Premiliminary Examination, Part I & II

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Dept. of Mathematics
University of Pennsylvania

Problem 1. *Let V be the real vector space of continuous real-valued functions on the closed interval $[0,1]$ and let $w \in V$. For $p, q \in V$, define $\langle p, q\rangle = \int_0^1 p(x)q(x)w(x)\,\mathrm{d}x$.*

(a) Suppose that $w(\alpha) > 0$ for all $\alpha \in [0,1]$. Does this follow that the above defines an inner product on V? Justify your assertion

(b) Does there exist a choice of w such that $w(1/2) < 0$ and such that the above defines an inner product on V? Justify your assertion.

Solution. (a) Yes, when $w(a) > 0$ on $[0,1]$, $\langle p, q\rangle$ is an inner product over $\mathbb{R}$. We need verify four properties:

1. $\langle f+g, q\rangle = \langle f, q\rangle + \langle g, q\rangle$, from the distributive property and linearity of integration
2. $\langle cf, q\rangle = c\langle f, q\rangle$, from linearity of integration
3. $\langle f, g\rangle = \langle g, f\rangle$, from commutativity
4. $\langle f, f\rangle \geq 0$ and zero if and only if $f \equiv 0$

To see why the fourth property is true recall that if f is continuous and $f \geq 0$ then $\int f = 0 \iff f \equiv 0$. So, $\langle f, f\rangle = 0 \iff f^2 w = 0 \iff f^2 = 0 \iff f = 0$. And of course $\langle f, f\rangle \geq 0$ since the integral of non-negative functions is non-negative.

(b) No $\langle p, q\rangle$ is not an inner product, as property (4) fails. To see this, we will find a non zero function f such that $\langle f, f\rangle \leq 0$. From continuity of w we can find a $\delta > 0$ such that $w(a) < 0$ for all $a \in (1/2-\delta, 1/2+\delta)$. Define $f(x) = \left(1 - \left|\frac{2(x-1/2)}{\delta}\right|\right) 1_{\left[\frac{1-\delta}{2}, \frac{1+\delta}{2}\right]}(x)$, where $1_A(x)$ is the indicator function of A. As $f(\frac{1-\delta}{2}) = f(\frac{1+\delta}{2}) = 0$, the function f is indeed continuous. Since the function $f^2 w$ is non-positive on $[0,1]$, we conclude that $\langle f, f\rangle \leq 0$. ■

Problem 2. *Let $\{x_n\}$ be a sequence of real numbers (indexed by $n \geq 0$), and let $0 < c < 1$ be a real number. Suppose that*

$$|x_{n+1} - x_n| \leq c|x_n - x_{n-1}|$$

for all $n = 1, 2, 3, \ldots$

(a) If $n \geq k$ are positive integers, show that

$$|x_{n+1} - x_k| < \frac{c^k}{1-c}|x_1 - x_0|$$

(Hint: First bound $|x_{n+1} - x_n|$ in terms of $|x_1 - x_0|$.)

(b) Prove that the sequence $\{x_n\}$ converges to a real number.

Solution. (a) By induction we get $|x_{n+1} - x_n| \leq c|x_n - x_{n-1}| \leq c^n|x_1 - x_0|$, for all n. Now, by applying the triangle inequality and the previous bound we obtain

$$\begin{aligned}
|x_{n+1} - x_k| &= |(x_{n+1} - x_n) + (x_n - x_{n-1}) + (x_{k+1} - x_k)| \\
&\leq |x_1 - x_0| \sum_{m=k}^{n} c^m \\
&= c^k |x_1 - x_0| \sum_{m=k}^{n} c^{m-k} \\
&\leq c^k |x_1 - x_0| \sum_{u \geq 0} c^u \\
&= \frac{c^k}{1-c} |x_1 - x_0|
\end{aligned}$$

(b) It suffices to show that $\{x_n\}$ is Cauchy. Let $\epsilon > 0$. We pick N such that $\frac{c^N}{1-c}|x_1 - x_0| < \epsilon$. Suppose $n, m \geq N$, then from part (a) we have

$$\begin{aligned}
|x_n - x_m| &\leq \frac{c^{\min\{n,m\}}}{1-c} |x_1 - x_0| \\
&\leq \frac{c^N}{1-c} |x_1 - x_0| \\
&< \epsilon
\end{aligned}$$

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Problem 3. *(a) In the polynomial ring $\mathbb{Q}[x]$ consider the ideal I generated by $x^4 - 1$ and $x^3 - x$. Does I have a generator $f(x) \in \mathbb{Q}[x]$? Either find one or explain why none exists.*

(b) In the polynomial ring $\mathbb{Q}[x, y]$, do the same for the ideal generated by the polynomials x and y.

Solution. (a)Since $\mathbb{Q}$ is a field $\mathbb{Q}[x]$ is a PID, there is $f \in \mathbb{Q}[x]$ such that $I = (f)$. By finding the common roots of $x^4 - 1$ and $x^3 - x$, we obtain that $\gcd(x^4 - 1, x^3 - x) = x - 1$. Hence, there are $p, q \in \mathbb{Q}[x]$ such that

$$p(x)(x^4 - 1) + q(x)(x^3 - x) = x - 1. \tag{1}$$

We claim that the ideal I generated by $x^4 - 1$ and $x^3 - x$ is equal to the ideal J generated by $x - 1$. Since $x - 1 | x^4 - 1, x^3 - x$ we have that $x^4 - 1, x^3 - x \in J$, therefore, by minimality of I, $I \subset J$. Similarly, using equation (1), we conclude that $J \subset I$.

(b) The ideal I generated by x, y is not principal. We will argue by contradiction, and assume $I = (f)$ for some $f \in \mathbb{Q}[x, y]$. First notice that $f \notin \mathbb{Q}$, as I does not contain constant polynomials. Since $x, y \in I$, f divides both x and y; but x and y are non-associated irreducible elements of the ring, therefore f must be a unit, which gives a contradiction. ■

Problem 4. *For each of the following, give either a proof or a counterexample.*

(a) Let f be a continuous real-valued function on the open interval $0 < x < 3$. Must f be uniformly continuous on the open interval $1 < x < 2$?

(b) Suppose instead that f is only assumed to be continuous on the open interval $0 < x < 2$. Must f be uniformly continuous on the open interval $1 < x < 2$?

Solution. (a) Yes. The restriction of f on $[1, 2]$ is uniformly continuous since $[1, 2]$ is compact. Therefore, the restriction of f on $1 < x < 2$ must be uniformly continuous as well.

(b) Not necessarily. A counterexample is $f(x) = \frac{1}{2-x}$. To show this we will argue by contradiction. Assume f is uniformly continuous. Take $x_n = 2 - \frac{1}{n}$, then $f(x_{n+1}) - f(x_n) = (n+1) - n = 1$. This is a contradiction, since uniformly continuous functions map Cauchy sequences to Cauchy sequences. ■

Problem 5. *Let V, W be two-dimensional real vectorspaces, and let $f_1, \cdots, f_5$ be linear transformations from V to W. Show that there exist real numbers $a_1, \ldots, a_5$, not all zero, such that $a_1 f_1 + \cdots + a_5 f_5$ is the zero transformation.*

Solution. The vector space of linear transformations from V to W has dimension 4. So, $f_1, f_2, \ldots, f_5$ must be linearly dependent.

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Problem 6. *Evaluate $\oint\limits_C \left(e^{x^2} + \sin(y^2)\right) \mathrm{d}x + \left(2xy\cos(y^2) + xy^3\right) \mathrm{d}y$, where C is the triangle with vertices $(0,0), (1,-1), (1,1)$ oriented counterclockwise.*

Solution. Define C_I to be the interior of C. We will calculate the line integral by applying Green's theorem.

$$\begin{aligned}
\oint_C \left(e^{x^2} + \sin(y^2)\right) \mathrm{d}x + \left(2xy\cos(y^2) + xy^3\right) \mathrm{d}y &= \iint_{C_I} \frac{\partial\left(2xy\cos(y^2) + xy^3\right)}{\partial x} - \frac{\partial\left(e^{x^2} + \sin(y^2)\right)}{\partial y} \, \mathrm{d}A \\
&= \int_0^1 \int_{-x}^{x} y^3 \, \mathrm{d}y \, \mathrm{d}x \\
&= 0
\end{aligned}$$

■

Problem 7. *Let $f : X \to Y$ be a continuous map between metric spaces. For each of the following, give either a proof or a counterexample, using just the definition of compactness.*

(a) If $A \subset X$ is compact, the so is $f(A) \subset Y$.

(b) If $B \subset Y$ is compact, the so is $f^{-1}(B) \subset X$.

Solution. (a)This is true. Take an open cover of $f(A) \subset \bigcup_\alpha U_\alpha$. Then,

$$\begin{aligned}
A &\subset f^{-1}(f(A)) \\
&\subset f^{-1}\left(\bigcup_\alpha U_\alpha\right) \\
&= \bigcup_\alpha f^{-1}(U_\alpha),
\end{aligned}$$

so, since A is compact, we can find a finite cover of A, i.e. $A \subset \bigcup_{i=1}^n f^{-1}(U_i)$. Finally, applying f to the previous containment we obtain $f(A) \subset \bigcup_{i=1}^n U_i$, hence, by definition, $f(A)$ is compact.

(b) This is not true. For instance, suppose that $X = \mathbb{R}$, then for any constant function $f : \mathbb{R} \to Y$, $x \mapsto c$, the preimage $f^{-1}(c) = \mathbb{R}$ is not compact. ■

Problem 8. *Find a continuous function $f : \mathbb{R} \to \mathbb{R}$ and a constant A such that*

$$\int_0^x f(t)(1 + t^2) \, \mathrm{d}t = \cos(x^2) + A. \tag{2}$$

Solution. Plugging $x = 0$ into (2) yields $A = -1$. Take $f(t) = \frac{\left(\cos(x^2)\right)'}{1+t^2}$. We will verify that f satisfies (2). Substituting f into (2), the LHS becomes $\int_0^x \left(\cos(t^2)\right)' \mathrm{d}t = \cos(x^2) - 1$. ■

Problem 9. *For every integer $n > 1$, let U_n be the group of invertible elements of $\mathbb{Z}/n\mathbb{Z}$ under multiplication.*

(a) Find the orders of U_8 and U_9. Explain.

(b) Determine whether the groups U_8 and U_9 are cyclic.

Solution. (a) The multiplicative group U_n consists of the invertible elements of $\mathbb{Z}_n$. The invertible elements of $\mathbb{Z}_n$ are the numbers that are relatively prime to n. The number of the relatively prime elements is equal to $\phi(n)$, where ϕ is Euler's totient function, given by $\phi(n) = n\prod_{p|n}\left(1-\frac{1}{p}\right)$. Therefore $|U_8| = 2^3 - 2^2 = 4$, and $|U_9| = 3^2 - 3 = 6$.

(b) Using the fundamental theorem for abelian groups (FTAG) an abelian group of 4 elements is either $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4$. However, since all the elements $\{1, 3, 5, 7\}$ have order 2 in U_8, we conclude that $U_8 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, so U_8 is not cyclic . By FTAG an abelian group of order 6, is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_3 \cong \mathbb{Z}_6$ since $\gcd(2, 3) = 1$. Therefore, U_9 is cyclic. ■

Problem 10. *Let $f(x, y) = x^2 - xy + y^2 - y$.*

(a) Does the function f achieve an absolute maximum on $\mathbb{R}^2$? an absolute minimum on $\mathbb{R}^2$? If so, find all points where this occurs.

(b) Do the same with $\mathbb{R}^2$ replaced by the square $0 \le x, y \le 1$.

Solution. (a) As $\lim_{x\to\pm\infty} f(x, 0) = \infty$, the function f does not attain a global maximum. Observe that

$$f(x, y) = \left(x - \frac{y}{2}\right)^2 + \frac{3}{4}\left(y - \frac{2}{3}\right)^2 - \frac{1}{3}$$

Therefore, the global minimum is attained when $y - \frac{2}{3} = 0$ and $x - \frac{y}{2} = 0$. So there is a unique global minimum achieved at $(\frac{1}{3}, \frac{2}{3})$.

(b) Denote D the unit square $0 \le x, y \le 1$ and ∂D its boundary. Since D is compact f attains a global minimum and maximum. The function f achieves its extrema either in the interior of D, at points where ∇f is vanishing, or on ∂D. To find the critical points we solve $\nabla f(x, y) = 0 \iff 2x - y = 0, -x - 1 + 2y = 0 \iff x = \frac{1}{3}, y = \frac{2}{3}$. From part (a) $f_{|D}$ must achieve its global minimum at $(\frac{1}{3}, \frac{2}{3})$, consequently, from the previous calculation, f must achieve its maximum somewhere on ∂D. We partition $\partial D = \bigcup_{i=1}^{4} D_i$ where D_i correspond to the different edges of D, and calculate the points where the maximum occurs.

- $D_1 = \{(x, y) \in \partial D | y = 0\}$, since $f_{|D_1} = x^2$ we find $\arg\max f_{|D_1} = \{(1, 0)\}$ and $\max f_{|D_1} = 1$.
- $D_2 = \{(x, y) \in \partial D | y = 1\}$, since $f_{|D_2} = x^2 - x \le 0$ there are no maxima on D_2
- $D_3 = \{(x, y) \in \partial D | x = 0\}$, since $f_{|D_3} = y^2 - y \le 0$ there are no maxima on D_3
- $D_4 = \{(x, y) \in \partial D | x = 1\}$, since $f_{|D_4} = (y - 1)^2$ we find $\arg\max f_{|D_4} = \{(1, 0)\}$ and $\max f_{|D_4} = 1$

From the previous analysis we see that the max f on D is 1, and it is attained at $(1, 0)$. ■

Problem 11. *Let a, b, c be real numbers, and consider the matrix $A = \begin{pmatrix} a & b & c \\ b & c & b \\ c & b & a \end{pmatrix}$*

(a) Explain why all the eigenvalues of A must be real.

(b) Show that some eigenvalue λ of A has the property that for every vector $v \in \mathbb{R}^3$, $v \cdot Av \le \lambda ||v||^2$. (Note: You are not being asked to compute the eigenvalue of A.)

Solution. (a) Every symmetric matrix has real eigenvalues. Indeed, let v be an eigenvector and λ its corresponding eigenvalue. Then,

$$\begin{aligned} \lambda\langle v, v\rangle &= \langle \lambda v, v\rangle = \langle Av, v\rangle \\ &= \langle v, Av\rangle = \langle v, \lambda v\rangle \\ &= \overline{\lambda}\langle v, v\rangle \end{aligned}$$

Hence, $(\lambda - \overline{\lambda})||v||^2 = 0$. Therefore, λ is real.

(b) Since A is symmetric we can find an orthonormal basis v_1, v_2, v_3 consisting of eigenvectors of A. Now, let λ_i be the corresponding eigenvalues (non necessarily distinct). So, using orthogonality, we obtain

$$\begin{aligned}
\langle Av, v\rangle &= \langle A\left(\sum_{i=1}^{3} a_i v_i\right), \sum_{i=1}^{3} a_i v_i\rangle \\
&= \langle\sum_{i=1}^{3} a_i\lambda_i v_i, \sum_{i=1}^{3} a_i v_i\rangle \\
&= \sum_{i=1}^{3} \lambda_i a_i^2 ||v_i||^2 \\
&\leq \max \lambda_i \sum_{i=1}^{3} a_i^2 ||v_i||^2.
\end{aligned}$$

And since, $||v||^2 = \sum_{i=1}^{3} a_i^2 ||v_i||^2$ we conclude. ■

Problem 12. *Consider the differential equation $y^{(4)} - y = ce^{2x}$ where c is a real constant.*

(a) Let S_c be the set of solutions of this equation. For which c is this set a vector space? Why?.

(b) For each such c, find this solution space explicitly, and find a basis for it.

Solution. (a) For $c \neq 0$ the space S_c is not a vector space since the zero function is not a member of S_c. For $c = 0$, notice that S_c is the kernel of the linear transformation $L(f) = f^{(4)} - f$, hence S_c is a vector space as the subspace of a vector space.

(b) The characteristic polynomial of the ODE

$$y^{(4)} - y = 0 \tag{3}$$

is $\chi(r) = r^4 - r$. Solving $\chi(r) = 0$ we find 4 roots, $r_1 = 0$ and $r_j = e^{\frac{2ji\pi}{3}}$ for $0 \leq j \leq 2$. Hence, the vectors $e^{r_i t}$ constitute the desired basis. A different basis, using real valued function is $1, e^t, e^{\frac{t}{2}} \cos\left(\frac{\sqrt{3}t}{2}\right), e^{\frac{t}{2}} \sin\left(\frac{\sqrt{3}t}{2}\right)$. ■