is $f'(x) = 3x^2$.

4. Prove that for all $k \in \mathbb{N}$ there exists $\varepsilon_k > 0$ such that all $n \times n$ matrices A with $\|A - \mathrm{Id}\| < \varepsilon_k$ have a k^{th} *root*, that is, an $n \times n$ matrix $\sqrt[k]{A}$ such that $(\sqrt[k]{A})^k = A$.

5. Let N be a positive integer. Prove that

$$\frac{1}{2} + \log N < \sum_{k=1}^{N} \frac{1}{k} \leq 1 + \log N.$$

6. For n a positive integer, let $\phi(n)$ denote the number of integers k, $1 \leq k < n$, which are relatively prime to n. Prove that

$$\phi(n) = n \prod_{p|n} (1 - \frac{1}{p}),$$

where the product is over all distinct primes p dividing n.

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7. Let

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 2 \\ 2 & -3 & 1 \\ 0 & 1 & 3 \end{bmatrix}.$$

Find an orthonormal basis of the column space of A.

8. Let A be a $n \times n$ matrix. Let $\{S_1, \ldots S_k\}$ be a collection of eigenvectors of A with $\lambda_1, \ldots \lambda_k$ as the corresponding eigenvalues. Prove that if $\lambda_i \neq \lambda_j$ for all $1 \leq i < j \leq k$, then $\{S_1, \ldots, S_k\}$ is linearly independent.

9. Let (X, d) be a compact metric space. Suppose that $f : X \to X$ satisfies $d(f(x), f(y)) < d(x, y)$ for $x \neq y$. Show that for any $x \in X$, the sequence defined by $x_0 = x$ and $x_{n+1} = f(x_n)$ converges to a unique fixed point of f.

10. A topological property is one that is invariant under homeomorphism, i.e. if two spaces are homeomorphic and one has the property, so does the other. Explain with a proof or counterexample which of the following properties of a metric space are or are not topological invariants: a. Compactness, b. Connectedness, c. Boundedness d. Completeness.

11. Suppose $f : [-1, 1] \to \mathbb{R}$ is continuous on the closed interval $[-1, 1]$, and twice differentiable on the open interval $(-1, 1)$. Suppose also that $f(-1) = 7, f(0) = 1$ and $f(1) = 1$. Prove that there exists $c \in (-1, 1)$ such that $f^{(2)}(c) = 6$.

12. Compute the following limit if it exists and justify your conclusion:

$$\lim_{n\to\infty} \int_0^1 (n+1)x^n(1-x^5)^{\frac{1}{5}}\,dx.$$

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