Preliminary Examination, Part I

August 28, 2007

This part of the examination consists of six problems. You should work all of the problems. Show all of your work in your workbook. Try to keep computations well-organized and proofs clear and complete. Justify the assertions you make. Be sure to write your name on each workbook you submit. All problems have equal weight.

1. Consider the system of linear equations

$$\begin{array}{rcrcrcrcl} x & + & y & + & 3z & + & w & = & a \\ x & + & 2y & + & 5z & + & 2w & = & b \\ 2x & + & y & + & 4z & + & w & = & c \end{array}$$

   a) Find the set of all triples of real numbers (a, b, c) for which the system has a solution (x, y, z, w).

   b) Find all solutions (x, y, z, w) if $a = b = c = 0$.

2. Let f be a twice continuously differentiable function on the real line. Suppose that $a < b < c$ and that $f(a) < f(c) < f(b)$. Show that there exist real numbers d, e in the interval $a < x < c$ such that $f'(d) = 0$ and $f''(e) < 0$.

3. Find all integers n that $\sqrt{n} \in \mathbb{Q}[\sqrt{2}]$. Justify your assertion.

4. a) Let $\{f_n\}$ be a sequence of continuous functions on the closed interval $[0, 1]$, such that for every $x \in [0, 1]$, $f_1(x) \geq f_2(x) \geq \ldots$ and $\lim_{n\to\infty} f_n(x) = 0$. Prove that $f_n \to 0$ uniformly on $[0, 1]$. (That is, show that

$$\sup\{|f_n(x)| : x \in [0, 1]\} \to 0 \text{ as } n \to \infty.)$$

   b) Show that if we instead consider functions on the *open* interval $(0, 1)$, then the corresponding assertion is false.

5. Find three complex numbers a, b, c such that $\mathbb{C}^3$ has an orthonormal basis consisting of eigenvectors of the linear transformation $T : \mathbb{C}^3 \to \mathbb{C}^3$ corresponding to the matrix

$$A = \begin{pmatrix} 1 & i & a \\ b & a & c \\ -a & b & -1 \end{pmatrix}.$$

   Explain.

6. Let R be the region in the first quadrant of the x, y-plane (where $x \geq 0, y \geq 0$) bounded between the axes and the curve $y = 1 - x^2$. Let C be the boundary of R, oriented counterclockwise. Evaluate the line integral $\oint_C (e^{x^2} + xy)dx + (\sin(e^y) + x^2)dy$.

Preliminary Examination, Part II

August 28, 2007

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7. Find all connected subsets of the rational numbers, $\mathbb{Q}$. Prove your assertion.

8. A certain smooth connected curve C in the x,y-plane has the property that it intersects *all* the hyperbolas $xy = a$ (for $a \in \mathbb{R}$, $a \neq 0$) at right angles, whenever they meet. If the curve C passes through the point $(1,1)$, at which point(s) does C meet the hyperbola $xy = 16$?

9. a) Find three non-isomorphic groups of order 42. Show that no two of your groups are isomorphic.

   b) Prove that *every* group G of order 42 has a normal subgroup other than G and the trivial group.

10. Using the *definition* of continuity, prove that the function $f(x,y) = xy$ is continuous at every point $(x,y) \in \mathbb{R}^2$.

11. Let A, B be $n \times n$ matrices over a field F.

    a) Show that the matrices AB and BA have the same trace.

    b) Deduce that if F has characteristic 0 and $AB - BA = cI$ for some $c \in F$, then $c = 0$.

    c) Does the conclusion in (b) necessarily hold if F has non-zero characteristic? Either give a proof that it does, or else give an example for which it does not.

12. Let $a < b$ be real numbers, and let f be a continuous function on the closed interval $[a,b]$. Prove that $\lim_{n\to\infty} \int_a^b f(x)\sin^n(x)dx = 0$. [Note: $\sin^n(x)$ means $(\sin(x))^n$.]