Spring 2015 Preliminary Exam Solutions

1. (a) By the Fundamental Theorem of Finitely Generated Abelian Groups, we must have that this group is the product of cyclic groups. Moreover, by noting that $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z} \simeq \mathbb{Z}/nm\mathbb{Z}$ if and only if $\gcd(n, m) = 1$, we may enumerate all possibilities:

   - $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/32\mathbb{Z}$
   - $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/16\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
   - $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$
   - $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^2$
   - $\mathbb{Z}/3\mathbb{Z} \times (\mathbb{Z}/4\mathbb{Z})^2 \times \mathbb{Z}/2\mathbb{Z}$
   - $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^3$
   - $\mathbb{Z}/3\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^5$

   There are thus $\boxed{\text{7 such groups.}}$

   (b) $\boxed{\text{Yes}}$, the group $S_3 \times \mathbb{Z}/16\mathbb{Z}$ is non-abelian since S_3 is non-abelian. To see that S_3 is non-abelian, note that

$$(1\ 2)(2\ 3) = (1\ 2\ 3) \neq (1\ 3\ 2) = (2\ 3)(1\ 2)\,.$$

2. This is a first order differential equation, so we must find an integrating factor. We set

$$\psi(x) = \exp\left(\int \frac{1}{x}\,dx\right) = x.$$

Multiplying both sides by $\psi(x)$ we see that the first equation is the same as

$$\frac{d}{dx}(xg) = x\sin x.$$

Taking the indefinite integral of both sides gives

$$xg(x) = \sin x - x\cos(x) + C$$

implying

$$g(x) = \frac{\sin x}{x} - \cos(x) + \frac{C}{x}\,.$$

Plugging in initial condition of $g(\pi) = 0$ gives $C = -\pi$, implying

$$\boxed{g(x) = \frac{\sin x}{x} - \cos(x) - \frac{\pi}{x}\,.}$$

3. Let $n := \dim(V)$. Since $\dim(\mathbb{R}) = 1$ and $\dim \operatorname{Im}(f) \neq 0$, this implies that $\operatorname{rank}(f) = \dim \operatorname{Im}(f) = 1$. By rank-nullity, this implies that the kernel B is of dimension $n-1$. Therefore, to see that $B \cup \{v\}$ is a basis of V, it is sufficient to show that it is linearly independent. Let $b_1, \ldots, b_{n-1}$ enumerate B and consider a linear combination

$$c_1b_1 + \cdots + c_{n-1}b_{n-1} + c_nv = 0\,.$$

Then applying f gives $c_n = 0$, and linear indepenence of B gives all other $c_i = 0$.

4. $\boxed{\text{Yes, } F \text{ must be a constant function}}$. Fix $x \in \mathbb{R}$. Then note that

$$|F'(x)| = \lim_{h \to 0} \frac{|F(x+h) - F(x)|}{|h|} \leq \lim_{h \to 0} 6|h^3| = 0\,.$$

This means that $F' \equiv 0$. By the mean value theorem, this implies that F is a constant function.

5. (a) The matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has eigenvalues $\pm i$, and therefore is not diagonalizable over $\mathbb{Q}$. However, since it has distinct eigenvalues, it is diagonalizable over $\mathbb{C}$.

(b) The matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has only the eigenvalue 0, and the eigenspace corresponding to 0 is of dimension 1, implying there does not exist a basis of eigenvectors over $\mathbb{C}$ and thus is not diagonalizable.

6. (a) This statement is $\boxed{\text{true}}$. Recall that for a set S we have

$$\overline{S} = \bigcap_{C \supset S : C \text{ is closed}} C.$$

Since $\overline{A} \cup \overline{B}$ is closed and contains $A \cup B$, we have that $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$. Moreover, since $A \subset A \cup B$, we have $\overline{A} \subset \overline{A \cup B}$ and similarly have $\overline{B} \subset \overline{A \cup B}$. This gives $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$.

(b) This statement is $\boxed{\text{not true}}$. Take $A = \mathbb{Q}$ and $B = \mathbb{R} \setminus \mathbb{Q}$. Then $A \cap B = \emptyset$, but $\overline{A} = \overline{B} = \mathbb{R}$ implying

$$\overline{A} \cap \overline{B} = \mathbb{R} \neq \emptyset = \overline{A \cap B}\,.$$

7. (a) By Green's Theorem, we have

$$\int_C \frac{1}{2}(x\,dy - y\,dx) = \int_D \frac{1}{2}(1+1)\,dA = \text{Area}(D)\,.$$

(b) Parameterize the ellipse by

$$r(t) = a\cos(t)\mathbf{i} + b\sin(t)\mathbf{j}$$

for $t \in [0, 2\pi)$. Then by the previous part, we have

$$\begin{aligned}
\text{Area} &= \int_C \frac{1}{2}(x\,dy - y\,dx) \\
&= \frac{1}{2}\int_0^2 \pi\left(a\cos(t)b\cos(t) - b\sin(t)(-a\sin(t))\right)\,dt \\
&= \frac{1}{2}\int_0^{2\pi} (ab\cos(t)^2 + ab\sin(t)^2)\,dt \\
&= \frac{1}{2}\cdot 2\pi \cdot ab \\
&= \boxed{ab\pi}\,.
\end{aligned}$$

8. We use the method of Lagrange multipliers, and define the function

$$\Lambda(x, y, \lambda) = x^2 + y^2 - \lambda(xy - 2)$$

and solve the equation $\nabla\Lambda = 0$. This gives three equations

$$\begin{aligned} 2x &= \lambda y \\ 2y &= \lambda x \\ xy &= 2. \end{aligned}$$

Dividing the first two equations gives $x^2 = y^2$, i.e. $|x| = |y|$. Plugging this into the final equation gives $x = y = \pm\sqrt{2}$. Plugging this to the function $x^2 + y^2$ gives the value of 4. This must be a minimum, as the function $x^2 + y^2$ is unbounded on the curve $xy = 2$. Therefore, the minimum of f on $xy = 2$ is $\boxed{4}$.

9. (a) Since $\mathbb{Q}$ is a field, $\mathbb{Q}[x]$ is a Euclidean domain, and therefore $\boxed{\text{is a PID}}$.

   (b) $\boxed{\text{This is not a PID}}$; we claim that the ideal (x, y) is not principal. Seeking a contradiction, suppose that $(x, y) = (f)$. Then this implies that $f|x$ and $f|y$. But since x and y are coprime and irreducible, this would imply that f is a unit. However, this would imply that $(x, y) = (f) = \mathbb{Q}[x, y]$ which is not true, since $1 \notin (x, y)$.

   (c) $\boxed{\text{This is not a PID}}$ since it is not an integral domain, as can be seen by

$$(1, 0) \cdot (0, 1) = (0, 0)\,.$$

10. Fix $\varepsilon > 0$. Set $\delta := \varepsilon$. Then for all x so that $|x - 1| < \delta = \varepsilon$, we have

$$|\sqrt{x} - 1| = \frac{|x - 1|}{|\sqrt{x} + 1|} \leq |x - 1| < \varepsilon$$

where the first inequality is by factoring the difference of squares and the second is by noting that $|\sqrt{x} + 1| > 1$.

11. (a) Note that

$$2(2, 1, 1) - (1, 2, 3) = (3, 0, -1)$$

implying that these three vectors span a two-dimensional subspace. We then note that the vectors $(1, 2, 3)$ and $(3, 0, -1)$ are linearly independent and orthogonal, implying that setting

$$\begin{aligned} v_1 &:= (1, 2, 3) \\ v_2 &:= (3, 0, -1) \end{aligned}$$

gives an orthogonal basis $\boxed{B = \{(1, 2, 3), (3, 0, -1)\}}$.

(b) We need to add another basis element, so we apply Gram-Schmidt to the above two vectors and the vector $(0,1,0)$ to get

$$\begin{aligned} v_3 &:= (0,1,0) - \frac{(1,2,3)\cdot(0,1,0)}{(1,2,3)\cdot(1,2,3)}(1,2,3) - \frac{(3,0,-1)\cdot(0,1,0)}{(3,0,-1)\cdot(3,0,-1)}(3,0,-1) \\ &= \left(-\frac{1}{7}, \frac{5}{7}, -\frac{3}{7}\right). \end{aligned}$$

The collection $\boxed{\{(1,2,3),(3,0,-1),(-1,5,-3)\}}$ is an orthogonal basis for $\mathbb{R}^3$ that contains B.

12. $\boxed{\text{We must have that } A \geq 0.}$ Seeking a contradiction, suppose that $A < 0$. Then there exists and N so that for all $n \geq N$,

$$|a_n - A| < |A|/2\,.$$

By triangle inequality, this would then imply that $a_n < A/2 < 0$ which is a contradiction.