Preliminary Examination, Part I

Tuesday morning, August 31, 2010

This part of the examination consists of six problems. You should work on all of the problems. Show all of your work in your workbook. Try to keep computations well-organized and proofs clear and complete. Justify the assertions you make. Be sure to write your name on each workbook you submit. All problems have equal weight.

1. Let M be a compact metric space and suppose $T : M \to M$ is a continuous function such that

$$d(T(x), T(y)) < d(x, y)$$

for every $x, y \in M$ such that $x \neq y$.

   (a) Prove that T has a unique fixed point. (Hint: Minimize $d(T(x), x)$.)

   (b) Prove that if $x_0 \in M$ is any point, then the sequence $\{x_n\}_{n=1}^{\infty}$, defined inductively by $x_1 = T(x_0)$, $x_{n+1} = T(x_n)$ converges to the fixed point.

2. Suppose $n \geq 2$ is an integer. Which of the following subsets of $G = GL(n, \mathbb{R})$ is a subgroup? Which subsets are normal subgroups? Explain.

   (a) $\{A \in G \,|\, \text{all entries of } A \text{ are integers}\}$

   (b) $\{A \in G \,|\, \det(A) > 0\}$

   (c) $\{A \in G \,|\, A = A^t\}$

   (d) $\{A \in G \,|\, A \cdot A^t = I\}$

3. Let $f(x) : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function.

   (a) Show that f can have at most countably many strict local maxima.

   (b) Assume that f is not monotone on any interval. Then show that the local maxima of f are dense in $\mathbb{R}$.

4. Show that the ring $\mathbf{Z}[[x]]$ (of formal power series $\Sigma_{i=0}^{\infty} a_i x^i$ with integers a_i) is an integral domain. Which elements in this ring are invertible (under multiplication)?

5. For any real number $\alpha \in \mathbb{R}$ define the sequence $\{x_n\}_{n \in \mathbb{N}}$ by $x_1 = \alpha$, $x_{n+1} = x_n^2 - 1$.

   (a) Prove that if the above sequence is convergent, then the limit has to be either $(1 + \sqrt{5})/2$ or $(1 - \sqrt{5})/2$.

   (b) Determine whether the sequence converges for *all* real numbers α , for *no* real numbers α, or for *some* but *not all* real numbers α.

6. Let $f : [0,1] \to [0,\infty)$ be a smooth function such that $f(0) = f(1) = 0$ and let C be the smooth curve from $(0,0)$ to $(1,0)$ given by the graph of f. Suppose that the line integral

$$\int_C (y+x)dx + ydy = \frac{3}{4}.$$

Find

$$\int_C y(e^{xy}+1)dx + xe^{xy}dy.$$

PRELIMINARY EXAMINATION, PART II

Tuesday afternoon, August 31, 2010

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7. Let D be a compact subset of $\mathbb{R}^2$ and let $f : D \to \mathbb{R}$ be a function. The graph of f is the set

$$G(f) = \{(x, f(x)) : x \in D\} \subset \mathbb{R}^3$$

Show that f is continuous if and only if $G(f)$ is compact.

8. For any real number c, let V_c be the set of solutiions to the differential equation $y'' - 4y' + 4y = c$.

   (a) Find all c such that V_c is a vector space. For each such c, find an explicit basis of V_c.

   (b) Show that if V_c is a vector space, then differentiation is a linear transformation $D : V_c \to V_c$. Find the matrix of D with respect to your basis in (a).

9. Suppose $x_1, x_2, x_3 \in \mathbb{R}^2$ are three points in the plane that do not lie on a line, and denote by T the closed triangle with vertices x_1, x_2 and x_3. Suppose $f : T \to \mathbb{R}$ is a continuous function which is differentiable on the interior of T and which vanishes on the boundary of T.

   Prove that there exists a point x in the interior of T such that the tangent plane to the graph of f at the point x is horizontal.

10. Let $R = (\mathbb{Z}/2)[x]$ and let I be the ideal $I = R \cdot (x^{17} - 1)$. Is there a non-zero element a of the quotient ring $A = R/I$ such that $a^2 = 0$?

11. Let $f : S^1 \to S^1$ be a continuous function from the unit circle to itself. Prove that if f is not onto, then f must have a fixed point.

12. Let T be the transpose map $T(A) = A^{tr}$ for $A \in M_2(\mathbb{R})$. Find the eigenvalues of T and a basis of $M_2(\mathbb{R})$ with respect to which T is diagonal.