Preliminary Examination, Part I

Wednesday morning, April 28, 2004

This part of the examination consists of six problems. You should work all of the problems. Show all of your work in your workbook. Try to keep computations well-organized and proofs clear and complete. Justify the assertions you make. Be sure to write your name on each workbook you submit. All problems have equal weight.

1. a) Find all solutions in integers to the equation $129x+291y = 1$.

   b) Do the same for the equation $129x + 291y = 3$.

   Justify your assertions.

2. Show $f(x) = x^2$ is not uniformly continuous as a function on the whole real line (i.e. show for some $\epsilon > 0$ there is no $\delta > 0$ so that $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta$).

3. For each of the following, either give an example or explain why none exists.

   a) A non-abelian group of order 20.

   b) Two non-isomorphic abelian groups of order 30.

   c) A finite field whose non-zero elements form a cyclic group of order 17 under multiplication.

   d) A non-trivial automorphism of a finite field.

4. Let f be a real-valued continuous function defined for all $0 \leq x \leq 1$, such that $f(0) = 1$, $f(1/2) = 2$ and $f(1) = 3$. Show that

$$\lim_{n\to\infty} \int_0^1 f(x^n)dx$$

   exists and compute this limit. Justify your assertions.

5. Given $\vec{v} = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ let A be the matrix

$$A = \begin{bmatrix} x_1x_1 & x_1x_2 & \cdots & x_1x_n \\ x_2x_1 & x_2x_2 & \cdots & x_2x_n \\ \cdots & \cdots & \cdots & \cdots \\ x_nx_1 & x_nx_2 & \cdots & x_nx_n \end{bmatrix}.$$

   (So $a_{ij} = x_ix_j$.)

   a) Compute the kernel of A, the range of A, the rank of A, and the eigenvalues of A.

   b) Repeat the previous part for $B = I + A$, where I is the identity matrix.

6. Give an explicit example of a vector field defined on the complement of the z-axis in 3-space whose curl is zero, but which is not the gradient of any function. Explain why this is such an example. (Hint: Give an example of a vector field $\vec{V}$ defined in the plane minus the origin so that

$$\vec{V} = P(x,y)\vec{\imath} + Q(x,y)\vec{\jmath} \quad \text{and} \quad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}\ ,$$

but $\vec{V}$ is not the gradient of a function u.)

Preliminary Examination, Part II

Wednesday afternoon, April 28, 2004

This part of the examination consists of six problems. You should work all of the problems. Show all of your work in your workbook. Try to keep computations well-organized and proofs clear and complete. Justify the assertions you make. Be sure to write your name on each workbook you submit. All problems have equal weight.

7. a) Let $S \subset \mathbb{R}$, and $a, b \in S$ with $a < b$. Suppose the set S is connected. Show that S contains the closed interval $a \leq x \leq b$ (i.e. $[a, b] \subset S$).

   b) Deduce that any non-empty connected subset of $\mathbb{Q}$ consists of just one element.

8. Let $\{a_n\}$ be a bounded sequence of real numbers. Consider the infinite series

$$f(x) = \sum_{n=1}^{\infty} \frac{a_n}{x^n}$$

where x is a real number. Prove that for any $c > 1$ this series converges uniformly on $\{x \in \mathbb{R} \mid x \geq c\}$.

9. Let A be the ring of continuous functions $f\colon \mathbb{R} \to \mathbb{R}$, under (pointwise) addition and multiplication.

   a) Determine whether A is a integral domain.

   b) Let $I \subset A$ be the subset consisting of functions f such that $f(0) = 0$. Is I an ideal? Is it a maximal ideal? What is A/I?

10. Suppose f is twice continuously differentiable in the closed interval $[0, 1]$. Suppose f is strictly concave upwards, so for all $0 \leq x < y \leq 1$ and $0 < \lambda < 1$ we have $f(\lambda x + (1 - \lambda)y) < \lambda f(x) + (1 - \lambda)f(y)$.

$$\text{Prove } \; f''(x) \geq 0 \; \text{ for all } \; x \in (0, 1).$$
$$\text{Prove } \; f''(x) > 0 \; \text{ for some } \; x \in (0, 1).$$

11. Let

$$M = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

a) Find the minimal and characteristic polynomials of M.

b) Is M similar to a diagonal matrix D over $\mathbb{R}$? If so, find such a D.

c) Repeat part (b) with $\mathbb{R}$ replaced by $\mathbb{C}$ and also by the field $\mathbb{Z}/5\mathbb{Z}$.

12. Let V be the vector space of C^∞ real-valued functions on $\mathbb{R}$. Consider the following maps $T_i \colon V \to V$.

$$T_1(f) = f'' - 6f' + 9f$$
$$T_2(f) = f' - xf$$
$$T_3(f) = ff'$$

a) Which of the maps T_i are linear transformations?

b) For each one that is, find a basis for the kernel.