Preliminary Examination, Part I

Thursday morning, April 30, 2009

This part of the examination consists of six problems. You should work all of the problems. Show all of your work in your workbook. Try to keep computations well-organized and proofs clear and complete. Justify the assertions you make. Be sure to write your name on each workbook you submit. All problems have equal weight.

1. Suppose that $f_1, f_2, \ldots$ is a sequence of real-valued continuous functions on $[0,1]$. Let f be a function on $[0,1]$.

(a) If the sequence $\{f_n\}$ converges uniformly to f, must f be continuous? Prove or give a counterexample.

(b) If $f_n(x) \to f(x)$ for every $x \in [0,1]$, must f be continuous? Prove or give a counterexample.

2. Find an orthonormal basis of the space of solutions to the system of equations

$$\begin{array}{ccccccccc} x & - & y & + & 2z & - & t & = 0 \\ x & + & y & & & - & 3t & = 0 \\ & & y & - & z & - & t & = 0 \\ x & & & + & z & - & 2t & = 0 \end{array}$$

3. For each of the following, either give an example or prove that no such example exists.

(a) A function $f : \mathbb{R} \to \mathbb{R}$ that is differentiable everywhere and is bounded above and below but has no critical points.

(b) A function $f : \mathbb{R} \to \mathbb{R}$ that is differentiable everywhere but such that f' is discontinuous at some point.

4. Let $G = GL_2(\mathbb{R})$ be the group of 2×2 real invertible matrices, under matrix multiplication. Which of the following are subgroups of G? Which are normal subgroups of G?

(a) $\{A \in GL_2(\mathbb{R}) | A \text{ is symmetric}\}$.

(b) $\{A \in GL_2(\mathbb{R}) | A^t = A^{-1}\}$. ($A^t$ denotes the transpose of A.)

(c) $\{A \in GL_2(\mathbb{R}) | \det A > 0\}$.

(d) $\{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R}) | c = 0\}$.

5. For each of the following, either give an example or explain why no such example exists.

(a) A sequence of real numbers $a_1, a_2, \dots$ and a bijection $\varphi : \mathbb{N} \to \mathbb{N}$ such that $\sum_{i=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} a_{\varphi(n)}$ does not converge. (Here $\mathbb{N}$ denotes the set of positive integers.)

(b) A sequence of real numbers $a_1, a_2, \cdots$ and a bijection $\varphi : \mathbb{N} \to \mathbb{N}$ such that $\sum_{i=1}^{\infty} |a_n|$ converges but $\sum_{n=1}^{\infty} a_{\varphi(x)}$ does not converge.

6. (a) Let C be the line segment from $(0,0)$ to $(1,1)$ in the plane. Evaluate the line integral

$$\int_C (3x^2y^2 + 1)dx + 2x^3y\ dy.$$

(b) If C' is some other smooth curve connecting $(0,0)$ to $(1,1)$, what can you say about the value of

$$\int_{C'} (3x^2y^2 + 1)dx + 2x^3y\ dy \ \ ?$$

Preliminary Examination, Part II

Thursday afternoon, April 30, 2009

This part of the examination consists of six problems. You should work all of the problems. Show all of your work in your workbook. Try to keep computations well-organized and proofs clear and complete. Justify the assertions you make. Be sure to write your name on each workbook you submit. All problems have equal weight.

7. Let K be a compact subset of $\mathbb{R}^n$, and suppose that $x \in \mathbb{R}^n$ does not lie in K. Let $d = \inf\{d(x,y) \mid y \in K\}$, where $d(x,y)$ is the distance from x to y. Prove that there is a point $z \in K$ such that $d(x,z) = d$.

8. (a) Show that if $a \in \mathbb{R}$ then the matrix

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ a-1 & -2 & a \end{pmatrix}$$

is similar to a real diagonal matrix D.
Find D explicitly in terms of a.

(b) For which $a \in \mathbb{R}$ is there a real matrix B such that $B^2 = A$?
For which $a \in \mathbb{R}$ is there a real matrix C such that $C^3 = A$?
(Here A is as in part (a).)

9. (a) Suppose that $f(x)$ is a continuous function from $[0,\infty)$ to $\mathbb{R}$. Show that for all $c \in [0,\infty)$,

$$\lim_{n\to\infty} \int_0^c f(x) \left(\frac{x^2}{1+x^2}\right)^n dx = 0.$$

(b) Find a continuous function $g(x)$ on $[0,\infty)$ such that

$$\int_0^\infty g(x) \left(\frac{x^2}{1+x^2}\right)^n dx = \infty$$

for all n.

10. Let $(A,+)$ be an abelian group and let $\mathrm{End}(A)$ be the set of endomorphisms of A (i.e. group homomorphisms $A \to A$). For $f,g \in \mathrm{End}(A)$ define $(f+g)(a) = f(a) + g(a)$ and $(f \cdot g)(a) = f(g(a))$.

(a) Show that $(\mathrm{End}(A),+,\cdot)$ is a ring. What are the additive and multiplicative identities?

(b) Show that for every ring R there is an abelian group A such that R is isomorphic to a subring of $(\mathrm{End}(A),+,\cdot)$.

11. Find the volume of the set of points (x, y, z) in $\mathbb{R}^3$ for which the distances from (x, y, z) to $(0, 0, 0)$ and from (x, y, z) to $(0, 0, 1)$ are both less than or equal to 1.

12. Let n be a positive integer.

(a) If A is an $n \times n$ real matrix and $A^m = 0$ for some positive integer m, what can you say about the eigenvalues of A? about the characteristic polynomial of A?

(b) For which positive integers m does there exist an $n \times n$ real matrix A such that $A^m = 0$ but $A^{m-1} \neq 0$?