MODULI SPACES OF RIEMANNIAN METRICS WITH POSITIVE AND NONNEGATIVE RICCI AND SECTIONAL CURVATURE

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ABSTRACT

MODULI SPACES OF RIEMANNIAN METRICS WITH POSITIVE AND NONNEGATIVE RICCI AND SECTIONAL CURVATURE

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We use invariants related to η invariants of Dirac operators to distinguish path components of moduli spaces of Riemannian metrics with positive and nonnegative Ricci and sectional curvature. In 7 dimensions, we calculate the Kreck-Stolz s invariant for metrics on spin total spaces of S^n bundles with nonnegative sectional curvature. We then apply it to show that the moduli spaces of metrics with nonnegative sectional curvature on certain 7-manifolds have infinitely many path components. These include certain positively curved Eschenburg and Aloff-Wallach spaces.

We next use the η invariant of spin^c Dirac operators to distinguish connected components of moduli spaces of Riemannian metrics with positive Ricci curvature. We find infinitely many non-diffeomorphic five dimensional manifolds for which these moduli spaces each have infinitely many components. The manifolds are total spaces of principal S^1 bundles over $\#^a\mathbb{C}P^2\#^b\overline{\mathbb{C}P^2}$ and the metrics are lifted from Ricci positive metrics on the base.

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Chapter 1

Introduction

The description of manifolds with positive and nonnegative scalar, Ricci, and sectional curvature is an important topic in Riemannian geometry. We know of few manifolds admitting metrics of positive sectional curvature, with only spheres and projective spaces in dimensions larger than 24 and infinite families only in dimensions 7 and 13.

More closed manifolds are known to admit Riemannian metrics of positive Ricci curvature exist, for example, all compact, simply connected homogeneous spaces, biquotients, and cohomogeneity one manifolds, see [Ber], [GZ2], [STu]. Systematic methods for constructing such metrics on certain connected sums and bundles have been explored in [CG], [GPT], [Na], [SW], [SY1], [W2].

On manifolds M^n which support such metrics, the next question is to describe the space of all such metrics. The proper space to study is the moduli space, or the quotient of the space of positive or nonnegative curvature metrics by the pullback action of the diffeomorphism group. We denote by $\mathfrak{M}$ the space of all metrics modulo the diffeomorphism group, and use a subscript to specify the curvature condition. The number of connected components of $\mathfrak{M}_{\mathrm{scal}>0}$, for instance, serves as a coarse quantification of positive scalar curvature metrics.

In dimension 2, where all three notions of positive curvature coincide, one sees using the uniformization theorem that the moduli space is connected. In dimension 3, the Ricci flow has been used to show that $\mathfrak{M}_{\mathrm{sec}>0}$,$\mathfrak{M}_{\mathrm{Ric}>0}$, and $\mathfrak{M}_{\mathrm{scal}>0}$ are connected when they are nonempty, see [H1], [H2], and [M].

The situation is different in higher dimensions. Kreck and Stolz [KS3] defined the s invariant of the path components of the moduli space of positive scalar curvature metrics $\mathfrak{M}_{\mathrm{scal}>0}(M)$ on certain spin manifolds. If two metrics on M yield different values of s, they cannot be connected by a path maintaining positive scalar curvature. Kreck and Stolz used the invariant to show that for a (4n+3)-manifold with vanishing rational Pontryagin classes and a unique spin structure the space of such metrics is either empty or has infinitely many components. Botvinnik and Gilkey used the relative η invariant to prove the same property for spin manifolds of odd dimension ≥ 5 with certain nontrivial fundamental groups and space forms in all dimensions ≥ 5, see [BG1], [BG2].

The s invariant can also be used to distinguish path components of the moduli spaces of metrics with stronger curvature conditions. Wraith showed that for a homotopy sphere σ^{4k-1} bounding a parallelisable manifold, $\mathfrak{M}_{\mathrm{Ric}>0}(\sigma)$ has infinitely many components. The procedure known as plumbing with disc bundles over sphere produces infinitely many parallelisable manifolds with boundaries diffeomorphic to σ. Wraith constructed metrics of positive Ricci curvature on each boundary in[W1] and calculated the s invariant of each metric in in [W3].

In [KS3] Kreck and Stolz studied the s invariant for two families of 7-manifolds. The first are the total spaces $N^7_{k,l}$ of principal S^1 bundles over $\mathbb{C}P^2 \times \mathbb{C}P^1$. These spaces are also described as the homogeneous spaces $S^5 \times S^3/S^1_{k,l}$, see [WZ]. Using the diffeomorphism invariants of [KS1] they show that each $N^7_{k,l}$, with k even and $\gcd(k,l) = 1$, is diffeomorphic to infinitely many manifolds in the same family. Calculating the s invariants for the Einstein metrics described in [WZ] and the homogeneous metrics induced from the product of two round metrics on $S^5 \times S^3$, it follows that $\mathfrak{M}_{\mathrm{Ric}>0}(N^7_{k,l})$ and $\mathfrak{M}_{\sec\geq 0}(N^7_{k,l})$ have infinitely many path components.

The second family are the Aloff-Wallach spaces $W^7_{k,l} = SU(3)/S^1_{k,l}$. Using the diffeomorphism classification in [KS2] they show that some $W^7_{k,l}$ are diffeomorphic to finitely many other manifolds in the family. As these spaces have homogeneous metrics of positive curvature when $kl(k+l) \neq 0$, they exhibit some examples where $\mathfrak{M}_{\sec>0}(W^7_{k,l})$ has more than one component. We note though that for all known families of manifolds admitting positive sectional curvature metrics, only finite subfamilies have the same cohomology ring, see [CEZ], [EZ].

The same methods are used in [DKT] to calculate the s invariants of homogeneous metrics on the total spaces of S^1 bundles $N^{4n+3}_{k,l}$ over $\mathbb{C}P^{2n} \times \mathbb{C}P^1$. As observed in [WZ], for fixed n and l this family contains only finitely many diffeomorphism types. It follows that for any fixed n and l there exists some k_0 such that $\mathfrak{M}_{\sec\geq 0}(N^{4n+3}_{k_0,l})$ has infinitely many path components. We note however that for $n \geq 2$ and $|l| > 2$ this does not identify any specific manifold having that property.

Dessai [D] used the s invariant to find several infinite families of 7-dimensional sphere bundles M^7 such that $\mathfrak{M}_{\mathrm{Ric}>0}(M)$ and $\mathfrak{M}_{\sec\geq 0}(M)$ have infinitely many path components. Grove and Ziller [GZ1, GZ3] constructed metrics on nonnegative sectional curvature on the manifolds in those families, and the diffeomorphism classifications in [CE] and [EZ] show that each manifold is diffeomorphic to infinitely many other members of the family.

More recently, Dessai and González-Álvaro [DG] showed that if M^5 is one of the five closed manifolds homotopy equivalent to $\mathbb{R}P^5$'s then $\mathfrak{M}_{\sec\geq 0}(M)$ and $\mathfrak{M}_{\mathrm{Ric}>0}(M)$ have infinitely many path components. López de Medrano [L] showed that each such M^5 admits infinitely many descriptions as a quotient of a Brieskorn variety, and Grove and Ziller showed the each quotient admits a metric of nonnegative sectional curvature [GZ2]. Dessai and González-Álvaro calculated the relative η invariant for those metrics to distinguish the path components.

This thesis is organized as follows. Below we summarize the main results, in dimensions 7 and 5. In Chapter 2 we develop invariants to detect path components of moduli spaces. In Chapter 3 and Chapter 4 we describe metrics and connections on certain 7- and 5-Manifolds respectively used to calculate the moduli space invariants, and combine the calculations with diffeomorphism classifications to prove the main results.

1.1 Main Results

1.1.1 7-Manifolds

We identify new 7-manifolds with $\mathfrak{M}_{\sec\geq 0}$ and $\mathfrak{M}_{\mathrm{Ric}>0}$ having infinitely many path components. As in previous examples, the manifolds are total spaces of $n-$sphere bundles. In the case of S^1 bundles, however, we develop new techniques to use metrics for which the orbits are not geodesics.

The first set of examples are total spaces $M_{m,n}$ of S^3 bundles over S^4. Such bundles are classified by pairs of integers $(m, n) \in \pi_3(SO(4)) \cong \mathbb{Z} \oplus \mathbb{Z}$. The second set are total spaces $S_{a,b}$ of S^3 bundles over $\mathbb{C}P^2$ which are sphere bundles of non-spin vector bundles. They are classified by two integers a, b describing the first Pontryagin class and the Euler class. Grove and Ziller [GZ1, GZ3] showed that $M_{m,n}$ and $S_{a,b}$ admit metrics of nonnegative sectional curvature.

Theorem A. *Let $m, n, a, b \in \mathbb{Z}$ with $n \neq 0$ and $a \neq b$. Then for $M = M_{m,n}$ or $S_{a,b}$ the moduli spaces $\mathfrak{M}_{\sec\geq 0}(M)$ and $\mathfrak{M}_{\mathrm{Ric}>0}(M)$ have infinitely many path components.*

Note that the family $M_{m,\pm 1}$ includes S^7 and the exotic Milnor spheres. The case of S^3 bundles over S^4 was obtained independently by Dessai in [D] using the relative index invariant of Gromov and Lawson [GL]. Those are the only previous non-homogeneous examples of this type. By [EZ] Proposition 6.7 the manifold $S_{-1,a(a-1)}$ is diffeomorphic to the Aloff-Wallach space $W^7_{a,1-a}$ discussed above. But in general $M_{m,n}$ and $S_{a,b}$ do not have the homotopy type of a 7-dimensional homogeneous space, e.g. when $|H^4(M_{m,n},\mathbb{Z})| = |n| \notin \{1,2,10\}$ or $|H^4(S_{a,b},\mathbb{Z})| = |a-b| = 2$ mod 3 respectively.

To describe the final set of manifolds, we start with S^2 bundles $\bar{N}_t$ and N_t over $\mathbb{C}P^2$ which are sphere bundles of spin, respectively non-spin, vector bundles and are classified by an integer t describing the first Pontryagin class. The 7-manifolds $\bar{M}^t_{a,b}$ and $M^t_{a,b}$ are the total spaces of S^1 bundles over $\bar{N}_t$ and N_t respectively, classified by two additional integers a and b, with $\gcd(a,b) = 1$, describing the Euler class. Escher and Ziller [EZ] showed that $M^t_{a,b}$ and $\bar{M}^{2t}_{a,b}$ admit metrics of non-negative sectional curvature such that S^1 acts by isometries.

Theorem B. *(a)* *Let $a, b, t \in \mathbb{Z}$ with gcd$(a,b) = 1$ and $t(a+b)^2 \neq ab$. Then $\mathfrak{M}_{\sec\geq 0}(M^t_{a,b})$ and $\mathfrak{M}_{\mathrm{Ric}>0}(M^t_{a,b})$ have infinitely many path components.*

(b) *Let $a, b, t \in \mathbb{Z}$ with gcd$(a, 2b) = 1$. Then $\mathfrak{M}_{\sec\geq 0}(\bar{M}^{2t}_{a,2b})$ and $\mathfrak{M}_{\mathrm{Ric}>0}(\bar{M}^{2t}_{a,2b})$ have infinitely many path components.*

In [EZ] Corollary 6.4 it was shown that the manifold $M^{-1}_{a,b}$ is the Eschenburg biquotient $F_{a,b} = S^1_{a,b,a+b}\backslash SU(3)/S^1_{0,0,2a+2b}$. These are the only Eschenburg biquotients admitting free S^1 actions and when $ab > 0$ they admit metrics of positive sectional curvature, see [Es]. Furthermore $M^1_{a,b}$ is the Aloff-Walach space $W_{a,b}$, which has positive sectional curvature if $ab(a+b) \neq 0$. We have thus as an immediate corollary

Corollary. *Let $a, b \in \mathbb{Z}$ with $gcd(a,b) = 1$. For $M = W_{a,b}$ or $M = F_{a,b}$ the moduli spaces $\mathfrak{M}_{\sec\geq 0}(M)$ and $\mathfrak{M}_{\mathrm{Ric}>0}(M)$ have infinitely many path components, and if $ab(a+b) \neq 0$, receptively $ab > 0$, then at least one of these components contains a metric of positive sectional curvature.*

Along with S^7, these are the first examples of this type. We note that in [EZ] one finds further examples of positively curved Eschenburg spaces which are diffeomorphic to some of the manifolds $S_{a,b}$ or $M^t_{a,b}$ and so the same conclusion holds.

In [KS3] Example 3.10, Kreck and Stolz showed that $\mathfrak{M}_{\sec>0}(W_{-4638661,582656})$, along with finitely many other examples, has more than one component. Thus with Theorem B we have $\mathfrak{M}_{\sec\geq 0}(W_{-4638661,582656})$ has infinitely many components, at least two of which are known to contain metrics of positive sectional curvature. We remark that so far, the Kreck Stolz examples are the only manifolds for which $\mathfrak{M}_{\sec>0}$ is known to have more than one path component.

By Corollary 7.8 of [EZ] $\bar{M}^0_{a,2b}$ is diffeomorphic to the homogeneous space $N^7_{2b,a}$, and hence Theorem B part (b) generalizes the Kreck-Stolz examples. But again in general $\bar{M}^{2t}_{a,2b}$ and $M^t_{a,b}$ do not have the homotopy type of a 7-dimensional homogeneous space, e.g. when $|H^4(\bar{M}^{2t}_{a,2b}, \mathbb{Z})| = |a^2 - 8tb^2| = 2 \bmod 3$ or $|H^4(M^t_{a,b}, \mathbb{Z})| =$

$|t(a+b)^2 - ab| = 2 \bmod 3$.

The strategy of the proof is as follows. We calculate the s invariant with topological data on the associated disc bundle of the sphere bundle. In Theorem 3.1.1 we extend the metric with $\sec \geq 0$ on each sphere bundle to a metric of positive scalar curvature on the associated disc bundle which is a product near the boundary. If the disc bundle is a spin manifold Kreck and Stolz [KS3] obtained a formula for the s invariant in terms of the index of the Dirac operator, which vanishes since the scalar curvature is positive, and topological data on a bounding manifold, see Theorem 2.1.4. Theorem A follows easily: the manifolds $M_{m,n}$ and $S_{a,b}$ are classified up to diffeomorphism in [CE] and [EZ] respectively. In particular each sphere bundle is diffeomorphic to infinitely many others. Their computations easily yield the formula for the s invariant as well. Theorem A follows since s is a polynomial in the integers a, b, m and n, where m, n satisfy $ma + nb = 1$.

Theorem B is more involved. For part (b) we observe that $\bar{M}^t_{a,b}$ is a spin manifold if and only if b is even, and in this case the associated disc bundle is a spin manifold as well. The proof then proceeds as before although the proof that the metrics have positive scalar curvature is more involved. We use the Kreck-Stolz invariants $s_1, s_2, s_3 \in \mathbb{Q}/\mathbb{Z}$ of [KS2] to obtain infinitely many circle bundles diffeomorphic to each manifold. For part (a) the manifolds $M^t_{a,b}$ are always spin but the disc bundles are not. Here we use another formula from [KS3], see Theorem 2.1.6, which does not require knowledge of a spin bounding manifold, but requires that the bundle be a circle bundle and that the fibers be geodesics. The latter condition does not hold for the metrics with $\sec \geq 0$, so we first deform the metrics, preserving positive scalar curvature, until the fibers are geodesics and such that S^1 still acts by isometries. Then the strategy proceeds in the same way.

We note that $\bar{M}^{2t}_{a,2b+1}$ and the S^3 bundles over $\mathbb{C}P^2$ which are sphere bundles of spin vector bundles also admit metrics with $\sec \geq 0$, but they are not spin manifolds so the methods do not apply. The conditions $a \neq b$ and $n \neq 0$ for $S_{a,b}$ and $M_{n,m}$ as well as $t(a+b)^2 \neq ab$ for $M^t_{a,b}$ are required to ensure the manifolds have the correct cohomology ring for the diffeomorphism classifications.

1.1.2 5-Manifolds

We identify an infinite family 5-manifolds M such that $\mathfrak{M}_{\mathrm{Ric}>0}(M)$ and $\mathfrak{M}_{\mathrm{scal}>0}(M)$ have infinitely many path components.

Theorem C. *Let $B = \#^a\mathbb{C}P^2\#^b\overline{\mathbb{C}P^2}$ with $a-b = 4$ mod 8 and let $S^1 \to M \to B$ be a principal bundle with first Chern class 2d, where $d \in H^2(B,\mathbb{Z})$ is primitive and $w_2(TB) = d$ mod 2. Then $\mathfrak{M}_{Ric>0}(M)$ and $\mathfrak{M}_{scal>0}(M)$ have infinitely many path components.*

The only other five dimensional manifolds known to have this property are the five homotopy real projective spaces recently discovered by Dessai and González-Álvaro [DG].

We will see that for a fixed base B the total spaces M satisfying the hypotheses of Theorem C are diffeomorphic, i.e. independent of the choice of d. M can be constructed by taking $a+b$ copies of $\mathbb{R}P^5$, removing equivariant tubular neighborhoods of Hopf orbits and gluing equivariantly along the boundaries of the tubular neighborhoods. The total spaces M have fundamental group $\mathbb{Z}_2$ and by the classification of Barden [Ba], the universal cover $\tilde{M}$ is diffeomorphic to $\#^{a+b-1} S^3 \times S^2$. But we do not know an explicit description of the deck group action by $\mathbb{Z}_2$ on $\tilde{M}$.

Those results (BG) do not apply in the case of Theorem C as the manifolds are not spin.

In Section 2.2 we show that in dimensions $4k+1$, the η invariant of a certain spin^c Dirac operator constructed for a positive scalar curvature metric g depends only on the class of g in $\mathfrak{M}_{\mathrm{scal}>0}$ or $\mathfrak{M}_{\mathrm{Ric}>0}$. In Section 4.2 we use a diffeomorphism result of Hambleton and Su [HS] to show that for a fixed base B all the total spaces M satisfying the hypothesis of Theorem C are diffeomorphic. Each base B admits a metric of positive Ricci curvature by a result of Sha and Yang [SY2]. The metrics can be lifted to metrics of positive Ricci curvature on the total spaces M by [GPT]. To complete the proof we calculate η for each metric and show that it obtains infinitely many values.

The standard method for calculating the η invariant of a spin Dirac operator on a manifold M with positive scalar curvature is to extend the metric over a manifold W with $\partial W = M$ such that the extension has positive scalar curvature as well. When M is not spin but spin^c, both the metric and a unitary connection on the complex line bundle associated to the spin^c structure must be extended. The desired condition then involves the curvatures of both metric and connection. In their work, Dessai and González-Álvaro passed to the universal cover to find a suitable W over which the connection could be extended to a flat connection. They use equivariant eta invariants on the cover to compute the eta invariant on the quotient. In this thesis, we work directly on M and use a manifold with boundary W with scal> 0 over which the connection cannot be extended to flat connection, but the curvature of the extension can be explicitly controlled.

To calculate η we extend the metric and connection on M to a metric h and connection ∇ on the disc bundle $W = M \times_{S^1} D^2$ associated to the S^1 bundle. We then use the Atiyah-Patodi-Singer index theorem [APS1] to obtain a formula for η in terms of the index of the spin^c Dirac operator on W and topological data on W. The index will vanish as long as

$$\mathrm{scal}(h) > 2|F^{\nabla}|_h$$

where F^{∇} is the curvature form of the connection ∇. We accomplish the extension for a general class of S^1 invariant metrics of positive scalar curvature in Section 4.3. This is more general than we need but may be of independent interest. In fact we construct h and ∇ such that

$$\mathrm{scal}(h) > \ell|F^{\nabla}|_h$$

where ℓ is a positive integer such that the first chern class of the S^1 bundle is ℓ times the canonical class of a spin^c structure on the quotient.

If we alter the hypotheses in Theorem C such that $a - b \neq 4 \mod 8$, the diffeomorphism result of [HS] determines the diffeomorphism type only up to a two-fold ambiguity. Thus the total spaces M for a fixed B fall into two diffeomorphism classes. Our methods imply that for one of those manifolds, $\mathfrak{M}_{\mathrm{Ric}>0}$ has infinitely many components, but we cannot say which one. See Theorem 4.2.7.

Sha and Yang also constructed metrics of positive Ricci curvature on the four manifolds $\#^m S^2 \times S^2$. One might expect our methods to yield a similar result in this case. The 5-manifolds, however, would be spin, and the eta invariant of the spin Dirac operator in dimension 4k+1 vanishes, even when twisted with certain complex line bundle, see [BG1].

Chapter 2

Moduli Space Invariants

We use the η invariant and related invariants of the spin^c Dirac operator, which we define in this section, to distinguish components of geometric moduli spaces. A manifold M is spin^c if there exists a complex line bundle λ over M such that the frame bundle of $TM \oplus \lambda$, a principal $SO(n) \times U(1)$ bundle, lifts to a principal $\mathrm{Spin}^c(n) = \mathrm{Spin}(n) \times_{\mathbb{Z}_2} U(1)$ bundle. A manifold is Spin^c if and only if the second Stiefel-Whitney class $w_2(TM)$ is the image of an integral class $c \in H^2(M, \mathbb{Z})$ under the map $H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{Z}_2)$. In this case c, which we call the canonical class of the Spin^c structure, is the first Chern class of λ, which we call the canonical bundle.

Using complex representations of $\mathrm{Spin}^c(n)$ we form Spin^c spinor bundles and equip them with actions of the complex Clifford algebra bundle $\mathbb{C}l(TM)$. When the dimension of M is even there is a unique irreducible such bundle S with a natural grading $S = S^+ \oplus S^-$. Given a metric g on M and a unitary connection

∇ on λ we can construct a spinor connection on ∇^s on S, compatible with Clifford multiplication, and a spin^c Dirac operator $D^c_{g,\nabla}$ acting on sections of S. See [LM] Appendix D for details. The Bochner-Lichnerowicz identity for this operator is

$$(D^c_{g,\lambda})^2 = (\nabla^s)^*\nabla^s + \frac{1}{4}\mathrm{scal}(g) + \frac{i}{2}F^\nabla. \tag{2.0.1}$$

Where the complex two-form F^∇ is the curvature of ∇. This form acts on the spinor bundle S by way of the vector bundle isomorphism $\Lambda T^*M \to \Lambda TM \to \mathbb{C}l(TM)$ given by g. The operator $(\nabla^s)^*\nabla^s$ is nonnegative definite with respect to the L^2 inner product on a closed manifold or a compact manifold with boundary on which the Atiyah-Patodi-Singer boundary conditions have been applied. See [APS1] Theorem 3.9 for details. The remaining term $\frac{1}{4}\mathrm{scal}(g) + \frac{i}{2}F^\nabla$ is positive definite if

$$\mathrm{scal}(g) > 2|F^\nabla|_g, \tag{2.0.2}$$

where the norm $|\cdot|_g$ is the operator norm on $\mathbb{C}l(TM)$ acting on S. In particular, $\ker(D^c_{g,\nabla}) = 0$ if (2.0.2) is satisfied. For a later purpose we note that for $\omega \in \Omega^2(M,\mathbb{C})$ and an orthonormal basis $\{e_i\}$ of TM with respect to g, we have

$$|\omega|_g \leq \sum_{i<j} |\omega(e_i, e_j)|. \tag{2.0.3}$$

Suppose W is a Spin^c manifold with boundary $\partial W = M$, with λ and c defined on W as above. W induces a Spin^c structure on M with canonical class $c|_{\partial W}$ and canonical bundle $\lambda|_{\partial W}$. Choose a metric h on W and a connection ∇ on λ which are product-like near ∂W, i.e.

$$h = h|_{\partial W} + dr^2$$

and

$$\nabla = \mathrm{proj}_M^*(\nabla|_{\partial W})$$

on a collar neighborhood $U \cong M \times I$ where I is an interval with coordinate r. Applying the Atiyah-Patodi-Singer boundary conditions, the Atiyah-Patodi-Singer index theorem [APS1] states that

$$\mathrm{ind}(D^c_{h,\nabla}|_{S^+}) = \int_W e^{c_1(\nabla)/2}\hat{A}(p(h)) - \frac{\dim(\ker(D^c_{h|_{\partial W},\nabla|_{\partial W}})) + \eta(D^c_{h|_{\partial W},\nabla|_{\partial W}})}{2}. \tag{2.0.4}$$

Here $c_1(\nabla)$ and $p(h)$ are the Chern-Weil Chern and Pontryagin forms constructed from the curvature tensors of the connection and metric respectively. $\hat{A}$ is Hirzebruch's polynomial in the Pontryagin forms and $D^c_{h|_{\partial W},\nabla|_{\partial W}}$ is the spin^c Dirac operator on M constructed using the induced metric and connection.

η is an analytic invariant of the spectrum of an elliptic operator defined in [APS1]. Given an elliptic differential operator D with spectrum $\{\lambda_i\}$ we define a complex function

$$\eta(D, s) = \sum_{\lambda \neq 0} \mathrm{sign}(\lambda_i)|\lambda_i|^{-s}.$$

One shows that the function is analytic when the real part of s is large and Atiyah, Patodi and Singer showed that it can be analytically continued to a meromorphic function which is analytic at 0. Thus we define $\eta(D) = \eta(D, 0)$. If a diffeomorphism ϕ preserves the spin^c structure, then $D^c_{\phi^* g, \phi^* \nabla}$ is conjugate to $D^c_{g,\nabla}$ and hence they have the same spectrum and the same values of η. We will use (2.0.4) to calculate η and related invariants for an operator $D_{g,\tilde{\nabla}}$ on a manifold M by finding a suitable W with $\partial W = M$ and extending g, $\tilde{\nabla}$ to product like h and ∇ on W.

2.1 Spin (4k+3)-Manifolds

A Spin manifold is a special case of a Spin^c manifold for which $w_2(TM)$ and c are zero. Thus λ is a trivial bundle and ∇ can be neglected in all expressions, and we use D_g to represent the Dirac operator with respect to a metric g in the spin case. Let (M, g) be a $4k-1$ dimensional Riemannian spin manifold with vanishing rational Pontryagin classes and $\text{scal}(g) > 0$. In order to define an intrinsic moduli space invariant on M, Kreck and Stolz consider a spin manifold W such that $\partial W = M$ and a metric h, product like near ∂W, and such that $h|_{\partial} W = g$.

In this context the Atiyah-Patodi-Singer signature theorem states

$$\text{sign}(W) = \int_W L(p(h)) - \eta(B_{h|_{\partial W}}) \tag{2.1.1}$$

where $\text{sign}(W)$ is the signature, L is Hirzebruch's L polynomial, and B is the signature operator acting on differential forms.

They multiply (2.1.1) by a factor $a_k = (2^{2k+1}(2^{2k-1}-1))^{-1}$ and add it to (2.0.4) in the spin case:

$$\operatorname{ind}(D_h) + a_k \operatorname{sign}(W) = \int_W (\hat{A} - a_k L)(p(h)) - \frac{\eta(D_g)}{2} - a_k \eta(B_g). \qquad (2.1.2)$$

Note that as (2.0.2) is satisfied on M (∇ can be neglected), $\dim(\ker(D_g))= 0$. Lemma 2.7 in [KS3] follows from Stokes' Theorem:

LEMMA 2.1.3. *[KS3] Let W be a manifold with boundary, and let α, β be closed forms on W such that $\alpha|_{\partial W} = d\hat{\alpha}$ and $\beta|_{\partial W} = d\hat{\beta}$. Then*

$$\int_W \alpha \wedge \beta = \int_{\partial W} \hat{\alpha} \wedge \beta + \left\langle j^{-1}(\alpha) \cup j^{-1}(\beta), [W, \partial W] \right\rangle$$

where j^{-1} represents any preimage under the long exact sequence map

$$j : H^*(W, \partial W; \mathbb{Q}) \to H^*(W, \mathbb{Q}).$$

The utility of this addition is that each summand of the form $(\hat{A} + a_k L)(p(h))$ is decomposable into a wedge product of Pontryagin forms. Since h is product-like, $p(h)|_M = p(h|_M) = p(g)$. By the assumption on the Pontryagin classes of M, these forms will be exact when restricted to M. Thus we can apply Lemma 2.1.3 to the integral

$$\int_W (\hat{A} + a_k L)(p(h)) = \int_M d^{-1}(\hat{A} + a_k L)(p(g)) + \left\langle j^{-1}(\hat{A} + a_k L)(p(TW)), [W, \partial W] \right\rangle$$

where d^{-1} represents replacing one factor of a decomposable form with an exterior anti-derivative as in Lemma 2.1.3. One checks that the choice of which form to replace does not affect the integral.

The Kreck-Stolz s invariant is defined intrinsically for a positive scalar curvature metric g on M :

$$s(M,g) = -\frac{\eta(D_g)}{2} - a_k\eta(B_g) + \int_M d^{-1}(\hat{A} + a_k L)(p(g)).$$

Then (2.1.2) becomes

$$\mathrm{ind}(D_h) + a_k \mathrm{sign}(W) = s(M,g) + \left\langle j^{-1}(\hat{A} + a_k L)(p(TW)), [W, \partial W] \right\rangle$$

They further showed that when $H^1(M, \mathbb{Z}_2) = 0$ the absolute value of the invariant depends only on the connected component of g in $\mathfrak{M}_{\mathrm{scal}>0}$. One proof is very similar to the proof of Theorem 2.2.1 below.

If W and h can be chosen such that $\mathrm{scal}(h) > 0$, then using (2.0.2) we see that $\mathrm{ind}(D_h) = 0$, proving the following theorem:

Theorem 2.1.4. *[KS3] Let W be a spin $(4k)$-manifold with a metric h of positive scalar curvature which is a product metric on a collar neighborhood of ∂W. If $\partial W = M$ has vanishing rational Pontryagin classes and $g = h|_M$ has positive scalar curvature then*

$$s(M,g) = -\left\langle j^{-1}(\hat{A}(TW) + a_k L(TW)), [W, \partial W] \right\rangle + a_k\ \mathrm{sign}(W) \qquad (2.1.5)$$

where $[W, \partial W]$ is the fundamental class and $\hat{A}$ and L are Hirzebruch's polynomials. Furthermore $j^{-1}p_i(W)$ is any preimage of the i^{th} Pontryagin class of W in $H^{4i}(W, \partial W; \mathbb{Q})$ and $\mathrm{sign}(W)$ is the signature of W.

In Theorem A and Theorem B part (b) the associated disc bundle to the sphere bundle is a spin manifold and hence we can apply Theorem 2.1.4 using the metrics constructed in Section 3.1. If however the disc bundle is not a spin manifold we use a different strategy. In the special case of an S^1 bundle with geodesic fibers, Kreck and Stolz use a cobordism argument to reduce to a case where another bounding manifold can be found and derive a correction term.

Theorem 2.1.6. *[KS3] Let $\pi : M \to B$ be a principal S^1 bundle such that M is a spin $(4k-1)$-manifold with vanishing rational Pontryagin classes. Suppose B is a spin manifold and M is given the spin structure induced by the vector bundle isomorphism $TM \cong \pi^*(TB) \oplus V$, where V is the trivial vector bundle generated by the action field of the S^1 action. Let g be a metric with scal$(g) > 0$ on M such that S^1 acts by isometries and the S^1 orbits are geodesics. Then*

$$s(M,g) = -\left\langle j^{-1}(\hat{A}(TW)\cosh(e/2) + a_k L(TW)), [W, \partial W]\right\rangle + a_k \mathrm{sign}(W). \quad (2.1.7)$$

Here W is the disc bundle associated to M. Furthermore j^{-1}, $\hat{A}$, L, a_k, $[W, \partial W]$ and sign(W) *are as in Theorem 2.1.4 and $e \in H^2(W, \mathbb{Z})$ is the image of the Euler class of the S^1 bundle under the isomorphism $H^2(W, \mathbb{Z}) \cong H^2(B, \mathbb{Z})$.*

Specializing to dimension 7, the formulas in (2.1.5) and (2.1.7) become

$$s(M,g) = -\frac{1}{2^7 \cdot 7} p_1^2 + \frac{1}{2^5 \cdot 7}\,\mathrm{sign}(W) \quad (2.1.8)$$

and

$$s(M,g) = -\frac{1}{2^7 \cdot 7} p_1^2 + \frac{1}{2^7 \cdot 3}\left(2p_1e^2 - e^4\right) + \frac{1}{2^5 \cdot 7}\mathrm{sign}(W) \tag{2.1.9}$$

where

$$p_1^2 = \left\langle (j^{-1}(p_1(TW)^2), [W, \partial W] \right\rangle,$$

$$p_1e^2 = \left\langle (j^{-1}(p_1(TW)e^2), [W, \partial W] \right\rangle \text{ and } e^4 = \left\langle (j^{-1}(e^4), [W, \partial W] \right\rangle.$$

In Sections 3 and 4 we find, for each manifold M in Theorems A and B, a sequence of metrics on manifolds diffeomorphic to M such that no two metrics yield the same value of $|s|$. These metrics can be pulled back to M, and by [KS3] Proposition 2.13, s is preserved under pullbacks. The following lemma of Dessai, Klaus, and Tuschmann ([DKT], Section 2.1) and Belegradek, Kwasik, and Schultz ([BKS], Proposition 2.7) shows that these sequences complete the proof of Theorems A and B.

Lemma 2.1.10. *[BKS, DKT] Let M be a simply connected spin (4k-1)-manifold with vanishing rational Pontryagin classes. Let g_1, g_2 be Riemannian metrics on M such that sec(g_i) ≥ 0, scal(g_i) > 0, and $|s|(M, g_1) \neq |s|(M, g_2)$. Then g_1, g_2 lie in different path components of $\mathfrak{M}_{\mathrm{sec}\geq 0}(M)$. Furthermore, there exist metrics $\hat{g}_1$ and $\hat{g}_2$ on M with $\mathrm{Ric}(\hat{g}_i) > 0$ and $s(M, \hat{g}_i) = s(M, g_i)$. Thus $\hat{g}_1$ and $\hat{g}_2$ lie in different path components of $\mathfrak{M}_{\mathrm{Ric}>0}(M)$.*

The proof relies on the result of Bohm and Wilking [BW] that metrics with nonnegative Ricci curvature on a simply connected manifold immediately evolve to have positive Ricci curvature under the Ricci flow.

2.2 Spin^c (4k+1)-Manifolds

In order to extend the tools of Kreck and Stolz to further examples, we prove that for certain $4n+1$ dimensional Spin^c manifolds the η invariant alone provides the desired invariant to distinguish connected components of $\mathfrak{M}_{\mathrm{scal}>0}$.

THEOREM 2.2.1. *Let M^{4n+1} be a closed spin^c manifold with canonical class $c \in H^2(M,\mathbb{Z})$ and canonical bundle λ. Suppose c and the Pontryagin classes $p_i(TM)$ are torsion and g_t, $t \in [0,1]$ is a smooth path of metrics on M with scal$(g_t) > 0$. If ∇_0 and ∇_1 are flat unitary connections on λ, then*

$$\eta(D^c_{g_0,\nabla_0}) = \eta(D^c_{g_1,\nabla_1}).$$

Proof. Modifying g_t if necessary we assume it is a constant path for t near 0 and 1. Given $L \in \mathbb{R}_{>0}$, define a smooth metric g on $M \times [0,1]$ by

$$g = g_t + L^2 dt^2.$$

Then g is product-like near $M \times \{0,1\}$. One sees that scal(g) differs from scal(g_t) by terms depending on the second fundamental form of each slice $M \times \{t\}$, but the second fundamental form tends to 0 as $L \to \infty$, so for large L we have scal$(g) > 0$.

The difference of unitary connections on a complex line bundle is an imaginary one form. Define $\alpha \in \Omega(M)$ such that

$$i\alpha = \nabla_1 - \nabla_0.$$

Since both connections are flat, $d\alpha = 0$. Let $\pi : M \times [0,1] \to M$ be the projection and let $f : M \times [0,1] \to [0,1]$ be the projection onto $[0,1]$ followed by be a smooth function which is 0 in a neighborhood of 0 and 1 in a neighborhood of 1. Define a connection on $\pi^*\lambda$ by

$$\nabla = \pi^*\nabla_0 + if\pi^*\alpha.$$

Then, since ∇_0 is flat,

$$F^\nabla = idf \wedge \pi^*\alpha.$$

Let e_i be an orthonormal frame for g at a point (p,t), such that $e_1 = \frac{1}{L}\partial_t$. Then

$$2\sum_{i<j} |(df \wedge \alpha)(e_i, e_j)| = \frac{2\partial_t f}{L} \sum_{i>1} \alpha(e_i).$$

Since e_i, $i > 2$, is tangent to $M \times \{t\}$, it does not depend on L. Using (2.0.3), for large L we have

$$\mathrm{scal}(g) > 2|F^\nabla|_g.$$

The definition of f ensures that ∇ is product-like near $\partial(M \times I)$. Then by (2.0.1) $D^c_{g,\nabla}$ has trivial kernel and $\mathrm{ind}(D^c_{g,\nabla}|_{S^+}) = 0$.

Since $F^{\nabla_i} = 0$ for $i = 1, 2$

$$\mathrm{scal}(g_i) > 0 = 2|F^{\nabla_i}|_{g_i}$$

and hence (2.0.1) implies $\ker D^c_{g_i,\nabla_i} = \{0\}$. We now apply the Atiyah-Patodi-Singer index theorem (2.0.4). The boundary of $M \times I$ is two copies of M with opposite orientations. The spectrum of the Dirac operator on $M \times \{0,1\}$ is the union of the spectra on $M \times \{0\}$ and $M \times \{1\}$, and the η invariant is the sum of the two η invariants. When we change the orientation of an odd dimensional manifold, the Dirac operator changes by a sign. Thus the Atiyah-Patodi-Singer theorem yields

$$\operatorname{ind}(D^c_{g,\nabla}|_{S^+}) = \int_{M\times[0,1]} e^{c_1(\nabla)/2}\hat{A}(p(g))$$
$$-\frac{1}{2}\left(\dim(\ker(D^c_{g_0,\nabla_0})) + \dim(\ker(D^c_{g_1,\nabla_1})) + \eta(D^c_{g_0,\nabla_0}) - \eta(D^c_{g_1,\nabla_1})\right)$$

and hence

$$\eta(D^c_{g_1,\nabla_1}) - \eta(D^c_{g_0,\nabla_0}) = 2\int_{M\times[0,1]} e^{c_1(\nabla)}\hat{A}(p(g)).$$

Since $\pi_1^* c$ is torsion, $c_1(\nabla)$ is exact. Because ∇ is flat near the boundary $c_1(\nabla)|_{\partial(M\times I)} = 0$. Furthermore g is product-like near the boundary so $p(g)|_{M\times\{i\}} = p(g_i)$. Since the real Pontryagin classes of M vanish $p_j(g_i)$ is exact for $j > 0$. By Stokes' theorem, and since the dimension of M is $4n+1$, the integral vanishes. $\square$

As a corollary we show how to use the η invariant to detect path components of moduli spaces of metrics with curvature conditions no weaker than positive scalar curvature.

Corollary 2.2.2. *Let M be as in Theorem 2.2.1. Let (g_i, ∇_i) be a sequence of Riemannian metrics g_i with $Ric(g_i) > 0$, and flat connections ∇_i on λ such that $\{\eta(D^c_{g_i,\nabla_i})\}_i$ is infinite. Then $\mathfrak{M}_{\mathrm{Ric}>0}(M)$ and $\mathfrak{M}_{\mathrm{scal}>0}(M)$ have infinitely many path components.*

Proof. Let $\mathrm{Diff}^c(M)$ be the set of diffeomorphisms of M which fix the spin^c structure. For $g \in \mathfrak{R}_{\mathrm{scal}>0}$ let $[g]$ represent the image in $\mathfrak{M}_{\mathrm{scal}>0}$ and $[g]^c$ the image in $\mathfrak{R}_{\mathrm{scal}>0}/\mathrm{Diff}^c(M)$. It follows from Ebin's slice theorem ([E], [Bo]), that if $[g_i], [g_j]$ are in the same connected component of $\mathfrak{R}_{\mathrm{scal}>0}/\mathrm{Diff}^c(M)$ then $g_i, \phi^* g_j$ are in the same path component of $\mathfrak{R}_{\mathrm{scal}>0}$ for some $\phi \in \mathrm{Diff}^c(M)$. Then there is a path between them maintaining positive scalar curvature, and by Theorem 2.2.1 and the spin^c diffeomorphism invariance of η we have $\eta(D^c_{g_i,\nabla_i}) = \eta(D^c_{\phi^* g_j, \phi^* \nabla_j}) = \eta(D^c_{g_j,\nabla_j})$. Since $\{\eta(D^c_{g_i,\nabla_i})\}$ is infinite, $\mathfrak{R}_{\mathrm{scal}>0}/\mathrm{Diff}^c(M)$ has infinitely many components.

Any diffeomorphism ϕ pulls back the spin^c structure to another one with canonical class $\phi^* c$, a torsion class in $H^2(M, \mathbb{Z})$. There are finitely many such classes. The finite group $H^1(M, \mathbb{Z}_2)$ indexes the Spin^c structures associated to each class. Thus the orbit of the spin^c structure under $\mathrm{Diff}(M)$ and the set $\mathrm{Diff}(M)/\mathrm{Diff}^c(M)$ are finite. The fibers of $\mathfrak{R}_{\mathrm{scal}>0}/\mathrm{Diff}^c(M) \to \mathfrak{M}_{\mathrm{scal}>0}$ are no larger than $\mathrm{Diff}(M)/\mathrm{Diff}^c(M)$, implying that $\mathfrak{M}_{\mathrm{scal}>0}$ has infinitely many components.

The proof is identical for $\mathfrak{M}_{\mathrm{Ric}>0}$ since $\mathrm{Ric} > 0$ implies $\mathrm{scal} > 0$. $\square$

In section Section 4.3 we show how to extend a metric g with $\mathrm{scal}(g) > 0$ and flat connection $\bar{\nabla}$ on certain S^1 principal bundles to h, ∇ on the associated D^2 bundles such that

$$\mathrm{scal}(h) > |F^{\nabla}|_h.$$

Then the index and dim(ker) terms in (2.0.4) vanish, and the η invariant can be calculated in terms of the integral. The integral is evaluated with Lemma 2.1.3

as in the previous section. Note that in this case only the cohomological term remains. Evaluating this term in the case of a disc bundle we prove the following in Section 4.2

Theorem 2.2.3. *Let S^1 act freely on a $4n+1$ manifold M by isometries of a Riemannian metric g with scal(g) > 0. Assume $\pi_1(M)$ is finite and let $B = M/S^1$ be the quotient. Suppose the first Chern class of the principal bundle $S^1 \to M \xrightarrow{\pi} B$ is ℓd where ℓ is a positive even integer and $w_2(TB) = d \bmod 2$. Finally assume the real Pontryagin classes of M vanish. Then M admits a spin^c structure with canonical class $\pi^* d$. If $\bar{\nabla}$ is a flat connection on the canonical bundle of this spin^c structure and $D^c_{g,\tilde{\nabla}}$ is the spin^c Dirac operator, then*

$$\eta(D^c_{g,\bar{\nabla}}) = \left\langle \frac{\sinh(d/2)\hat{A}(TB)}{\sinh(\ell d/2)}, [B] \right\rangle.$$

When $n = 1$,

$$\eta(D^c_{g,\bar{\nabla}}) = \left\langle -\frac{(\ell^2-1)d^2 + p_1(TB)}{24\ell}, [B] \right\rangle. \tag{2.2.4}$$

Chapter 3

7-Manifolds

3.1 Metrics on Sphere and Disc Bundles

In [GZ1] and [GZ3] one finds many examples of metrics with nonnegative sectional curvature on principal $SO(n)$ bundles such that $SO(n)$ acts by isometries. Hence the associated sphere bundles admit such metrics as well. We will apply Theorem 2.1.4 to appropriate metrics constructed on the associated sphere and disc bundles.

THEOREM 3.1.1. *Let P be a principal $SO(n+1)$ bundle admitting a metric g_P, invariant under the $SO(n+1)$ action, with $sec(g_P) \geq 0$. In the case $n = 1$ assume in addition that at each point $x \in P$ there exists a 2-plane $\sigma_x \subset T_xP$ with $sec_{g_P}(\sigma_x) > 0$ which is orthogonal to the orbit of $SO(2)$. Then there exists a metric g_M on the associated sphere bundle $M = P \times_{SO(n+1)} S^n$ with $sec(g_M) \geq 0$ and $scal(g_M) > 0$ that extends to a metric g_W on the associated disc bundle $W = P \times_{SO(n+1)} D^{n+1}$*

with $scal(g_W) > 0$*. Furthermore* g_W *is a product near the boundary of* W*.*

Proof. Let g_{S^n} be the standard metric on the sphere of radius $1/2$. We define the metric g_M such that the product metric $g_P + g_{S^n}$ and g_M make the projection

$$\rho : P \times S^n \to P \times_{SO(n+1)} S^n = M$$

into a Riemannian submersion. By the O'Neill formula g_M has nonnegative sectional curvature.

To show g_M has positive scalar curvature we must check that each point of M has a 2-plane of positive sectional curvature. First assume $n > 1$. Consider $(p, x) \in P \times S^n$. Let $X, Y \in \mathfrak{so}(n+1)$ be such that the action fields $X^*, Y^* \in T_x S^n$ are linearly independent. The vertical space of the $SO(n+1)$ action on $P \times S^n$ is the set of vectors $(Z^*, -Z^*)$ for all $Z \in \mathfrak{so}(n+1)$, where we repeat notation for the action fields on P and S^n. It follows that the projections of $(0, X^*), (0, Y^*) \in T_{(p,x)} P \times S^n$ onto the horizontal space are $A = (aX^*, bX^*)$ and $B = (cY^*, dY^*)$ for some $a, b, c, d \neq 0$ and hence $\sec(A \wedge B) > 0$ in the product metric $g_P + g_{S^n}$.Since the plane is horizontal the O'Neill formula implies that $\sec_{g_M}(\rho_* A \wedge \rho_* B) > 0$. Thus g_M has positive scalar curvature.

In the case of $n = 1$ we have by assumption a 2-plane $\sigma_x \subset T_x P$ in the horizontal space of the $SO(2)$ action on P. It follows that $(\sigma_x, 0)$ lies in the horizontal space of the $SO(2)$ action on $P \times S^1$, and by the O'Neill formula $\sec_{g_M}(\rho_*(\sigma_x, 0)) \geq \sec_{g_P}(\sigma_x)$. So g_M has a 2-plane of positive sectional curvature at each point and hence $\mathrm{scal}(g_M) > 0$.

We next show that g_M extends to a metric g_W on W with positive scalar curvature. Let $f : [0, 1] \to \mathbb{R}$ be a concave function with $f(0) = 0$, $f'(0) = 1$, $f'(r) < 1$ for $r \in (0, 1]$ and $f(r) = 1/2$ for $r \in [R, 1]$ for some $R < 1$. Then

$$g_{D^{n+1}} = dr^2 + f(r)^2 g_{S^n}$$

is a smooth metric on D^{n+1} with $\sec(g_{D^{n+1}}) \geq 0$. Define the metric g_W on W such that $g_P + g_{D^{n+1}}$ and g_W make the projection

$$\pi : P \times D^{n+1} \to P \times_{SO(n+1)} D^{n+1} = W$$

into a Riemannian submersion.

The assumption that f is concave and $f'(r) < 1$ when $r > 0$ ensure $\sec(g_{D^{n+1}}) \geq 0$. Furthermore, when $n > 1$, planes tangent to the spheres of constant radius r will have positive sectional curvature, and we repeat the argument above to conclude $\mathrm{scal}(g_W) > 0$. For $n = 1$, the same argument as for g_M implies $\mathrm{scal}(g_W) > 0$.

For $r \in [R, 1]$ the projection π can be regarded as

$$\pi : (P \times S^n) \times [R, 1] \to (P \times_{SO(n+1)} S^n) \times [R, 1] \cong M \times [R, 1].$$

The image is a collar neighborhood of the boundary of W. Since $f = 1/2$ in this region, the metric on the left is $g_P + g_{S^n} + dr^2$ and the metric induced on the quotient is $g_M + dr^2$. So g_W is a product metric near the boundary with $g_W|_{\partial W} = g_M$.

$\square$

We note that by replacing g_{S^n} by $\frac{1}{\lambda} g_{S^n}$ in the proof and considering $\lambda \in [0, 1]$, one sees that g_M lies in the same path component of $\mathfrak{M}_{\sec \geq 0}(M)$ as the metric induced by g_P under the submersion $P \to P/SO(n) \cong M$. This is of particular interest when $n = 1$ in which case $P = M$.

In the case of an S^1 bundle with totally geodesic fibers, Theorem 2.1.6 applies without requiring the disc bundle to be spin. The following theorem shows that some S^1 invariant metrics with nonnegative sectional curvature can be deformed to metrics with geodesic fibers while maintaining positive scalar curvature.

THEOREM 3.1.2. *Let M be a manifold admitting a free S^1 action and a metric g of nonnegative sectional curvature, invariant under that action. Suppose that for each $x \in M$ there is a 2-plane $\sigma_x \subset T_xM$ orthogonal to the S^1 orbit with $sec(\sigma_x) > 0$. Then M admits a metric h of positive scalar curvature such that S^1 acts by isometries, the S^1 orbits are geodesics, and h and g are in the same path component of $\mathfrak{M}_{\mathrm{scal}>0}(M)$.*

Proof. Since the set of 2-planes orthogonal to the S^1 orbits is compact, the maximum sectional curvature of such a plane at each point is a positive continuous function, and hence there exists $C > 0$ such that we can choose σ_x with $\sec(\sigma_x) > C$. Let X be the action field of the S^1 action on M and $u = |X|_g$. We fix $0 < \epsilon < \inf_M(u)$ such that

$$\sup_{x\in M,\ |Y|_g=1} \left(\epsilon^2 \left| \frac{3Y(u)^2}{(u^2-\epsilon^2)^2} - \frac{\mathrm{Hess}_u(Y,Y)}{u(u^2-\epsilon^2)} \right| \right) < \frac{C}{n-1}.$$

For each $\lambda \in (0,1]$ we define $v : M \to \mathbb{R}_{>0}$ by

$$v_\lambda = \frac{\epsilon u}{\lambda\sqrt{u^2-\epsilon^2}}$$

and a warped product metric g_λ on $M \times S^1$:

$$g_\lambda = g + v_\lambda^2 d\theta^2.$$

Next define the metric h_λ on M such that g_λ and h_λ make the projection

$$\pi : M \times S^1 \to M \times_{S^1} S^1 \cong M$$

into a Riemannian submersion. The action

$$z \cdot (x, y) \to (x, yz)$$

of S^1 on $M \times S^1$ is by isometries of g_λ, commutes with the quotient action, and induces an action on $M \times_{S^1} S^1$ which makes the diffeomorphism $M \times_{S^1} S^1 \cong M$ equivariant. Thus S^1 acts on M by isometries of h_λ.

We now show that $\mathrm{scal}(h_\lambda) > 0$ for all $\lambda \in (0, 1]$. For a point $x \in M$ let $\{X_1, ..., X_{n-1}, X\}$ be an orthogonal basis of T_xM with $|X_i|_g = 1$ and $\sigma_x = \mathrm{span}(X_1, X_2)$. Then we can find $a, b \in \mathbb{R}$ and $Z = (aX, b\partial_\theta)$ such that $\{(X_1, 0), ..., (X_{n-1}, 0), Z\}$ is an orthonormal basis of the horizontal space of π at $(x, y) \in M \times S^1$. By the O'Neill formula

$$\mathrm{scal}(h_\lambda) \geq \sec_{g_\lambda}((X_1, 0) \wedge (X_2, 0))$$
$$+ \sum_{(i,j)\neq(1,2)} \sec_{g_\lambda}((X_i, 0) \wedge (X_j, 0)) + \sum_{k=1}^{n-1} \sec_{g_\lambda}((X_k, 0) \wedge Z).$$

Since $(M \times \{y\}, g)$ is totally geodesic,

$$\sec_{g_\lambda}((X_1, 0) \wedge (X_2, 0)) = \sec_g(\sigma_x) > C$$

and

$$\sec_{g_\lambda}((X_i,0)\wedge(X_j,0)) = \sec_g(X_i\wedge X_j) \geq 0.$$

Furthermore, using the basis $\{(X_k,0), \frac{1}{b}Z\}$ for the plane $(X_k,0)\wedge Z$

$$\sec_{g_\lambda}((X_k,0)\wedge Z) = \frac{\left\langle R_g(X_k,\frac{a}{b}X)\frac{a}{b}X, X_k\right\rangle_g + \langle R_{g_\lambda}((X_k,0),(0,\partial_\theta))(0,\partial_\theta),(X_k,0)\rangle_{g_\lambda}}{\frac{a^2}{b^2}u^2+v_\lambda^2}$$

$$\geq -\left|\sec_{g_\lambda}((X_k,0)\wedge(0,\partial_\theta))\right| = -\left|\frac{1}{v_\lambda}\mathrm{Hess}_{v_\lambda}(X_k,X_k)\right|.$$

For details on the sectional curvatures of a warped product, used in the last equality, see [Ba] Section 9J. Applying the definition of v_λ we have

$$\mathrm{scal}(h_\lambda) \geq C - \sum_{k=1}^{n-1}\epsilon^2\left|\frac{3X_k(u)^2}{(u^2-\epsilon^2)^2} - \frac{\mathrm{Hess}_u(X_k,X_k)}{u(u^2-\epsilon^2)}\right| > 0.$$

So h_λ, $\lambda\in(0,1]$, is a continuous path of metrics with positive scalar curvature. Each h_λ is identical to g on the orthogonal complement of X, while

$$|X|^2_{h_\lambda} = \frac{u^2v_\lambda^2}{u^2+v_\lambda^2} = \frac{\epsilon^2u^2}{\lambda^2(u^2-\epsilon^2)+\epsilon^2}.$$

Since X is a Killing vector field and $|X|_{h_1}=\epsilon$ is constant, the integral curves of X, which are the orbits of the S^1 action, are geodesics in h_1. Furthermore $|X|_{h_0}=u$, so $h_0=g$. Thus $h=h_1$ and g are in the same path component of $\mathfrak{M}_{\mathrm{scal}>0}(M)$.

□

3.2 S^3 bundles

In this section we prove Theorem A, starting with the simplest case.

3.2.1 S^3 bundles over S^4

S^3 bundles over S^4 are classified by elements of $\pi_3(SO(4)) = \mathbb{Z} \oplus \mathbb{Z}$. We use the basis for $\pi_3(SO(4))$ given by the maps $\mu(q)(v) = qvq^{-1}$ and $\nu(q)(v) = qv$. Here $v \in \mathbb{R}^4$ viewed as the quaternions and $q \in S^3$ viewed as the unit quaternions. Let $M_{m,n}$ be the bundle classified by $m\mu + n\nu \in \pi_3(SO(4))$. In [GZ1] it is shown that the $SO(4)$ principal bundle of every S^3 bundle over S^4 admits an $SO(4)$ invariant metric of nonnegative sectional curvature, and hence the sphere bundles do as well.

Assume $n \neq 0$. From the homotopy long exact sequence one sees that $H^4(M_{m,n}, \mathbb{Z}) = \mathbb{Z}_n$, so the rational Pontryagin classes of $M_{m,n}$ vanish. Let $W_{m,n}$ be the associated disc bundle. Then $H^2(W_{m,n}, \mathbb{Z}_2) = H^2(S^4, \mathbb{Z}_2) = 0$ and hence $W_{m,n}$ is a spin manifold. Theorem 3.1.1 and Theorem 2.1.4 imply that $M_{m,n}$ has a metric $g_{M_{m,n}}$ of nonnegative sectional and positive scalar curvature with s invariant given by (2.1.8). Crowley and Escher [CE] computed the invariants

$$p_1^2(W_{m,n}) = \frac{4(n+2m)^2}{n}$$

and $\mathrm{sign}(W_{m,n}) = 1$. So

$$s(M_{m,n}, g_{M_{m,n}}) = \frac{-(n+2m)^2 + n}{2^5 \cdot 7 \cdot n}.$$

Corollary 1.6 of [CE] shows that $M_{m',n}$ and $M_{m,n}$ are diffeomorphic if $m' = m$ mod $56n$. So the manifolds in the sequence $\{M_{m+56ni,n}\}_i$ are all diffeomorphic to $M_{m,n}$. Since n is constant in the sequence, the s invariant is a polynomial in i. It follows that there is an infinite subsequence of metrics with distinct s invariants.

Lemma 2.1.10 completes the proof of the first part of Theorem A.

Remark 3.2.1. Comparison to the 7-dimensional homogeneous spaces in [Ni] shows that $M_{m,n}$ has the cohomology ring of such a space only when $|n| = 1$, 2 or 10. The homogeneous candidates are S^7, T_1S^4 and the Berger space $SO(5)/SO(3)$ with $H^4 = 0, \mathbb{Z}_2$ and $\mathbb{Z}_{10}$.

3.2.2 S^3 bundles over $\mathbb{C}P^2$

In [GZ3] it is shown that every principal $SO(4)$ bundle over $\mathbb{C}P^2$ with $w_2 \neq 0$ admits an $SO(4)$ invariant metric of nonnegative sectional curvature. Such bundles are classified by two integers a, b describing the first Pontryagin and Euler classes $p_1 = (2a + 2b + 1)x^2$ and $e = (a - b)x^2$, where x is the generator of $H^*(\mathbb{C}P^2, \mathbb{Z})$. Let $\pi : S_{a,b} \to \mathbb{C}P^2$ be the S^3 bundle over $\mathbb{C}P^2$ with these characteristic classes. If $a \neq b$ then the Gysin sequence implies that $H^4(S_{a,b}, \mathbb{Z}) = \mathbb{Z}_{|a-b|}$, so the rational Pontryagin classes vanish.

Let $E^4 \to \mathbb{C}P^2$ be the 4-plane bundle associated to $S_{a,b}$ and $W_{a,b} \subset E^4$ the associated disc bundle with projection $\rho : W_{a,b} \to \mathbb{C}P^2$. Then $TW_{a,b} \cong \rho^*(E^4 \oplus T\mathbb{C}P^2)$ and $w_2(TW_{a,b}) = \rho^*(w_2(E^4) + w_2(T\mathbb{C}P^2)) = 0$. So $W_{a,b}$ is a spin manifold. Theorem 3.1.1 and Theorem 2.1.4 imply that $S_{a,b}$ has a metric $g_{S_{a,b}}$ of nonnegative sectional and positive scalar curvature with s invariant given by (2.1.8).

It is shown in [EZ] Proposition 4.3 that

$$p_1^2(W_{a,b}) = \frac{1}{a-b}(2a+2b+4)^2$$

and

$$\operatorname{sign}(W_{a,b}) = \operatorname{sgn}(a-b).$$

So

$$s(S_{a,b}, g_{S_{a,b}}) = -\frac{(a+b+2)^2}{2^5 \cdot 7 \cdot (a-b)} + \frac{\operatorname{sgn}(a-b)}{2^5 \cdot 7}.$$

Corollary 4.5 of [EZ] implies that $S_{a,b}$ and $S_{a',b'}$ are diffeomorphic if $a-b = a'-b' > 0$ and $a = a' \bmod \lambda = 2^3 \cdot 3 \cdot 7 \cdot |a-b|$. Thus the manifolds in the sequence $\{S_{a+i\lambda,b+i\lambda}\}_i$ are all diffeomorphic to $S_{a,b}$. Here we may assume $a-b>0$ as $S_{a,b}$ is diffeomorphic to $S_{b,a}$. Since $a-b$ is constant for the sequence, the s invariant is a polynomial in i. So there is an infinite subsequence of metrics in this sequence with distinct s invariants. Lemma 2.1.10 completes the proof of the second part of Theorem A.

Remark 3.2.2. (a) The only 7-dimensional homogeneous spaces with the same cohomology ring as any $S_{a,b}$ are the families $N^7_{k,l}$ and $W^7_{k,l}$ described in the introduction, see [Ni]. The quantities $|H^4(N^7_{k,l},\mathbb{Z})| = l^2$ and $|H^4(W^7_{k,l},\mathbb{Z})| = k^2+l^2+kl$ are always equal to 0 or 1 mod 3, so if $|a-b| = 2 \bmod 3$, $S_{a,b}$ does not have the homotopy type of a 7-dimensional homogeneous space.

(b) By [EZ] Proposition 6.7, $S_{-1,a(a-1)}$ is diffeomorphic to the homogeneous Aloff-Wallach space $W^7_{a,1-a}$. There also exist infinitely many positively curved Eschenburg spaces and many other Aloff-Wallach spaces which are diffeomorphic to S^3 bundles over $\mathbb{C}P^2$, see [EZ] Theorem 8.1.

3.3 S^1 bundles

We will use the diffeomorphism classification of [KS2] for the S^1 bundles described in this section. It applies to spin 7-manifolds with $\pi_1(M) = 0$, $H^2(M,\mathbb{Z}) = \mathbb{Z}$, $H^3(M,\mathbb{Z}) = 0$ and $H^4(M,\mathbb{Z})$ finite cyclic and generated by the square of a generator of $H^2(M,\mathbb{Z})$. For such manifolds Kreck and Stolz defined three invariants $s_1(M), s_2(M), s_3(M) \in \mathbb{Q}/\mathbb{Z}$ and proved that two such spin manifolds M and M' are diffeomorphic if and only if $|H^4(M,\mathbb{Z})| = |H^4(M',\mathbb{Z})|$ and $s_i(M) = s_i(M')$ for $i = 1, 2, 3$ ([KS2] Theorem 3.1).

Escher and Ziller [EZ] defined two families of 7-manifolds as follows. Let x be the generator of $H^*(\mathbb{C}P^2,\mathbb{Z})$. Define $p : N_t \to \mathbb{C}P^2$ as the S^2 bundle with Pontryagin and Stiefel-Whitney classes $p_1(N_t) = (1-4t)x^2$ and $w_2(N_t) \neq 0$. They showed that N_t is diffeomorphic to the projectivization $P(E_t)$ of the complex line bundle E_t over $\mathbb{C}P^2$ with Chern classes $c_1(E_t) = x$ and $c_2(E_t) = tx^2$. Furthermore if P_t is the principal $U(2)$ bundle corresponding to E_t, N_t is diffeomorphic to P_t/T^2, where $T^2 \subset U(2)$.

Let y be the first Chern class of the dual of the tautological line bundle over $P(E)$. By the Leray-Hirsch theorem

$$H^*(N_t) = \mathbb{Z}[x,y]/(x^3, y^2 + xy + tx^2).$$

For simplicity, we denote $p^*(x)$ again by x. Finally define the principal S^1 bundle

$$S^1 \to M^t_{a,b} \to N_t \qquad \text{with Euler class} \quad e = ax + (a+b)y \quad \text{and } \gcd(a,b) = 1.$$

Proposition 6.1 in [EZ] shows that the bundle $S^1 \to M^t_{a,b} \to N_t$ is equivalent to $T^2/S^1_{a,b} \to P_t/S^1_{a,b} \to P_t/T^2 \cong N_t$ where $S^1_{a,b} = \{\mathrm{diag}(e^{ia\theta}, e^{ib\theta})\} \subset U(2)$. Since $\gcd(a,b) = 1$, the total space is simply connected, and from the Gysin sequence it follows that the cohomology ring of $M^t_{a,b}$ is of the form required by the diffeomorphism classification of [KS2] as long as $0 \neq |t(a+b)^2 - ab| = |H^4(M^t_{a,b}, \mathbb{Z})|$.

Next define $\bar{p} : \bar{N}_t \to \mathbb{C}P^2$ as the S^2 bundle with Pontryagin and Stiefel-Whitney classes $p_1(\bar{N}_t) = 4tx^2$ and $w_2(\bar{N}_t) = 0$. In this case $\bar{N}_t$ is diffeomorphic to the projectivization $P(\bar{E}_t)$ of the complex line bundle $\bar{E}_t$ over $\mathbb{C}P^2$ with Chern classes $c_1(\bar{E}_t) = 2x$ and $c_2(\bar{E}_t) = (1-t)x^2$. If $\bar{P}_t$ is the principal $U(2)$ bundle associated to $\bar{E}_t$, $\bar{N}_t$ is diffeomorphic to $\bar{P}_t/T^2$. Let y be the first Chern class of the dual of the tautological line bundle over $P(\bar{E}_t)$. Then

$$H^*(\bar{N}_t) = \mathbb{Z}[x,y]/(x^3, y^2 + 2xy + (1-t)x^2).$$

Again we denote $\bar{p}^*(x)$ by x. Finally define the principal S^1 bundle

$$S^1 \to \bar{M}^t_{a,b} \to \bar{N}_t \qquad \text{with Euler class} \quad e = (a+b)x + by \quad \text{and } \gcd(a,b) = 1.$$

In this case, one sees that $\pi_1(\bar{P}_t) = \mathbb{Z}_2$ and $\bar{P}_t$ has a two-fold cover $\bar{P}'_t$ which is a principal $S^1 \times S^3$ bundle over $\mathbb{C}P^2$. Furthermore $\bar{N}_t \cong \bar{P}'_t/T^2$, with $T^2 = \{(e^{i\theta}, e^{i\phi})\} \subset S^1 \times S^3$. Proposition 7.5 in [EZ] shows that the bundle defining $\bar{M}^t_{a,b}$ is equivalent to $T^2/S^1_{-b,a} \to \bar{P}'_t/S^1_{-b,a} \to \bar{P}'_t/T^2$ where $S^1_{-b,a} = \{(e^{-ib\theta}, e^{ia\theta})\}$. As before, $\bar{M}^t_{a,2b}$ is simply connected and has the cohomology necessary for the diffeomorphism classification of [KS2], since a is odd and so $|H^4(M^t_{a,2b})| = |a^2 - 4tb^2| \neq 0$.

Escher and Ziller showed that $M^t_{a,b}$ and $\bar{M}^{2t}_{a,b}$ admit S^1 invariant metrics $g^t_{a,b}$ and $\bar{g}^{2t}_{a,b}$ respectively with nonnegative sectional curvature. In order to apply Theorem 3.1.1 and Theorem 3.1.2 we prove the following lemma.

Lemma 3.3.1. *At each point x of $(M^t_{a,b}, g^t_{a,b})$ and $(\bar{M}^{2t}_{a,b}, \bar{g}^{2t}_{a,b})$ there exists a 2-plane σ_x orthogonal to the S^1 orbit with sec$(\sigma_x) > 0$.*

Proof. The metrics are constructed using cohomogeneity one actions, and we first recall the general description of such manifolds. We consider actions of a compact Lie group G on a manifold M such that the orbit space is the interval $[-1, 1]$. Let $\pi : M \to [-1, 1]$ be the projection onto the orbit space. Let $H \subset G$ be the isotropy subgroup of a point in the principal orbit $\pi^{-1}(0)$ and $K^{\pm}$ the isotropy groups of points in the singular orbits $\pi^{-1}(\pm 1)$. The slice theorem implies that $\pi^{-1}([-1, 0])$ is equivariantly diffeomorphic to the disc bundle $D_- = G \times_{K_-} D^{d_-}$ where K_- acts linearly on D^{d_-} and K_-/H is diffeomorphic to the sphere $S^{d_- -1}$. Here d_- is the codimension of the singular orbit. Furthermore, the boundary of D_- is G/H, diffeomorphic to the principal orbit $\pi^{-1}(0)$. D_+ is described equivalently with the same boundary. Then M is diffeomorphic to the union $D_- \cup_{G/H} D_+$. Conversely, given Lie groups $H \subset K_\pm \subset G$ with $K_\pm/H \cong S^{d_\pm - 1}$, the action of $K_\pm$ on $S^{d_\pm-1}$ extends to a linear action on $D^{d_\pm}$. We can then define $M = D_- \cup_{G/H} D_+$ as above, and M will admit a cohomogeneity one action by G with isotropy groups $H \subset K_\pm$.

If $d_\pm = 2$, it is shown in [GZ1] that one can define a metric with $\sec \geq 0$ on M as follows. Let $\mathfrak{g}, \mathfrak{k}, \mathfrak{h}$ be the Lie algebras of G, K_-, H respectively and Q a bi-invariant metric on G. Choose $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{k}$ and $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{p}$ to be Q-orthogonal decompositions and $\overline{Q}_a$ the left invariant metric on G defined by

$$\overline{Q}_a = Q|_{\mathfrak{m}\oplus\mathfrak{h}} + aQ|_{\mathfrak{p}}.$$

Let $f(r)$ be a concave function with $f(0) = 0$, $f'(0) = 1$ and $f(r) = \sqrt{\frac{al^2}{a-1}}$ for r near the boundary of D^2, where $2\pi l$ is the length of K_-/H with respect to Q. In [GZ1] it is shown that the metric

$$\overline{g} = \overline{Q}_a + dr^2 + f(r)d\theta^2$$

on $G \times D^2$ has nonnegative curvature as long as $1 < a \leq 4/3$ and hence induces a G invariant metric g_- of nonnegative curvature on the quotient D_-. Furthermore g_- is a product near the boundary G/H, with the induced metric on G/H the same as that induced by Q. A similar metric can be put on D_+, and because of the boundary condition the two can be glued to form a smooth G invariant metric g of nonnegative sectional curvature on $D_- \cup_{G/H} D_+ \cong M$.

In order to prove the claim, we need to describe the manifolds and metrics in a slightly different way than in [GZ3]. For $p_-, p_+, q \in \mathbb{Z}$, p_+ odd and $p_- \neq q \bmod 2$, let $P_{p_-,p_+,q}$ be the cohomogeneity one manifold defined by the following Lie groups:

$$G = U(2) \times S^3$$

$$H = \mathbb{Z}_4 = \left\langle \left(\pm i^q \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, j \right) \right\rangle$$

$$K_- = H \cdot \left\{ \left(\operatorname{diag}\left(e^{ip_-\theta}, e^{-ip_-\theta}\right), e^{i\theta} \right) \right\}$$

$$K_+ = \left\{ \left(e^{iq\theta} R(p_+\theta), e^{j\theta} \right) \right\}$$

where $R(\phi)$ represents a 2×2 rotation matrix and the sign in H is chosen to make H a subgroup of K_+. One easily sees that $U(2)$ acts freely on $P_{p_+,p_-,q}$. Since $U(2)$ commutes with S^3, the quotient $P_{p_+,p_-,q}/U(2)$ admits an action by S^3 which is cohomogeneity one with the same isotropy groups as the action of S^3 on $\mathbb{C}P^2$ (see [GZ3] Figure 2.2.) Thus $P_{p_+,p_-,q}$ is the total space of a principal $U(2)$ bundle over $\mathbb{C}P^2$.

Suppose P is a principal $U(2)$ bundle over $\mathbb{C}P^2$. From the spectral sequence of the fibration $U(2) \to P \to \mathbb{C}P^2$, one sees that $H^2(P, \mathbb{Z}) \cong \mathbb{Z}_{|c_1|}$, where c_1 denotes the coefficient of x in the first Chern class $c_1(P) \in H^2(\mathbb{C}P^2, \mathbb{Z})$. Applying the Seifert-Van Kampen theorem to $P_{p_-,p_+,q} = D_- \cup D_+$, one shows that $\pi_1(P_{p_-,p_+,q}) = \mathbb{Z}_q$. By the universal coefficient theorem we conclude that $H^2(P_{p_+,p_-,q}, \mathbb{Z}) = \mathbb{Z}_q$ and hence $c_1(P_{p_+,p_-,q}) = qx$.

Let Z be the center of $U(2)$. Since $U(2)/Z \cong SO(3)$, P/Z is a principal $SO(3)$ bundle over $\mathbb{C}P^2$ with first Pontryagin class $p_1(P/Z) = c_1(P)^2 - 4c_2(P)$, see [EZ], 2.5, 2.6. In particular, $P_{p_-,p_+,q}/Z$ admits a cohomogeneity one action by $SO(3) \times S^3$ and one easily shows that the isotropy groups are

$$H = \mathbb{Z}_4 = \langle (R_{1,3}(\pi), j) \rangle$$

$$K_- = H \cdot \left\{ \left(R_{2,3}(2p_-\theta), e^{i\theta} \right) \right\}$$

$$K_+ = \left\{ \left(R_{1,3}(2p_+\theta), e^{j\theta} \right) \right\}.$$

Here $R_{n,m} \in SO(3)$ is a rotation in the n, m plane of $\mathbb{R}^3$. By [GZ3] Theorem 4.7, this bundle has first Pontryagin class $p_1(P_{p_-,p_+,q}/Z) = (p_+^2 - p_-^2)x^2$. It follows that $c_2(P_{2t}) = \frac{1}{4}(q^2 - p_+^2 + p_-^2)x^2$.

The description of the action on $P_{p_-,p_+,q}$ has $d_\pm = 2$, so we can construct a $U(2)$-invariant metric g with $\sec \geq 0$ as above. We check that g has a 2-plane with $\sec > 0$ orthogonal to the orbit of $T^2 \subset U(2)$ at each point. We do this on each half $D_\pm = G \times_{K_\pm} D^2$ separately. By the O'Neill formula it is necessary to find such a 2-plane orthogonal to the orbit of $T^2 \times K_\pm$ at each point of $G \times D^2$. For D_- we have $\mathfrak{g} = \mathfrak{u}(2) \oplus \mathfrak{su}(2)$, $\mathfrak{k} = \mathfrak{p} = \mathrm{span}\{(p_- i, i)\}$ and $\mathfrak{h} = \{0\}$. Here $\{i, j, k\}$ is the standard basis of $\mathfrak{su}(2)$ and $\{l, i, j, k\}$ is the standard basis of $\mathfrak{u}(2)$ with l the generator of the center.

Since T^2 and K_- act on G on the left and right respectively, the tangent space to the orbit at each point $(y, z) \in G \times D^2$ is contained in

$$dL_y(\mathfrak{k}) + dR_y(\mathfrak{u}(2) \oplus \{0\}) + T_z D^2.$$

Here L_y and R_y designate left and right translation on G. Since $(0,j)$ and $(0,k)$ are orthogonal to $\mathfrak{k}$ and $\mathfrak{u}(2)\oplus\{0\}$ with respect to the left invariant metric $\overline{Q}_a$ and $\mathfrak{u}(2)\oplus\{0\}$ is Ad-invariant, $dL_y(0,j)$ and $dL_y(0,k)$ are orthogonal to the orbit of $T^2\times K_\pm$. Choose $\tau_{[y,z]}$ to be the image of $dL_y(0,j)\wedge dL_y(0,k)$. By the O'Neill formula

$$\sec_g(\tau_{[y,z]}) \geq \frac{3}{4}|[dL_y(0,j),dL_y(0,k)]^V|^2_{\bar{g}} \geq 3|dL_y(0,i)^V|^2_{\bar{g}} > 0$$

where $dL_y(0,i)^V$ is the projection of $dL_y(0,i)$ onto $dL_y(\mathfrak{k})$. The same argument can be made on D_+ using $dL_y(0,i)\wedge dL_y(0,k)$.

To summarize, $P_{p_-,p_+,q}$ is the $U(2)$ principal bundle over $\mathbb{C}P^2$ with Chern classes $c_1 = qx$ and $c_2 = \frac{1}{4}(q^2 - p_+^2 + p_-^2)x^2$ and it admits a $U(2)$ invariant metric g and a 2-plane $\tau_{\bar{x}}$ at each point $\bar{x}\in P_{p_-,p_+,q}$ with $\tau_{\bar{x}}\perp T^2\cdot\bar{x}$ and $\sec g(\tau_{\bar{x}})>0$. In particular, $P_{2t,1-2t,1} = P_t$ and $P_{2t-1,2t+1,2} = \bar{P}_{2t}$. The metric $g^t_{a,b}$ is defined such that g and $g^t_{a,b}$ make $P_t\to P_t/S^1_{a,b}$ into a Riemannian submersion. Let g' be the locally isometric lift of g to the universal cover $\bar{P}'_{2t}$ of $\bar{P}_{2t}$. Note that the $T^2\subset U(2)$ action on $\bar{P}_{2t}$ lifts to the $T^2\subset S^1\times S^3$ action on $\bar{P}'_{2t}$. Then $\bar{g}^{2t}_{a,b}$ is defined such that g' and $\bar{g}^{2t}_{a,b}$ make $\bar{P}'_{2t}\to\bar{P}'_{2t}/S^1_{-b,a}$ into a Riemannian submersion.

On each manifold, the image σ_x of $\tau_{\bar{x}}$ under the S^1 quotient will be orthogonal to the orbits of T^2/S^1. Using the O'Neill formula once more it follows that $\sec(\sigma_x)>0$ with respect to $g^t_{a,b}$ and $\bar{g}^{2t}_{a,b}$. We note that these metrics are invariant under the centralizer of S^1, which is isomorphic to $S^1\times S^3$ in each case. The groups acting effectively by isometries are $S^1\times SO(3)$ and $U(2)$ respectively.

$\square$

Lemma 3.3.1 yields the metrics required to calculate the s invariant for the two families of S^1 bundles.

3.3.1 S^1 bundles over spin S^2 bundles over $\mathbb{C}P^2$

Let $\pi : \bar{E}^3 \to \mathbb{C}P^2$ be the 3-plane bundle associated to $S^2 \to \bar{N}_t \xrightarrow{\bar{p}} \mathbb{C}P^2$ and i the inclusion $i : \bar{N}_t \to \bar{E}^3$. Then $T\bar{N}_t$ and the normal bundle V to $\bar{N}_t$ span $i^*(T\bar{E}^3)$. Since $T\bar{E}^3 \cong \pi^*(\bar{E}^3 \oplus T\mathbb{C}P^2)$ and V is trivial we have

$$w_2(T\bar{N}_t) = \bar{p}^*(w_2(\bar{E}^3) + w_2(\mathbb{C}P^2)) = x.$$

Next let E^2 be the 2-plane bundle associated to $\bar{M}^t_{a,b}$ and $\bar{W}^t_{a,b} \subset E^2$ the disc bundle with projection $\sigma : \bar{W}^t_{a,b} \to \bar{N}_t$. We have the bundle isomorphism $T\bar{W}^t_{a,b} \cong \sigma^*(E^2 \oplus T\bar{N}_t)$ and second Stiefel Whitney class

$$w_2(T\bar{W}^t_{a,b}) = w_2(E^2) + w_2(T\bar{N}_t) = e(\bar{M}^t_{a,b}) + w_2(T\bar{N}_t) = (a+b+1)x + by \mod 2.$$

Here the notation σ^* is repressed since it is an isomorphism on cohomology. Thus when b is even, a is odd, and $\bar{W}^t_{a,b}$ is a spin manifold. From the Gysin sequence one sees that $H^4(\bar{M}^t_{a,b}, \mathbb{Z})$ is torsion so all the rational Pontryagin classes vanish.

Therefore for t and b even, using Lemma 3.3.1 $(\bar{M}^t_{a,b}, \bar{g}^t_{a,b})$ satisfies the hypotheses of Theorem 3.1.1 and Theorem 2.1.4, and $s(\bar{M}^t_{a,b}, \bar{g}^t_{a,b})$ is given by (2.1.8).

In [EZ] it was shown that for $\bar{W}^t_{a,b}$ we have

$$p_1^2 = b\left(-\frac{(3+4t)^2}{a^2 - b^2 t} + 6 + 8t + 3a^2 + tb^2\right)$$

and

$$\operatorname{sign}(\bar{W}^t_{a,b}) = \begin{cases} 0 & \text{if } a^2 - tb^2 > 0 \\ 2 & \text{if } a^2 - tb^2 < 0 \text{ and } b(1+t) > 0 \\ -2 & \text{if } a^2 - tb^2 < 0 \text{ and } b(1+t) < 0 \end{cases}.$$

Thus for t and b even

$$s(\bar{M}^t_{a,b}, \bar{g}^t_{a,b}) = \frac{b}{2^7 \cdot 7}\left(\frac{(3+4t)^2}{a^2 - b^2 t} - 6 - 8t - 3a^2 - tb^2\right) + \frac{1}{2^5 \cdot 7}\operatorname{sign}(\bar{W}^t_{a,b}). \quad (3.3.2)$$

When b is even, $\bar{M}^t_{a,b}$ has the cohomology appropriate to calculate the Kreck-Stolz diffeomorphism invariants $s_1, s_2, s_3 \in \mathbb{Q}/\mathbb{Z}$ [KS2]. These invariants are calculated in [EZ]:

$$s_1(\bar{M}^t_{a,b}) = s(\bar{M}^t_{a,b}, g^t_{a,b}) \mod \mathbb{Z}$$

$$\begin{aligned} s_2(\bar{M}^t_{a,b}) = &-\frac{1}{2^4 \cdot 3}(b(n^2 + tm^2) - 2anm) \\ &-\frac{1}{2^4 \cdot 3 \cdot (a^2 - tb^2)}(4nm(an^2 + atm^2 + 2tbnm) \\ &-(3 + 4t - 2n^2 - 2tm^2)(bn^2 + btm^2 + 2anm)) \end{aligned}$$

and

$$\begin{aligned} s_3(\bar{M}^t_{a,b}) = &-\frac{1}{2^2 \cdot 3}(b(n^2 + tm^2) - 2anm) \\ &-\frac{1}{2^3 \cdot 3 \cdot (a^2 - tb^2)}(16nm(an^2 + atm^2 + 2tbnm) \\ &-(3 + 4t - 8n^2 - 8tm^2)(bn^2 + btm^2 + 2anm)) \end{aligned}$$

where m, n are such that $ma + nb = 1$.

Now set $r = a^2 - 8tb^2$, $\lambda = 2^7 \cdot 3 \cdot 7r$ and choose m, n such that $ma + 2nb = 1$. We then define

$$a_k = a + 16b^2\lambda k,\ b_k = b,\ t_k = t + 4a\lambda k + 32b^2\lambda^2 k^2,\ m_k = m,\ n_k = n - 8b\lambda mk.$$

We see that $a_k^2 - 8t_k b_k^2 = r$, $m_k a_k + 2n_k b_k = 1$, and each of a_k, b_k, m_k, n_k, t_k is equal to the corresponding a, b, m, n, t mod λ. When $r < 0$ we have $t, t_k > 0$, so $2b_k(1+2t_k)$ has the same sign as $2b(1+2t)$. It follows that sign$(\bar{W}^{2t_k}_{a_k,2b_k})$ =sign$(\bar{W}^{2t}_{a,2b})$. This is enough to ensure the numerators of $s_i(M^{2t}_{a,2b})$ and $s_i(M^{2t_k}_{a_k,2b_k})$ are equal modulo the denominators so $s_i(M^{2t}_{a,2b}) - s_i(M^{2t_k}_{a_k,2b_k}) \in \mathbb{Z}$. Thus the invariants $s_i \in \mathbb{Q}/\mathbb{Z}$ and $|H^4(M,\mathbb{Z})|$ are equal and $M^{2t_k}_{a_k,2b_k}$ is diffeomorphic to $M^{2t}_{a,2b}$ by [KS2] Theorem 3.1. Since $a_k^2 - 8t_k b_k^2$ and sign$(\bar{W}^{2t_k}_{a_k,2b_k})$ are constant for the sequence $\{M^{2t_k}_{a_k,2b_k}\}_k$, the s invariant is a polynomial in k, and there is an infinite subsequence of metrics with distinct s invariants. Lemma 2.1.10 completes the proof of Theorem B part (b).

3.3.2 S^1 bundles over non-spin S^2 bundles over $\mathbb{C}P^2$

Let $\pi : E^3 \to \mathbb{C}P^2$ be the 3-plane bundle associated to $p : N_t \to \mathbb{C}P^2$. By the bundle isomorphism of the previous section we have

$$w_2(TN_t) = p^*(w_2(E^3) + w_2(\mathbb{C}P^2)) = p^*(2x) = 0 \text{ mod } 2.$$

Thus we can give $M^t_{a,b}$ the spin structure induced from the bundle isomorphism $TM^t_{a,b} \cong \rho^* TN_t \oplus V'$, where V' is the bundle generated by the S^1 action field and $\rho : M^t_{a,b} \to N_t$. From the Gysin sequence one sees that $H^4(M^t_{a,b},\mathbb{Z}) = \mathbb{Z}_{|t(a+b)^2 - ab|}$, so the rational Pontryagin classes vanish when $t(a+b)^2 - ab \neq 0$.

By Lemma 3.3.1 $g^t_{a,b}$ satisfies the conditions of Theorem 3.1.2. It follows that $s(M^t_{a,b}, g^t_{a,b}) = s(M^t_{a,b}, h)$ for an S^1 invariant metric h with geodesic fibers. Then the circle bundle $M^t_{a,b} \to N_t$ and h satisfy the hypotheses of Theorem 2.1.6 and $s(M^t_{a,b}, g^t_{a,b})$ is given by (2.1.9).

In [EZ] the terms p_1^2, $p_1 e^2$ and e^4 are calculated for $W^t_{a,b}$ and we have

$$s(M^t_{a,b}, g^t_{a,b}) = \frac{(a+b)(1-t)^2}{2^3 \cdot 7 \cdot (t(a+b)^2 - ab)}$$

$$+\frac{1}{2^5 \cdot 3 \cdot 7}(-3ab + (1-t)(8 + (a+b)^2)) + \frac{1}{2^5 \cdot 7}\mathrm{sign}(W^t_{a,b})$$

where

$$\mathrm{sign}(W^t_{a,b}) = \begin{cases} 0 & \text{if } ab - t(a+b)^2 < 0 \\ 2 & \text{if } ab - t(a+b)^2 > 0 \text{ and } a+b > 0 \\ -2 & \text{if } ab - t(a+b)^2 > 0 \text{ and } a+b < 0 \end{cases} .$$

When $t(a+b)^2 \neq ab$, $M^t_{a,b}$ also has the cohomology ring necessary to define the diffeomorphism invariants s_i. They are calculated in [EZ] Proposition 5.2. Just as for $\bar{M}^t_{a,b}$ they are given by rational functions with numerators depending on a, b, m, n, t and $\mathrm{sign}(W^t_{a,b})$, where m, n are such that $ma+nb = 1$. The denominators divide $2^5 \cdot 3 \cdot 7 \cdot |t(a+b)^2 - ab|$. As these are the only relevant details, we omit the equations for brevity.

Let $r = t(a+b)^2 - ab \neq 0$, $\lambda = 2^5 \cdot 3 \cdot 7r$ and choose m, n such that $ma+nb = 1$. We set

$$a_k = a + (a+b)^2\lambda k,\ b_k = b - (a+b)^2\lambda k,\ t_k = t - (a-b)\lambda k - (a+b)^2\lambda^2 k^2,$$

$$m_k = m + (n-m)(a+b)\lambda k,\ \ n_k = n + (n-m)(a+b)\lambda k.$$

One checks that $t_k(a_k+b_k)^2 - a_k b_k = r$, $m_k a_k + n_k b_k = 1$, $a_k + b_k = a + b$ and each of a_k, b_k, m_k, n_k, t_k is equal to the corresponding a, b, m, n, t mod λ. It follows that $M^{t_k}_{a_k,b_k}$ is diffeomorphic to $M^t_{a,b}$ while $s(M^{t_k}_{a_k,b_k}, g^{t_k}_{a_k,b_k})$ is a polynomial in k. This completes the proof of Theorem B.

Remark 3.3.3. (a) One easily sees that $W^7_{a,b}$ is diffeomorphic to only finitely many other $W^7_{k,l}$, and no other homogeneous spaces. By [STa] Proposition 1.1 the space of G invariant metrics with nonnegative sectional curvature on a homogeneous space G/H is connected. Thus $\mathfrak{M}_{\sec\geq 0}(W^7_{a,b})$ has infinitely many components by the corollary, but only finitely many of them contain homogeneous metrics. Each of those in turn contains a positively curved metric, except in the case of $W_{1,0}$. There are examples due to [KS3] where one has two components containing metrics with $\sec > 0$.

(b) One sees from the diffeomorphism invariants that no two of the Eschenburg spaces $F_{a,b}$ are diffeomorphic, so we cannot use this set of metrics to prove that any $\mathfrak{M}_{\sec>0}(F_{a,b})$ is not path connected.

(c) We saw in the proof Lemma 3.3.1 that $S^1 \times SO(3)$ and $U(2)$ respectively act by isometries on $g^t_{a,b}$ and $\bar{g}^{2t}_{a,b}$, and we suspect each is the full identity component of the isometry group.

(d) The same argument as in Remark 3.2.2 shows that $M^t_{a,b}$ and $\bar{M}^{2t}_{a,b}$ do not have the homotopy type of a 7-dimensional homogeneous space if $|t(a+b)^2 - ab|$ or

$|a^2 - 2tb^2| = 2 \bmod 3.$

Chapter 4

5-Manifolds

4.1 Diffeomorphism Classification of 5 Manifolds with $\pi_1 = \mathbb{Z}_2$

Our proof relies on a diffeomorphism classification of five manifolds with fundamental group $\mathbb{Z}_2$ carried out by Hambleton and Su [HS]. Given a manifold M with fundamental group $\mathbb{Z}_2$, a characteristic submanifold $P \subset M$ is defined as follows. Let $f : M \to \mathbb{R}P^N$ be a classifying map of the universal cover, for N sufficiently large. We can choose f to be transverse to $\mathbb{R}P^{N-1}$, and hence $P = f^{-1}(\mathbb{R}P^{N-1})$ is a smooth manifold. If furthermore $w_2(TM) \neq 0$ and $w_2(T\tilde{M}) = 0$, where $\tilde{M}$ is the universal cover of M, then P is well defined up to $\mathrm{Pin}^+(4)$ cobordism. Here $\mathrm{Pin}^{\pm}(4)$ is the extension of $O(4)$ by $\mathbb{Z}_2$ such that a preimage of a reflection squares to ± 1. and $\Omega_n^{\mathrm{Pin}^{\pm}}$ is the cobordism group of $n-$manifolds with $\mathrm{Pin}^{\pm}(n)$ structures.

For details, see [HS], [GT].

Theorem 4.1.1. *[HS] Let M_1, M_2 be 5 manifolds such that $\pi_1(M_i) = \mathbb{Z}_2$, $\pi_1(M_i)$ acts trivially on $\pi_2(M_i)$, $H_2(M_i, \mathbb{Z})$ is torsion free, $w_2(M_i) \neq 0$, and $w_2(\tilde{M}_i) = 0$ for $i = 1, 2$, where $\tilde{M}_i$ is the universal cover of M_i. Then M_1 is diffeomorphic to M_2 if and only if*

$$\mathrm{rank}(H_2(M_1, \mathbb{Z})) = \mathrm{rank}(H_2(M_2, \mathbb{Z}))$$

and

$$[P_1] = \pm[P_2] \in \Omega_4^{\mathrm{Pin}^+}$$

where P_i is the characteristic submanifold of M_i.

We are intersted in 5-manifolds which arise as the total space of principal circle bundles. Let B be a simply connected 4-manifold, d a primitive element of $H^2(B, \mathbb{Z})$ such that $w_2(TB) = d \bmod 2$, and $S^1 \to M \xrightarrow{\pi} B$ the principal S^1 bundle with first chern class $2d$. From the long exact sequence of a fibration and the Gysin sequence one sees that $\pi_1(M) = \mathbb{Z}_2$, $\pi_1(M)$ acts trivially on $\pi_2(M)$, and $H_2(M, \mathbb{Z})$ is torsion free with $\mathrm{rank}(H_2(M, \mathbb{Z}) = \mathrm{rank}(H_2(B, \mathbb{Z})) - 1$. Furthermore, $w_2(TM) = \pi^*(w_2(TB)) = \pi^* d \bmod 2 \neq 0$. The universal cover $\tilde{M}$ is the total space of the S^1 bundle with first chern class d, and by a similar argument $w_2(T\tilde{M}) = 0$. Thus we can apply Theorem 4.1.1 to M.

$\Omega_4^{\mathrm{Pin}^+}$ is isomorphic to $\mathbb{Z}_{16}$ with generator $[\mathbb{R}P^4]$, while $\Omega_2^{\mathrm{Pin}-}$ is isomorphic to

$\mathbb{Z}_8$ with generator $[\mathbb{R}P^2]$. There is a short exact sequence

$$0 \to \mathbb{Z}_2 \to \Omega_4^{\mathrm{Pin}^+} \xrightarrow{\phi} \Omega_2^{\mathrm{Pin}^-} \to 0$$

where ϕ is given by taking the cobordism class of a submanifold dual to w_1; see [HS] and [KT] for details. Hambleton and Su proved the following under this isomorphism.

THEOREM 4.1.2. *[HS] Let B be a closed simply connected four manifold and $S^1 \to M \xrightarrow{\pi} B$ the principal S^1 bundle with first chern class $2d$, where d is a primitive element such that $w_2(TB) = d$ mod 2. If P is a characteristic submanifold of M, then*

$$\phi([P]) = \pm \left\langle d^2, [B] \right\rangle \in \mathbb{Z}_8.$$

4.2 S^1 bundles

In this section we prove Theorem C. We want to use the Atiyah-Patodi-Singer index theorem to calculate the η invariant of a metric on M. Many authors have computed η and related invariants on spin manifolds M by extending metrics to manifolds W with boundary diffeomorphic to M. If the extension has positive scalar curvature, the index of the Dirac operator will vanish. In the spin^c case, we must also extend an auxiliary connection. A difficulty arises when the extended connection cannot be flat because the canonical class on the spin^c structure on W is not torsion. Then the metric and connection must satisfy (2.0.2). The following theorem, which we prove in Section 4.3, illustrates how to use certain free S^1 actions on M to accomplish this.

THEOREM 4.2.1. *Let S^1 act freely on M by isometries of a Riemannian metric g_M with scal(g_M) > 0 and assume $\pi_1(M)$ is finite. Let $B = M/S^1$ the quotient and $\rho : W = M \times_{S^1} D^2 \to B$ the associated disc bundle. Suppose the first Chern class of the principal S^1 bundle $\pi : M \to B$ is ℓd for $d \in H^2(B, \mathbb{Z})$ and $\ell \in \mathbb{Z}$. If λ is the complex line bundle over W with first Chern class $\rho^* d$, then there exists a metric g_W on W and a connection ∇ on λ such that*

$$\mathrm{scal}(g_W) > l|F^{\nabla}|_{g_W}. \tag{4.2.2}$$

Furthermore there is a collar neighborhood $V \cong M \times [0, N]$ of $\partial W \cong M$ such that for $t \in [0, N]$ near 0, g_W is a product metric

$$g_W \cong g_M + dt^2 \tag{4.2.3}$$

and

$$\nabla \cong proj^*_{V,M} \bar{\nabla} \tag{4.2.4}$$

where $\bar{\nabla}$ is any flat unitary connection on $\lambda|_{\partial W}$.

Notice that here there are no restrictions on the dimension of M or Pontryagin classes of M, d need not be primitive, and no spin^c structure is required. We next use Theorem 4.2.1 and (2.0.4) to calculate η for S^1 invariant metrics on certain spin^c manifolds in dimensions $4n+1$, proving Theorem 2.2.3 from Section 2, which we restate here.

THEOREM. *Let S^1 act freely on a $4n+1$ manifold M by isometries of a Riemannian metric g with scal$(g) > 0$. Assume $\pi_1(M)$ is finite and let $B = M/S^1$ be the quotient. Suppose the first Chern class of the principal bundle $S^1 \to M \xrightarrow{\pi} B$ is ℓd where ℓ is a positive even integer and $w_2(TB) = d \bmod 2$. Finally assume the real Pontryagin classes of M vanish. Then M admits a spin^c structure with canonical class $\pi^* d$. If $\bar{\nabla}$ is a flat connection on the canonical bundle of this spin^c structure and $D^c_{g,\tilde{\nabla}}$ is the spin^c Dirac operator, then*

$$\eta(D^c_{g,\tilde{\nabla}}) = \left\langle \frac{\sinh(d/2)\hat{A}(TB)}{\sinh(\ell d/2)}, [B] \right\rangle.$$

When $n = 1$,

$$\eta(D^c_{g,\tilde{\nabla}}) = \left\langle -\frac{(\ell^2-1)d^2 + p_1(TB)}{24\ell}, [B] \right\rangle.$$

Proof. Since TM is the direct sum of $\pi^* TB$ and a trivial bundle generated by the action field of the S^1 action,

$$w_2(TM) = \pi^* w_2(TB) = \pi^* d \bmod 2$$

Let μ be the complex line bundle over B associated to $\pi : M \to B$. Let $W = M \times_{S^1} D^2$ and let $\rho : W \to B$ be the disc bundle associated to $\pi : M \to B$. Then $TW = \rho^*(TB \oplus \mu)$ and since ℓ is even

$$w_2(TW) = \rho^*(d + \ell d) \bmod 2 = \rho^* d \bmod 2.$$

It follows that W admits a spin^c structure with canonical class ρ^*d. We call the canonical bundle λ. The spin^c structure on W induces one on M with canonical class π^*d.

Then M, W, and λ satisfy the hypotheses of Theorem 4.2.1. We construct the metric g_W on W and connection ∇ on λ as in the theorem such that $g_W|_M = g_M$ and $\nabla|_M = \bar{\nabla}$. Define the spin^c Dirac operator $D^c_{g_W,\nabla}$ on W and $D^c_{g_M,\bar{\nabla}}$ as in Section 2. Given that g_W and ∇ are product-like near ∂W, we can apply (2.0.4). Since $\bar{\nabla}$ is flat

$$\mathrm{scal}(g_M) > 2|F^{\bar{\nabla}}|_{g_M} = 0$$

and by (4.2.2)

$$\mathrm{scal}(g_W) > \ell|F^{\nabla}|_{g_W} \geq 2|F^{\nabla}|_{g_W}$$

Then (2.0.1) implies that $\mathrm{ind}(D^c_{g_W,\nabla}) = 0$ and $\ker(D^c_{g_M,\bar{\nabla}}) = \{0\}$. It follows from (2.0.4) that

$$\eta(D^c_{g_M,\bar{\nabla}}) = 2\int_W e^{c_1(\nabla)/2}\hat{A}(p(g_W)). \tag{4.2.5}$$

To evaluate that integral, we use Lemma 2.1.3. Let $\alpha = e^{c_1(\nabla)/2}$ and $\beta = \hat{A}(p(g_W))$. Since g_W is product like near the boundary, $p_i(g_W)|_{\partial W} = p_i(g_M)$. For $i > 0$ $p_i(g_M)$ is exact by the assumption on the Pontryagin classes of M. Since $c_1(\nabla)|_{\partial W} = c_1(\bar{\nabla})$ and $\bar{\nabla}$ is flat, we can choose $\hat{\alpha} = 0$. The form $c_1(\nabla)$ represents the cohomology class $c_1(\lambda) = \rho^*d$. Thus

$$\eta(D^c_{g,\bar{\nabla}}) = 2\left\langle j^{-1}\left[e^{\rho^*d/2}\right] \cup j^{-1}\left[\hat{A}(TW)\right], [W, \partial W]\right\rangle$$

The following cup product diagram commutes:

$$\begin{array}{ccc} H^s(W,\partial W)\oplus H^t(W,\partial W) & \xrightarrow{\cup} & H^{s+t}(W,\partial W) \\ \big\downarrow{\scriptstyle (\mathrm{Id},j)} & & \big\downarrow \\ H^s(W,\partial W)\oplus H^t(W) & \xrightarrow{\cup} & H^{s+t}(W,\partial W) \end{array}$$

Thus

$$\eta(D^c_{g,\bar\nabla}) = 2\left\langle j^{-1}\left[e^{\rho^*d/2}\right]\cup\left[\hat{A}(TW)\right], [W,\partial W]\right\rangle. \tag{4.2.6}$$

Since the terms of $\hat{A}(TW)$ have degree $4k$, $k\in\mathbb{Z}$, and the dimension of W is $4n+2$, only terms of degree $4k+2$ in $e^{\rho^*d/2}$ will contribute. In those degrees, $e^{\rho^*d/2}=\sinh(\rho^*d/2)$ as power series.

Since $TW=\rho^*(TB\oplus\mu)$, $\hat{A}(TW)=\rho^*(\hat{A}(TB)\hat{A}(\mu))$. For the complex line bundle μ, we have

$$\hat{A}(\mu)=\frac{c_1(\mu)/2}{\sinh(c_1(\mu)/2)}=\frac{\ell d}{2\sinh(\ell d/2)}$$

as a formal power series. The series $\sinh(d/2)$ is divisible by d, so

$$\rho^*\left(\frac{\sinh(d/2)}{\ell d}\right)\in H^*(W,\mathbb{Q}).$$

Let $\Phi\in H^2(W,\partial W,\mathbb{Z})$ be the Thom class of $\rho: W\to B$. Then $j(\Phi)=\rho^*c_1(\mu)=\rho^*(\ell d)$. By means of another commutative diagram

$$\begin{array}{ccc} H^*(W,\partial W)\oplus H^*(W) & \xrightarrow{\cup} & H^*(W,\partial W) \\ \big\downarrow{\scriptstyle (j,\mathrm{Id})} & & \big\downarrow{\scriptstyle j} \\ H^*(W)\oplus H^*(W) & \xrightarrow{\cup} & H^*(W) \end{array}$$

we see

$$j\left(\Phi\cup\rho^*\left(\frac{\sinh(d/2)}{\ell d}\right)\right)=\rho^*\left(\ell d\cup\frac{\sinh(d/2)}{\ell d}\right)=\rho^*\sinh(d/2).$$

Substituting into (4.2.6)

$$\eta(D^c_{g,\bar\nabla}) = 2\left\langle \Phi \cup \rho^*\left(\frac{\sinh(d/2)}{\ell d}\right) \cup \rho^*\left(\frac{\hat{A}(TB)\cdot \ell d}{2\sinh(\ell d/2)}\right), [W,\partial W]\right\rangle.$$
$$= \left\langle \Phi \cup \rho^*\left(\frac{\sinh(d/2)\hat{A}(TB)}{\sinh(\ell d/2)}\right), [W,\partial W]\right\rangle.$$

The Thom isomorphism yields

$$\eta(D^c_{g,\bar\nabla}) = \left\langle \frac{\sinh(d/2)\hat{A}(TB)}{\sinh(ld/2)}, [B]\right\rangle.$$

When $n = 1$ the dimension of B is four and we have, as series in $H^*(B,\mathbb{Z})$

$$\hat{A}(TB) = 1 - \frac{p_1(TB)}{24}$$
$$\frac{\sinh(d/2)}{\sinh(ld/2)} = \frac{1}{\ell}\left(1 - \frac{(\ell^2-1)d^2}{24}\right).$$

Multiplying and isolating terms of degree 4 yields (2.2.4). $\square$

To prove Theorem C, let Let

$$\{z_1, ..., z_a, w_1, ..., w_b\} \subset H^2(B,\mathbb{Z}) \cong \mathbb{Z}^{a+b}$$

be a basis such that $\{z_i\}$, and $\{y_i\}$ each correspond to the standard basis of a factor $H^2(\mathbb{C}P^2,\mathbb{Z})$ or $H^2(\overline{\mathbb{C}P^2},\mathbb{Z})$ respectively. Then

$$w_2(TB) = \sum_{i=1}^{a} z_i + \sum_{j=1}^{b} w_j \bmod 2$$

and so

$$d = \sum_{i=1}^{a} u_i z_i + \sum_{j=1}^{b} v_j w_j$$

for some odd integers u_i, v_j. For each $q \in \mathbb{Z}$ let $d_q = d + 4qz_1$ (if $a = 0$, replace z_1 with w_1 and adjust the proof accordingly). Note that $w_2(TB) = d_q \bmod 2$ for all q. Let $S^1 \to M_q \xrightarrow{\pi_q} B$ be the principal bundle with first Chern class $2d_q$, so $M_0 = M$.

The discussion after Theorem 4.1.1 indicates that Theorem 4.1.1 applies to each M_q. Using the Gysin sequence and Universal Coefficient Theorem, we see that

$$\operatorname{rank}(H_2(M_q, \mathbb{Z})) = a + b - 1.$$

If P_q is a characteristic submanifold of M_q, Theorem 4.1.2 implies that

$$\phi([P_q]) = \pm \left\langle d_q^2, [B] \right\rangle \bmod 8 = \pm \left(\sum_{i=1}^{a} u_i^2 + 8u_1 q + 16q^2 - \sum_{j=1}^{b} v_j^2 \right) \bmod 8$$
$$= \pm(a - b) \bmod 8 = 4 \bmod 8,$$

since each z_i^2, w_k^2 generates $H^4(B, \mathbb{Z})$, $z_i^2 = -w_j^2$, $z_i z_j = w_i w_j = 0$ for $i \neq j$, and $z_i w_j = 0$. Thus, since $\phi : \mathbb{Z}_{16} \to \mathbb{Z}_8$ is the quotient map, $[P_q] = \pm 4 \in \mathbb{Z}_{16}$. It follows from Theorem 4.1.1 that the manifolds M_q are all diffeomorphic. Let $\phi_q : M \to M_q$ be such a diffeomorphism.

The tangent bundle TM is isomorphic to the direct sum of $\pi^* TB$ and a trivial bundle generated by the S^1 action, so

$$w_2(TM) = \pi^* w_2(TB) = \pi^* d \bmod 2.$$

Thus M admits a spin^c structure with canonical class $\pi^* d$. Using ϕ_q we give each M_q a spin^c structure such that ϕ_q is a spin^c diffeomorphism. One sees from the Gysin sequence that $\pi^* d$ and $\pi_q^* d_q$ are the unique nontrivial torsion elements of $H^2(M, \mathbb{Z})$

and $H^2(M_q, \mathbb{Z})$ respectively. Then $\phi_q^*\pi_q^* d_q = \pi^* d$ and $\pi_q^* d_q$ is the canonical class of the spin^c structure on M_q.

Sha and Yang [SY2] constructed a metric with Ric> 0 on B. Gilkey, Park, and Tuschmann [GPT] showed that if B^n admits Ric> 0 and M is the total space of a principal bundle with compact connected structure group G and $\pi_1(M)$ is finite, then M admits a G invariant metric with Ric> 0. Thus each M_q admits an S^1 invariant metric g_q such that $\mathrm{Ric}(g_q) > 0$ and hence $\mathrm{scal}(g_q) > 0$.

Using the Gysin sequence again it follows that $H^4(M_q, \mathbb{R}) = 0$ and M_q, g_q, and B satisfy the hypotheses of Theorem 2.2.3 with $M = M_q$, $g_M = g_q$, $d = d_q$, $\ell = 2$, and $\bar{\nabla} = \nabla^q$ where ∇^q is any flat connection on the canonical bundle of the spin^c structure. Then, using the spin^c diffeomorphism invariance and (2.2.4) we have

$$\eta(D^c_{\phi_q^* g_q, \phi_q^* \nabla^q}) = \eta(D^c_{g_q, \nabla_q}) = -\frac{1}{16}\left(\sum_{i=1}^{a} u_i^2 + 8qu_1 + 16q^2 - \sum_{j=1}^{b} v_i^2 + a - b\right)$$

using the fact that $\langle p_1(TB)/3, [B]\rangle = \langle L(TB), [B]\rangle$ is equal to the signature of B.

Thus $\eta(D^c_{\phi_q^* g_q, \phi_q^* \nabla^q})$ is a nontrivial polynomial in q and takes on infinitely many values. Theorem 2.2.2 implies that $\mathfrak{M}_{\mathrm{Ric}>0}(M)$ and $\mathfrak{M}_{\mathrm{scal}>0}(M)$ have infinitely many components, completing the proof of Theorem C. □

If $a - b \neq 4 \bmod 8$, the diffeomorphism result of [HS]does not uniquely determine the diffeomorphism type, but a manifold with the same properties nonetheless exists.

THEOREM 4.2.7. *Let* $B = \#^a\mathbb{C}P^2\#^b\overline{\mathbb{C}P^2}$ *with* $a + b > 1$. *Then there exists a* *principal bundle $S^1 \to M \to B$ such that $\mathfrak{M}_{Ric>0}(M)$ and $\mathfrak{M}_{scal>0}(M)$ have infinitely many path components.*

Proof. We proceed as in the proof of Theorem C. Hence $\phi([P_q]) = \pm(a-b)$ mod 8. Since $a - b \neq 4$ mod 8, however, this does not uniquely identify $\pm[P_q]$, which is $\pm(a-b)$ or $\pm(a-b)+8$ in $\mathbb{Z}_{16}$. It follows that the manifolds M_q fall into two diffeomorphism classes. Thus some M_{q_0} is diffeomorphic to infinitely many other M_q, and the remainder of the proof of Theorem C implies that $\mathfrak{M}_{\mathrm{Ric}>0}(M_{q_0})$ and $\mathfrak{M}_{\mathrm{scal}>0}(M_{q_0})$ have infinitely many components. □

4.3 Metric and Connection on Disc Bundles

In this section we prove Theorem 4.2.1. We first set up notation for the tangent space to W. We consider D^2 to be the unit disc in $\mathbb{C}$. Let $\sigma : M \times D^2 \to W$ be the quotient map so $\sigma(p, x) = [p, x]$. Then $\rho([p, x]) = \pi(p)$. The metric g_M and the S^1 action induce an orthogonal splitting $T_pM = \bar{H}_p \oplus \bar{V}_p$ into horizontal space $\bar{H}_p$ and vertical space $\bar{V}_p$ of π. Define horizontal and vertical spaces of ρ to be

$$H_{[p,x]} = \sigma_*(\bar{H}_p \oplus \{0\})$$

and

$$V_{[p,x]} = \sigma_*(\{0\} \oplus T_x D^2)$$

for $p \in M$ and $x \in D^2$.

These is well defined since for $z \in S^1$, $\bar{H}_{zp} = z_*\bar{H}_p$ and $T_{zx}D^2 = z_*T_xD^2$. One can use a local section of σ to see that $H_{[p,x]}$ and $V_{[p,x]}$ are smooth distributions on W. Note that $V_{[p,x]}$ is the tangent space to the fiber $\rho^{-1}(\pi(p)) = \sigma(\{p\} \times D^2)$ and $T_{[p,x]}W = H_{[p,x]} \oplus V_{[p,x]}$. Away from the zero section of ρ, $V_{[p,x]}$ is spanned by

$$W_r = \sigma_*(0, \partial_r) \text{ and } W_\theta = \sigma_*(0, \partial_\theta).$$

These are well defined smooth vector fields since ∂_θ, ∂_r are S^1 invariant vector fields on D^2.

Fix $0 < L < 1$ and define a diffeomorphism

$$\tau : M \times [L, 1] \hookrightarrow M \times D^2 \xrightarrow{\sigma} W$$

of $M \times [L, 1]$ to a collar neighborhood U of ∂W. Let t be the coordinate on $[L, 1]$ and, in a slight abuse of notation, let $\mathrm{proj}_{U,M} : M \times [L, 1] \to M$ be the projection. Thus

$$\rho \circ \tau = \pi \circ \mathrm{proj}_{U,M}$$

$$\tau_*(\bar{H}_p \oplus \{0\}) = H_{[p,x]}$$

$$\tau_*(0, \partial_t) = W_r$$

Let $X^*(p) = \frac{d}{dt}\big|_{t=0} e^{it} \cdot p$ be the action field of the S^1 action on M, which spams $\bar{V}_p$. Then, since $\sigma_*(X^*, \partial_\theta) = 0$,

$$\tau_*(X^*, 0) = -W_\theta.$$

Furthermore $\tau|_{M\times\{1\}}$ identifies M and ∂W, sending $\bar{H}_p$ to $H_{[p,1]}$ and X^* to $-W_\theta$.

We keep track of the maps in the following diagram.

$$\begin{tikzcd}
 & M \times D^2 \arrow[d, "\sigma"] \\
M \times I \arrow[ur, hook] \arrow[r, "\tau"] \arrow[d, "\mathrm{proj}_{U,M}"'] & W \arrow[d, "\rho"] \\
M \arrow[r, "\pi"] & B
\end{tikzcd}$$

To construct g_W and ∇ we will use two smooth functions on the interval $[0,1]$. Let $f_1 : [0,1] \to [0,1]$ be a smooth monotone function which is 0 in a neighborhood of 0 and 1 in a neighborhood of $[L,1]$.

For a constant $\epsilon > 0$, let

$$f_2(r) = -\frac{1}{2}\int_0^r f_1(t)dt - \epsilon r^3 + r.$$

One easily sees that $f_2 > 0$ on $(0,1]$ for small ϵ.

4.3.1 Metric

We define a Riemannian metric at a point $(p,(r,\theta)) \in M \times D^2$, where r,θ are polar coordinates on D^2, by

$$g_{M\times D^2}(p,(r,\theta)) = g_M(p) + \epsilon^2 |X^*(p)|^2_{g_M}\left(dr^2 + \frac{f_2(r)^2}{1-\epsilon^2 f_2(r)^2}d\theta^2\right).$$

By converting to Cartesian coordinates on D^2 , one sees that $g_{M\times D^2}$ is smooth as long as

$$\frac{1}{r^4}\left(\frac{f_2^2}{1-\epsilon^2 f_2^2} - r^2\right)$$

is a smooth function of $r \in [0,1]$. This is easily seen to hold since for r near 0, $f_2(r) = r - \epsilon r^3$. Since $g_{M\times D^2}$ is invariant under the diagonal action of S^1 on $M \times D^2$, it induces a metric g_W on W such that $g_{M\times D^2}$ and g_W make σ into a Riemannian submersion. Similarly, let g_B be the metric on B such that g_M and g_B make π into a Riemannian submersion.

LEMMA 4.3.1. *g_W and g_B make ρ into a Riemannian submersion.*

Proof. With respect to $g_{M\times D^2}$, $\bar{H}_p \oplus \{0\}$ is orthogonal to X^* and TD^2. Thus $\bar{H}_p \oplus \{0\}$ is orthogonal to the vertical space of σ, which is spanned by (X^*, ∂_θ), and to the horizontal projection of TD^2 as well. It follows that with respect to g_W, $H_{[p,x]}$ is orthogonal to $V_{[p,x]}$ and is the horizontal space of ρ. Finally, we have

$$g_W|_{H_{[p,x]}} \cong g_{M\times D^2}|_{\bar{H}_p\oplus\{0\}} \cong g_M|_{\bar{H}_p} \cong g_B|_{T_{\pi(p)}B}.$$

$\square$

We first describe the induced metric on the D^2 fibers of ρ.

LEMMA 4.3.2.

$$g_W|_{\rho^{-1}(\pi(p))} \cong \epsilon^2 |X^*(p)|_{g_M} \left(dr^2 + f_2(r)^2 d\theta^2\right)$$

Proof. $\sigma|_{\{p\}\times D^2} : D^2 \to \rho^{-1}(\pi(p))$ is a diffeomorphism such that $\partial_r, \partial_\theta$ are mapped to W_r, W_θ. Since σ is a Riemannian submersion with vertical space generated by (X^*, ∂_θ), we calculate

$$|W_r|^2_{g_W} = |(0,\partial_r)|^2_{g_{M\times D^2}} = \epsilon^2 |X^*|^2_{g_M}$$

$$\begin{aligned}
|W_\theta|^2_{g_W} &= |(0,\partial_\theta)|^2_{g_{M\times D^2}} - \frac{\langle (0,\partial_\theta),(X^*,\partial_\theta)\rangle^2_{g_{M\times D^2}}}{\langle (X^*,\partial_\theta),(X^*,\partial_\theta)\rangle_{g_{M\times D^2}}} \\
&= \epsilon^2 |X^*|^2_{g_M}\left(\frac{f_2(r)^2}{1-\epsilon^2 f_2(r)^2}\right) \\
-\epsilon^4 |X^*|^4_{g_M} &\left(\frac{f_2(r)^2}{1-\epsilon^2 f_2(r)^2}\right)^2 \left(\frac{1}{|X^*|^2_{g_M} + \epsilon^2 |X^*|^2_{g_M}\left(\frac{f_2(r)^2}{1-\epsilon^2 f_2(r)^2}\right)}\right) \\
&= \epsilon^2 |X^*|^2_{g_M} f_2(r)^2
\end{aligned}$$

$$\langle W_r, W_\theta\rangle_{g_W} = \langle (0,\partial_r),(0,\partial_\theta)\rangle_{g_{M\times D^2}} = 0.$$

□

We next modify g_W to have the desired product structure near ∂W. We use a technique of Wraith, which allows deformations of metrics with positive mean curvature at the boundary.

Lemma 4.3.3. *∂W has positive mean curvature with respect to an inward normal vector.*

Proof. Let $\bar{X}_i$ be local S^1-invariant vector fields extending an orthonormal frame of $\bar{H}_p$ and define $X_i = \sigma_*(\bar{X}_i, 0)$. At a point $[p,1]$, $\{X_i, \frac{1}{\epsilon |X^*|_{g_M} f_2} W_\theta\}$ is an orthonormal basis of $T\partial W$ and $-\frac{1}{\epsilon |X^*|_{g_M}} W_r$ is an inward pointing unit normal vector. Since

$$[X_i, W_r] = [\sigma_*(\bar{X}_i,0), \sigma_*(0,\partial_r)] = \sigma_*[(\bar{X}_i,0),(0,\partial_r)] = 0$$

and $|X_i| = 1$,

$$\frac{1}{\epsilon|X^*|_{g_M}} \langle \nabla_{X_i} X_i, -W_r \rangle = \frac{1}{\epsilon|X^*|_{g_M}} \langle X_i, \nabla_{X_i} W_r \rangle = \frac{1}{\epsilon|X^*|_{g_M}} \langle X_i, \nabla_{W_r} X_i \rangle = 0.$$

Thus

$$\frac{1}{\epsilon^3|X^*|^3_{g_M} f_2(1)^2} \langle \nabla_{W_\theta} W_\theta, -W_r \rangle = \frac{1}{2\epsilon^3|X^*|^3_{g_M} f_2(1)^2} W_r(|W_\theta|^2) = \frac{f_2'(1)}{\epsilon|X^*|_{g_M} f_2(1)}.$$

Evaluating that quantity at $r = 1$ we see the the mean curvature is

$$\frac{1/2 - 3\epsilon}{\epsilon|X^*|_{g_M} f_2(1)} > 0$$

for sufficiently small ϵ. □

We see that $g_W|_{\partial W}$ is obtained from g_M by shrinking the S^1 fibers of π, a process which preserves positive scalar curvature.

Lemma 4.3.4. *There exists a smooth path of metrics $g_M(s)$ on M, $s \in [\epsilon^2 f_2(1)^2, 1]$, such that $g_M(\epsilon^2 f_2(1)^2) = g_W|_{\partial W}$, $g_M(1) = g_M$, and scal$(g_M(s)) > 0$ for all s.*

Proof. We recall that $\tau|_{M\times\{1\}} : M \to \partial W$ is a diffeomorphism. We see that

$$\left(\left(\tau|_{M\times\{1\}}\right)^* g_W\right)|_{\bar{H}_p} = g_W|_{H_{[p,1]}} = g_M|_{\bar{H}_p}$$

and

$$|X^*(p)|^2_{\left(\tau|_{M\times\{1\}}\right)^* g_W} = |W_\theta([p,1])|^2_{g_W} = \epsilon^2 f_2(1)^2 |X^*(p)|^2_{g_M}.$$

Thus defining

$$g_M(s) = g_M|_{\bar{H}_p} + s g_M|_{\bar{V}_p}$$

we have, for ϵ small enough, that $\epsilon^2 f_2(1)^2 < 1$, $g_M(\epsilon^2 f_2(1)^2) = \left(\tau|_{M\times\{1\}}\right)^* g_W$, and $g_M(1) = g_M$. Since the metric is not changing on the horizontal space of π, each $g_M(s)$ makes π into a Riemannian submersion with g_B. The O'Neil formula [Bes] then implies

$$\mathrm{scal}(g_M(s)) = \mathrm{scal}(g_B) - s|A_\pi|^2 - |T_\pi|^2 - |N_\pi|^2 - 2\delta N_\pi \geq \mathrm{scal}(g_M) > 0$$

where A_π, T_π, N_π are the tensors defined for the Riemannian submersion π with respect to g_M. $\square$

Use the normal exponential map from ∂W to define a collar neighborhood $V \cong M \times [0, N]$, where $t \in [0, N]$ is the distance to ∂W. We choose N small such that $V \subset U$. Using this identification, g_W has the form

$$g_W = g(t) + dt^2$$

where $g(t) = g_W|_{M\times t}$ is a smooth path of metrics on M. Since $g(0) = g_W|_{\partial W}$ has positive scalar curvature, we can choose N small such that $\mathrm{scal}(g(t)) > 0$ for all $t \in [0, N]$.

LEMMA 4.3.5. *We can alter g_W inside of V such that it is product like near ∂W with $g_W|_{\partial W} = g_M$ and scal$(g_W|_V) > 0$*

Proof. We use the paths $g_M(s)$ and $g(s)$ and the following lemma from [W3] to replace g_W near the boundary with a product metric restricting to g_M at the boundary.

LEMMA 4.3.6. *[W3] Let $g(t) + dt^2$ be a metric of positive scalar curvature on*

$M \times [0, N]$ such that $scal(g(t)) > 0$ and $M \times \{0\}$ has positive mean curvature with respect to the inward normal vector ∂_t. Let $\bar{g}(t)$ be a smooth path of metrics on M such that $scal(\bar{g}(t)) > 0$ for $t \in [0, N]$ and $\bar{g}(t) = g(t)$ for t in a neighborhood of N. Then there exists a function $\beta : [0, N] \to \mathbb{R}_+$ such that $\beta = 1$ for t in a neighborhood of N, $\beta = \beta(0)$ is constant for t in a neighborhood of 0, and $\bar{g}(t)+\beta(t)dt^2$ has positive scalar curvature.

To define our replacement path $\bar{g}$, we define two smooth functions.

$$\chi_1 : [0, N/2] \to [\epsilon^2 f_2(1)^2, 1]$$

such that $\chi_1(t) = 1$ for t near 0 and $\chi_1(t) = \epsilon^2 f_2(1)^2$ for t near $N/2$, and

$$\chi_2 : [N/2, N] \to [0, 1]$$

such that $\chi_2(t) = 0$ for$t$ near $N/2$ and $\chi_2(t) = t$ for t near N. We then define a smooth path of metrics

$$\bar{g}(t) = \begin{cases} g_M \circ \chi_1(t) & t \in [0, N/2] \\ g \circ \chi_2(t) & t \in [N/2, N] \end{cases} .$$

By Lemma 4.3.4 and the definition of g, $\mathrm{scal}(\bar{g}(t)) > 0$ for all t. Then Lemma 4.3.3 and Lemma 4.3.6 imply that $\bar{g}(t) + \beta(t)dt^2$ has positive scalar curvature for the function $\beta(t)$ given by Lemma 4.3.6. For t near N, $\bar{g}(t) = g(t)$ and $\beta(t) = 1$, so $\bar{g}(t) + \beta(t)dt^2 = g_W$. Thus replacing $g_W|_V$ with this metric results in a new smooth metric, for which we reuse the notation g_W. Since $\bar{g}(t) = g$ and $\beta(t)$ is constant for t near 0, $\bar{g}(t) + \beta(t)dt^2$ has the desired product structure (4.2.3). This proves Lemma 4.3.5.

□

4.3.2 Connection

Let $\beta \in \Omega^2(B)$ represent the image of ℓd in $H^2(B, \mathbb{R})$. The Gysin sequence for an S^1 bundle shows that $\pi^* ld = 0$, so we can choose $\alpha \in \Omega^1(M)$ such that $\pi^*\beta = d\alpha$. Since $\pi^*\beta$ is S^1 invariant, we can choose α to be S^1 invariant by averaging.

Lemma 4.3.7. $\alpha(X^*) = -\frac{1}{2\pi}$

Proof. Let $\Phi \in \Omega^2(W)$ be a Thom form of the disc bundle $\rho : W \to B$. Since

$$[\Phi] \mapsto \rho^* ld$$

under the long exact sequence map $H^2(W, \partial W) \to H^2(W)$, We have

$$\rho^*\beta - \Phi = d\bar{\alpha}$$

for some $\bar{\alpha} \in \Omega^1(W)$. Since Φ vanishes near ∂W,

$$d\bar{\alpha}|_M = \rho^*\beta|_M = \pi^*\beta = d\alpha.$$

Since $\pi_1(M)$ is finite, $\bar{\alpha}|_M - \alpha$ is exact. By the defining property of the Thom form, for any point $q \in B$, $\int_{\rho^{-1}(q)} \Phi = 1$. We use Stokes' theorem to compute

$$-1 = \int_{\rho^{-1}(q)} \rho^*\beta - \Phi = \int_{\rho^{-1}(q)} d\bar{\alpha} = \int_{\pi^{-1}(q)} \bar{\alpha} = \int_{\pi^{-1}(q)} \alpha = 2\pi\alpha(X^*).$$

□

We next construct a form $\gamma \in \Omega^1(W)$ extending $2\pi\alpha/\ell$. We first define a form $\bar{\gamma} \in \Omega(M \times D^2)$.

At $(p, x) \in M \times D^2$, $x \neq 0$, set

$$\bar{\gamma}|_{\bar{H}_p \times \{0\}} = \frac{2\pi}{\ell}\alpha_{\bar{H}_p} \quad \bar{\gamma}(X^*, 0) = -\frac{f_1(r)}{\ell}$$

$$\bar{\gamma}(0, \partial_r) = 0 \quad \bar{\gamma}(0, \partial_\theta) = \frac{f_1(r)}{\ell}.$$

where r is the radial coordinate on D^2. This form extends smoothly to the origin of D^2 since f_1 is zero in a neighborhood of $r = 0$. Since r, $\bar{H}_p \oplus \{0\}$, α, ∂_r, ∂_θ, X^* are all preserved by the S^1 action, $\bar{\gamma}$ is S^1 invariant. The vertical space of σ is generated by (X^*, ∂_θ), and so $\bar{\gamma}$ vanishes on the vertical space. It follows that there is a unique form $\gamma \in \Omega(W)$ such that $\sigma^*\gamma = \bar{\gamma}$.

Lemma 4.3.8. $\tau^*\gamma = \frac{2\pi}{\ell}\mathrm{proj}^*_{U,M}\alpha$

Proof. Recall that $f_1(r) = 1$ for r in the image of τ and note that $\tau^*\gamma = (\sigma^*\gamma)|_{M\times[L,1]} = \bar{\gamma}|_{M\times[L,1]}$. Thus:

$$\tau^*\gamma|_{\bar{H}_p\oplus\{0\}} = \bar{\gamma}|_{\bar{H}_p\oplus\{0\}} = \frac{2\pi}{\ell}\alpha_{\bar{H}_p}, \qquad \tau^*\gamma(X^*, 0) = \bar{\gamma}(X^*, 0) = -\frac{f_1(r)}{\ell} = \frac{2\pi}{\ell}\alpha(X^*)$$

$$\text{and} \quad \tau^*\gamma(0, \partial_t) = \bar{\gamma}(0, \partial_r) = 0 = \frac{2\pi}{\ell}\alpha(\mathrm{proj}_{M*}(0, \partial_t)).$$

□

Let λ_B be the complex line bundle with $c_1(\lambda_B) = d$. Given a differential form in the de Rahm cohomology class of $2\pi i$ times the first Chern class of a complex line bundle, there is a unitary connection on the line bundle whose curvature is that differential form. Thus, since β represents ℓd, let ∇_B be a unitary connection on λ_B with curvature

$$F^{\nabla_B} = \frac{2\pi i}{\ell}\beta.$$

We now define a connection on λ

$$\nabla = \rho^*\nabla_B - i\gamma.$$

LEMMA 4.3.9. *∇ is flat on U.*

Proof. We need to show that $F^{\tau^*\nabla} = 0$. Using Lemma 4.3.8 it follows that

$$\tau^*\nabla = \tau^*\rho^*\nabla_B - i\tau^*\gamma$$

$$= \mathrm{proj}^*_{U,M}\left(\pi^*\nabla_B - \frac{2\pi i}{\ell}\alpha\right)$$

and hence the curvature of the term in the parentheses is

$$\frac{2\pi i}{\ell}\pi^*\beta - \frac{2\pi i}{\ell}d\alpha = 0.$$

$\square$

We finish the construction of ∇ by modifying it so that it is product like near ∂W and restricts to $\bar{\nabla}$ at ∂W. Let $\mathrm{proj}^*_{V,M} : V \to M$ be the projection defined by the identification $V \cong M \times [0, N]$ from Section 4.3.1. Note that while $V \subset U$, $\mathrm{proj}_{V,M}$ and $\mathrm{proj}_{U,M}$ will not in general agree (the later was defined independently of h and the former using h.) Since $V \subset U$, ∇ is flat on V. Since $\mathrm{proj}_{V,M}$ and the inclusion of $\partial W \cong M \times \{0\}$ are homotopy inverses, $\mathrm{proj}^*_{M,V}(\lambda|_M) = \lambda|_V$. Thus $\nabla|_V$ and $\mathrm{proj}^*_{V,M}\bar{\nabla}$ are both flat unitary connections on $\lambda|_V$ and

$$\mathrm{proj}^*_{V,M}(\bar{\nabla}) - \nabla|_V = i\delta$$

for some closed form $\delta \in \Omega^1(V)$. Since $\pi_1(V) = \pi_1(M)$ is finite, $\delta = df$ for a smooth function f on V. We modify f to a function $\bar{f}$ which is equal to f near $\partial W \cong M \times \{0\}$ and equal to 0 near $M \times \{N\}$. We then replace ∇ with $\nabla + id\bar{f}$ on V. We see that ∇ is still smooth, flat on V, and near ∂W, $\nabla = \mathrm{proj}^*_{V,M}\bar{\nabla}$, satisfying (4.2.4)

4.3.3 Curvature

We complete the proof of Theorem 4.2.1 by showing that (4.2.2) holds. On V, ∇ is flat and by Lemma 4.3.5 $\mathrm{scal}(g_W) > 0$, so the inequality is satisfied. For the remainder of the proof we consider $W \backslash V$. Then $\mathrm{scal}(g_W)$ is given by Lemma 4.3.1 and the O'Neil formula for the scalar curvature of Riemannian submersion.

$$\mathrm{scal}(g_W) = \mathrm{scal}\left(g_W|_{\rho^{-1}(\pi(p))}\right) + \mathrm{scal}(g_B) - |A_\rho|^2 - |T_\rho|^2 - |N_\rho|^2 - 2\delta N_\rho.$$

As $\epsilon \to 0$, $|A_\rho| \to 0$, while the final three terms remain constant. By Lemma 4.3.2,

$$\mathrm{scal}\left(g_W|_{\rho^{-1}(\pi(p))}\right) = -\frac{2}{\epsilon^2 |X^*|^2_{g_M}} \left(\frac{f_2''}{f_2}\right).$$

Therefore, as $\epsilon \to 0$,

$$\mathrm{scal}(g_W) = -\frac{2}{\epsilon^2 |X^*|^2_{g_M}} \left(\frac{f_2''}{f_2}\right) + O(1).$$

Let $\bar{X}_i$ be an orthonormal basis of $\bar{H}_p$ with respect to g_M. Let $X_i = \sigma_*(\bar{X}_i, 0)$. Then $\{X_i\}$ is an orthonormal basis of $H_{[p,x]}$ with respect to g_W outside of V. Away from the zero section $\{\frac{1}{\epsilon|X^*|_{g_M}} W_r, \frac{1}{\epsilon|X^*|_{g_M} f_2} W_\theta\}$ is an orthonormal basis of $V_{[p,x]}$. Neither the $\bar{X}_i$ nor ∇ depend on ϵ. Then as $\epsilon \to 0$, using (2.0.3)

$$\left|F^\nabla\right|_{g_M} \leq \frac{1}{\epsilon^2 |X^*|^2_{g_M} f_2} |F^\nabla(W_r, W_\theta)| + \sum_i \frac{1}{\epsilon |X^*|_{g_M}} |F^\nabla(W_r, X_i)|$$

$$+ \frac{1}{\epsilon |X^*|_{g_M} f_2} |F^\nabla(W_\theta, X_i)| + O(1).$$

LEMMA 4.3.10. $F^\nabla(W_r, W_\theta) = -i f_1'(r)/\ell$, $F^\nabla(W_r, X_i) = F^\nabla(W_\theta, X_i) = 0$.

Proof. Since $\rho_* W_r = \rho_* W_\theta = 0$,

$$F^\nabla(W_r, W_\theta) = -i d\gamma(W_r, W_\theta) = -i d\gamma(\sigma_*(0, \partial_r), \sigma_*(0, \partial_\theta))$$

$$= -i \sigma^* d\gamma((0, \partial_r), (0, \partial_\theta)) = -i d\bar{\gamma}((0, \partial_r), (0, \partial_\theta)) = -i \partial_r \bar{\gamma}(0, \partial_\theta) = -i \frac{f_1'(r)}{\ell}$$

similarly

$$\mathrm{F}^\nabla(W_r, X_i) = -i d\bar{\gamma}((0, \partial_r), (\bar{X}_i, 0)) = -i \left(\partial_r \left(\frac{2\pi}{\ell} \alpha(\bar{X}_i) \right) - \bar{X}_i \left(\frac{f_1(r)}{\ell} \right) \right) = 0$$

and

$$\mathrm{F}^\nabla(W_\theta, X_i) = -i d\bar{\gamma}((0, \partial_\theta), (\bar{X}_i, 0)) = -i \left(\partial_\theta \left(\frac{2\pi}{\ell} \alpha(\bar{X}_i) \right) \right) = 0.$$

□

Lemma 4.3.10 implies that as $\epsilon \to 0$,

$$\mathrm{scal}(g_W) - \ell |F^\nabla|_{g_M} = \frac{1}{\epsilon^2 |X^*|^2_{g_M}} \left(\frac{-2f_2'' - f_1'}{f_2} \right) + O(1)$$

$$= \frac{12}{\epsilon|X^*|^2_{g_M}} \left(\frac{r}{f_2}\right) + O(1)$$

From the definition of f_2 one sees that $r/f_2 \to 1$ as $r \to 0$. It follows that we can choose ϵ small enough that (4.2.2) holds, completing the proof of Theorem 4.2.1. $\square$

In [KS3] Lemma 4.2, Kreck and Stolz constructed positive scalar curvature metrics on associated disc bundles in order to calculate their invariant for spin manifolds with free S^1 actions. In their proof, they needed to assume that the S^1 orbits were geodesics. The metric g_W constructed in Theorem 4.2.1 generalizes their method to a free isometric S^1 action without the geodesic condition.

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