CLASS OF 1880 EXAM

(Math competition ONLY for UPenn Freshmen.)

April 18, 2017

Solve the problems in the space provided. Show your work and give justification for your answers as complete as possible. If you run out of room for an answer, continue on the back of the page, or on the last 2 pages – Extra Space.

Time available: 2 hours!

Full name: _____

Penn ID: _____

E-mail: _____

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

1. (10 points) Let a be a real number. Consider the equation (where x is the unknown)

$$4^{x^2-x} - a2^{x^2} - 4^x = 0.$$

Find the value(s) of a, for which this equation has exactly one real root.

2. (10 points) Let $a_1 = \frac{5}{20.17}\pi$ and define the sequence (a_n) via $a_{n+1} = (\sin a_n)^2$ for $n \geq 1$. Show that this sequence $(a_n)_{n\geq 1}$ converges and find its limit.

3. (10 points) Let $a_1, \ldots, a_{2017}$ be real numbers, such that

$$a_1 + a_2 + \cdots + a_{2017} = 2017.$$

Find the maximal possible number of pairs (i, j) $(1 \le i < j \le 2017)$, such that $a_i + a_j < 2$.

4. Let n be a positive integer less than 2014). Alice and Bob play the following game:

Alice chooses n distinct numbers among $2, 3, \ldots, 2017$, and then Bob chooses another 2 distinct numbers from the remaining ones, and all $n + 2$ so chosen numbers are written down in increasing order:

$$a_1 < a_2 < \cdots < a_{n+2}$$

Bob wins if there are two consecutive numbers in this sequence, i.e. a_i and a_{i+1}, for which a_i divides a_{i+1}. Alice wins if there are no such two consecutive numbers.

Example: say $n = 4$, Alice choses $2, 3, 40, 70$, Bob chooses $5, 90$, then the numbers in increasing order are $2 < 3 < 5 < 40 < 70 < 90$, and since $5 | 40$ are consecutive we have that Bob wins this round. If Bob chose $90, 100$ instead then the numbers are $2 < 3 < 40 < 70 < 90 < 100$ and there no two consecutive numbers where the smaller divides the larger, so Alice wins.

(a) (5 points) Show that if $n \geq 10$, Alice has a winning strategy.

(b) (5 points) Show that if $n \leq 9$, Bob can always win the game.

5. (10 points) Let M be a set of 2017 positive integers, and for each subset $A \subset M$ define a function $f(A)$ as the set of elements $m \in M$, for which the number of elements $a \in A$ dividing m, i.e. $\#\{a \in A : a|m\}$, is odd. For example, if $M = \{1, \ldots, 2017\}$ and $A = \{500, 1000, 2017\}$, then $f(A) = \{500, 1500, 2017\}$. Find the smallest number k of colors, for which we can color each nonempty subset $A \subset M$ in one of these k colors, such that whenever $A \neq f(A)$, then A and $f(A)$ always have distinct colors.

(We are looking for the universal k, that works for any set M.)

Extra space

Extra space