Preliminary Examination, Part I

Tuesday morning, August 31, 2004

This part of the examination consists of six problems. You should work all of the problems. Show all of your work in your workbook. Try to keep computations well-organized and proofs clear and complete. Justify the assertions you make. Be sure to write your name on each workbook you submit. All problems have equal weight.

1. Show *by induction* that for every positive integer n,

$$\sum_{k=1}^{n} \frac{1}{4k^2 - 1} = \frac{n}{2n+1}$$

2. Suppose f is twice continuously differentiable with positive second derivative (i.e. $f''(x) > 0$) in the closed interval $[0, 2]$. Suppose $f(0) = 1$, $f(1) = -1$ and $f(2) = 2$. Prove f has exactly two zeros in the open interval $(0, 2)$ (i.e. there are exactly two numbers $x_1, x_2 \in (0, 2)$ so that $f(x_1) = f(x_2) = 0$).

3. Let V be the real vector space consisting of polynomials $f(x) \in \mathbb{R}[x]$ having degree at most 5 (including the 0 polynomial).
   a) Find a basis for V, and determine the dimension of V.
   b) Define $T\colon V \to \mathbb{R}^6$ by $T(f) = (f(0), f(1), f(2), f(3), f(4), f(5))$. Show T is a linear transformation and find its kernel.
   c) Deduce that for every choice of $a_0, \cdots, a_5 \in \mathbb{R}$ there is a unique polynomial $f(x) \in \mathbb{R}[x]$ of degree at most 5 such that $f(j) = a_j$ for $j = 0, 1, \cdots, 5$.

4. Give an ϵ-δ proof that $f(x) = 1/x$ is continuous at $x = 1/2$.

5. For each of the following, either find an example or explain why none exists.
   a) A commutative ring R and two elements $a, b \in R$ whose only common divisors are units, but which do not generate the unit ideal of R.
   b) A positive integer n and a quadratic polynomial

   $$f(x) = x^2 + ax + b \in (\mathbb{Z}/n\mathbb{Z})[x]$$

   having more than two roots.
   c) A cubic polynomial $g(x) = x^3 + ax^2 + bx + c \in \mathbb{Z}[x]$ that has a root in $\mathbb{Q}$ but not in $\mathbb{Z}$.

6. a) Is there a metric space structure on the set $\mathbb{Z}$ such that the open sets are precisely the subsets $S \subset \mathbb{Z}$ such that $\mathbb{Z} - S$ is finite, and also the empty set?

b) Is there a metric space structure on the set $\mathbb{Z}$ such that every subset is open?

Justify your assertions.

PRELIMINARY EXAMINATION, PART II

Tuesday afternoon, August 31, 2004

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7. Let $\vec{F}$ be a vector field defined in $\mathbb{R}^3$ minus the origin defined by

$$\vec{F}(\vec{r}) = \frac{\vec{r}}{\| \vec{r} \|^3} = \frac{x\vec{i} + y\vec{j} + z\vec{k}}{(x^2 + y^2 + z^2)^{3/2}}$$

for $\vec{r} \neq 0$.

a) Compute $div\ \vec{F}$.

b) Let S be the sphere of radius 1 centered at $(x, y, z) = (2, 0, 0)$. Compute

$$\iint_S \vec{F} \cdot \vec{n} \quad dS.$$

8. Suppose f is a continuous function defined on the whole real line which is periodic with period one (so $f(x+1) = f(x)$ for all real x). Suppose

$$\int_0^1 f(x)dx = 1 \qquad \text{and} \qquad f(0) = 2.$$

Compute the limits

$$\lim_{c \to \infty} \int_0^1 f(cx)dx \qquad \text{and} \qquad \lim_{c \to 0} \int_0^1 f(cx)dx,$$

and justify your assertions.

9. Let G be a group of order 63.

a) Show that G contains a non-trivial normal subgroup N (i.e. $N \neq 1, G$).

b) Deduce that G is solvable.

10. Suppose $\{a_n \colon n = 1, 2, \cdots\}$ is a sequence of real numbers so that

$$\sum_{n=1}^{\infty} |a_n| = 1.$$

Let $f(x)$ be given by the cos series

$$f(x) = \sum_{n=1}^{\infty} a_n \cos(nx).$$

Prove that the series for f converges and that f is continuous.

11. Consider the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^4$ corresponding to the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 2 & -8 \\ 3 & 1 & 3 \\ 1 & -2 & 8 \end{bmatrix}$$

That is, if $\vec{v} = (x, y, z) \in \mathbb{R}^3$ then $T(\vec{v}) = \vec{w} = (a, b, c, d)$ where

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}.$$

a) Find the dimension of the range (image) of T, and find a basis for this range.

b) Find the dimension of the kernel of T and find a basis for this kernel.

12. Let V be the vector space of real-valued C^∞ functions on the closed interval $0 \le x \le 1$. If $f, g \in V$ define

$$f \cdot g = \int_0^1 f(x)g(x)dx.$$

a) Show that the pairing $f \cdot g$ defines an inner product on V.

b) Let $W \subset V$ be the subspace spanned by the functions 1, e^x and e^{-x}. Find an orthonormal basis for W with respect to the above inner product.