Signature | Printed Name

Preliminary Examination, Part I

Monday, August 28, 2017 9:30am-12:00pm

This part of the examination consists of six problems. You should work all of the problems. Show all of your work. Try to keep computations well-organized and proofs clear and complete. Be sure to write your name both on the exam and on any extra sheets you may submit.

All problems have equal weight of 10 points.

1. Find an orthogonal basis of $\mathbb{R}^3$ that contains a basis of the span of $(1, 2, 3)$ and $(4, 5, 6)$.

2. For each positive integer n, define $f_n(x) = x^n$ for $0 \leq x \leq 1$.

   a) Is each function f_n uniformly continuous?

   b) Is the sequence of functions $\{f_n\}$ uniformly convergent?

   Justify your assertions.

3. a) How many abelian groups of order 108 are there, up to isomorphism?
   b) Are there any non-abelian groups of order 108? Either show that there aren't any or else give an example of one.

4. Let $\{a_1, a_2, \ldots\}$ be a sequence of real numbers such that $\sum_{n=1}^{\infty} a_n$ converges, but $\sum_{n=1}^{\infty} |a_n|$ does not converge. Let $b_1, b_2, \ldots$ be the positive terms among the a_n's, and let $c_1, c_2, \ldots$ be the negative terms.

   a) Prove that there are infinitely many terms b_i and infinitely many terms c_i.

   b) Prove that the series $\sum_{i=1}^{\infty} b_i$ diverges to ∞, and $\sum_{i=1}^{\infty} c_i$ diverges to $-\infty$.

   c) Let α be a real number. Show that there is some rearrangement of the terms a_n such that the sum of the rearranged series converges to α.

5. Let A denote the matrix

$$A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}.$$

a) Determine if there is a basis of $\mathbb{R}^2$ consisting of eigenvectors of A. If there is, find one.

b) Compute $A^{2017}u_0$, where $u_0 = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$.

(Hint: Do not try to compute this directly.)

6. a) Show that a closed subset of a compact topological space is compact.
   b) Show that a compact subset of a Hausdorff space is closed.

Extra page for work

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PRELIMINARY EXAMINATION, PART II

Monday, August 28, 2017 1:30pm-4:00pm

This part of the examination consists of six problems. You should work all of the problems. Show all of your work. Try to keep computations well-organized and proofs clear and complete. Please write your name on both the exam and any extra sheets you may submit.

All problems have equal weight.

7. Evaluate the contour integral

$$\oint_C \big(y^3 + 3x^2y + \cos(x^2)\big)\,dx + \big(x + e^{y^3}\big)\,dy,$$

where C is the unit circle $x^2 + y^2 = 1$ oriented counterclockwise.
(Hint: Some ways are easier than others.)

8. Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function such that $0 < f(x) < 1$ for all real numbers x. Show that $f''(x) = 0$ for some real number x.

9. Let $T : \mathbb{R}^n \to \mathbb{R}^k$ be a linear transformation, corresponding to a matrix A. Let T^* be the adjoint operator of T, corresponding to the transpose of A. Show that

$$\ker(T^*T) = \ker(T).$$

Here $\ker(T)$ denotes the kernel of T.
(Hint: Consider $\|Tx\|$.)

10. Using just the definition of the derivative, prove that every differentiable function $f : \mathbb{R} \to \mathbb{R}$ is continuous.

11. a) For which integers n is there a finite field whose additive group is cyclic of order n?

    b) For which integers n is there a finite field whose multiplicative group of invertible elements is cyclic of order n?

    Justify your assertions.

12. Find orthogonal trajectories for the family of plane curves E_c given by $4x^2+9y^2 = c$, for $c > 0$. That is, find a non-constant one-parameter family of curves D_t such that each D_t intersects each E_c orthogonally, wherever they meet.

Extra page for work