Preliminary Examination, Part I

Thursday, April 30, 2015

This part of the examination consists of six problems. You should work all of the problems. Show all of your work in your workbook. Try to keep computations well-organized and proofs clear and complete. Be sure to write your name on each workbook you submit. All problems have equal weight.

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1. (a) How many abelian groups of order 96 are there, up to isomorphism?

   (b) Do there exist any non-abelian groups of order 96?

   Explain.

2. Solve the following initial value problem for the function $g(x)$:

$$\begin{cases} \frac{d}{dx}g + \frac{1}{x}g &= \sin x \qquad \text{for } x > \pi \\ g(\pi) &= 0. \end{cases}$$

3. Let $f : V \to \mathbb{R}$ be a non-zero linear functional on a finite dimensional real vector space V (i.e. f is a linear map to $\mathbb{R}$ that is not identically zero). Let B be a basis of the kernel (i.e. nullspace) of f, and let $v \in V$ be a vector such that $f(v) \neq 0$. Prove that $B \cup \{v\}$ is a basis of V.

4. Let $F : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that

$$|F(x) - F(y)| \leq 6(x-y)^4 \qquad \text{for all } x, y \in \mathbb{R}.$$

   Must F be a constant function? Give a proof or a counterexample.

   (Hint: Using the definition of the derivative, how big can $|F'(x)|$ be?)

5. (a) Find a 2×2 matrix with entries in $\mathbb{Q}$ that is diagonalizable over $\mathbb{C}$ but not over the field $\mathbb{Q}$.

   (b) Find a 2×2 matrix with entries in $\mathbb{Q}$ that is not diagonalizable over $\mathbb{C}$, or explain why none exists.

6. Let M be a metric space, and write $\bar{A}$ for the closure of a subset $A \subseteq M$.

   For each of the following assertions, determine whether the statement is true, and give a proof or a counterexample.

   a) $\bar{A} \cup \bar{B} = \overline{A \cup B}$, for all subsets A, B of M.

   b) $\bar{A} \cap \bar{B} = \overline{A \cap B}$, for all subsets A, B of M.

Preliminary Examination, Part II

Thursday, April 30, 2015

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7. (a) Suppose C is a smooth closed simple curve which surrounds a (simply connected) region D in $\mathbb{R}^2$, oriented counterclockwise. Show that the line integral
$$\int_C \frac{1}{2}(xdy - ydx)$$
computes the area of D.

   (b) Use the integral above to compute the area of the region enclosed by the ellipse $x^2/a^2 + y^2/b^2 \leq 1$, where $a, b > 0$.

8. Minimize the function $f(x,y) = x^2 + y^2$ on $\mathbb{R}^2$ subject to the constraint that $xy = 2$.

9. Which of the following rings are principal ideal domains? Justify your answers.

   a. The ring $\mathbb{Q}[x]$.

   b. The ring $\mathbb{Q}[x,y]$.

   c. The ring $\mathbb{Z} \times \mathbb{Z}$.

10. *Using* the ε-δ definition of limit, prove directly that the function $f(x) = \sqrt{x}$ is continuous at $x = 1$.

11. (a) Find an orthogonal basis B for the subspace of $\mathbb{R}^3$ spanned by the vectors $(1,2,3)$, $(3,0,-1)$, $(2,1,1)$. (Here we use the standard dot product on $\mathbb{R}^3$.)

    (b) Determine whether B is a basis of $\mathbb{R}^3$. If it is not, find an orthogonal basis of $\mathbb{R}^3$ that contains B.

12. Say $a_n > 0$ are a sequence of positive real numbers and $a_n \to A$ as $n \to \infty$. Either prove that $A \geq 0$ or provide a counterexample.