Preliminary Examination, Part I

Tuesday morning, August 25, 2009

This part of the examination consists of six problems. You should work all of the problems. Show all of your work in your workbook. Try to keep computations well-organized and proofs clear and complete. Justify the assertions you make. Be sure to write your name on each workbook you submit. All problems have equal weight.

1. Define: $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } \ x = \frac{p}{q}, \text{ and } \frac{p}{q} \text{ is in lowest terms} \\ 0 & \text{if } \ x \ \text{ is irrational} \end{cases}$$

Prove that f is continuous at c if and only if c is irrational.

2. Let V be the real vector space of real polynomials of degree ≤ 5, and define $T : V \to \mathbb{R}^3$ by $T(f) = (f(0), f(1), f(2))$.

(a) Show that T is a linear transformation.

(b) Find the dimensions of the kernel and image of T, and find a basis for each.

3. Suppose that f and g are real-valued functions on $\mathbb{R}$ having period 1 (i.e. $f(x+1) = f(x)$ and $g(x+1) = g(x)$ for all x) and having continuous first derivatives.

(a) Prove that $f'(c) = g'(c)$ for some non-negative $c \in \mathbb{R}$.

(b) Prove that there is a smallest such value of c.

4. For each of the following, either give an example or prove that no such example exists.

(a) Three non-isomorphic abelian groups of order 60.

(b) A non-abelian group of order 95.

5. Show that the series

$$f(x) = \sum_{n=1}^{\infty} \sin(\frac{x}{n^2})$$

converges for every $x \in [0, \frac{\pi}{2}]$ and that it defines a function f that is continuous and monotone on this interval.

6. Let R be the region in the (x, y)-plane that lies below the graph of $y = \sin x$ and over the x-axis, between $x = 0$ and $x = \frac{\pi}{2}$. Let C be the boundary of R, oriented counterclockwise. Compute $\oint y(e^{xy} + 1)dx + xe^{xy}dy$.

Preliminary Examination, Part II

Tuesday afternoon, August 25, 2009

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7. Let (X, d) be a metric space, let $0 < c < 1$, and let $f : X \to X$ be a map with the property that

$$d(f(x), f(y)) \leq cd(x, y) \quad \text{for all} \quad x, y \in X.$$

(a) If (X, d) is complete, show that there is a unique $x_0 \in X$ such that $f(x_0) = x_0$.

(b) Does the conclusion of part (a) hold even without completeness? Give either a proof or a counterexample.

8. (a) Show that the matrix

$$A = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & c \end{pmatrix}$$

is diagonalizible over $\mathbb{C}$ for all θ, $c \in \mathbb{R}$.

(b) For which $\theta, c \in \mathbb{R}$ is A diagonalizible over $\mathbb{R}$?

9. For each of the following either give an example or prove that no such example exists.

(a) A continuous, bounded function on the open interval $(0, 1)$ that is not uniformly continuous.

(b) A continuous, periodic function on $\mathbb{R}$ that is not uniformly continuous.

10. Let R be an integral domain and let $R[[x]]$ be the ring of formal power series

$$\sum_{i=0}^{\infty} a_i x^i,$$

with $a_i \in R$.

(a) Prove that $R[[x]]$ is an integral domain.

(b) Prove that

$$\sum_{i=0}^{\infty} a_i x^i$$

is a unit in $R[[x]]$ if and only if a_0 is a unit in R.

11. Find a smooth curve in the plane passing through the point $(1, 0)$ such that for every point (a, b) on the curve, the tangent line to the curve at (a, b) intersects the y-axis at the point $(0, a)$.

12. For any $n \times n$ real matrix $A = (a_{ij})$, define $\|A\| = \max\{|a_{ij}| \mid 1 \leq i, j \leq n\}$.

(a) Show that $\|AB\| \leq n\|A\|\|B\|$ for all A and B; and show that $\|A^r\| \leq n^{r-1}\|A\|^r$ for all A and all $r \geq 1$.

(b) Deduce that for all A, the series

$$I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots$$

converges to an $n \times n$ matrix $e(A)$. (That is, the $(i,j)^{th}$ entry of the series converges for all i, j.)

(c) Suppose that $v \in \mathbb{R}^n$ is an eigenvector of A with eigenvalue λ. Show that v is an eigenvector of $e(A)$ and find its eigenvalue.