NONABELIANIZATION, SPECTRAL DATA AND CAMERAL DATA

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ABSTRACT

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This thesis surveys parts of the forthcoming joint work [23] in which the non-abelianization map of [16] was extended from the case of $G = SL(n)$ and $G = GL(n)$ to the case of arbitrary reductive algebraic groups. The non-abelianization map is an algebraic map from a moduli space of certain N-local systems on the complement of a divisor P in a punctured Riemann surface X, to the moduli space of G-local systems on X.

Contents

Chapter 1

Introduction

Non-abelianization refers to the "co-ordinate charts" in the form of maps from the moduli of certain $\mathbb{G}_m$-local systems on a curve $\overline{X}\backslash R$ (for a divisor $R \subset \overline{X}$) to the moduli of $GL(n)$ or $SL(n)$ local systems on X, where $\overline{X} \to X$ is an $n:1$ cover, and X is a punctured Riemann surface. These are referred to as coordinate charts because complex analytically one can describe the moduli space of $\mathbb{G}_m$-local systems via it's monodromy as $(\mathbb{G}_m)^a/\mathbb{G}_m$ for some number a, with the action of $\mathbb{G}_m$ being trivial[1]. These co-ordinate charts are conjectured to be hyperkähler [16, 13, 17] when restricted to holomorphic symplectic leaves of the Poisson moduli spaces of local systems, and allow one to describe wall crossing phenomena in the BPS spectrum of a $4d$ $\mathcal{N} = 2$ field theories of class S with gauge group $SL(n)$. These wall crossing phenomena were related to coordinate charts in the series of papers [18, 17, 16, 14, 13, 15]. This thesis surveys the extension of the non-abelianization construction to the setting of reductive algebraic groups, as developed in the joint work [23].

[1]We consider the stacky quotient, so this is not simply equivalent to $(\mathbb{G}_m)^a$

In the case where we consider non-abelianization for the group $SL(2)$, subject to a modification that involves working on the unit tangent bundle of a punctured Riemann surface, these coordinates correspond [11, 21, 17] to the Fock–Goncharov coordinates [12] on the complexification of Teichmuller space. An extension of these methods to compact surfaces for $G = SL(2), \mathbb{R})$ in [11] recovers Thurston's shear-bend coordinates [35].

The covers $\overline{X} \to X$ are *spectral covers* in the sense of [20], and are constructed from points in the Hitchin base for the group $GL(n)$ or $SL(n)$. Fibers of the Hitchin moduli space are described in terms of $\mathbb{G}_m$-bundles (line bundles) on the spectral curve in this setting. This was generalized in the series of works [8, 32, 10, 9]. The paper [9] describes fibers of the Hitchin system in terms of N-shifted, weakly W-equivariant T-bundles on a curve called the *cameral cover* $\tilde{X} \to X$ in the case where $\tilde{X}$ is smooth. The curve $\tilde{X}$ is constructed from a point in the Hitchin base.

Analogously to the case of the description of Hitchin fibers, we can extend the non-abelianization map of [16] to the setting of arbitrary reductive algebraic groups G by replacing the moduli space of certain $\mathbb{G}_m$-local systems on[2] $\overline{X} \backslash R_\rho$ to the moduli space of N-shifted weakly W-equivariant T-local systems on $\tilde{X} \backslash R$ (for $R \subset \tilde{X}$ the ramification divisor of $\tilde{X} \to X$). We have to reduce to a subset of such T-local systems that satisfy a condition which we call the S-monodromy condition introduced in 4.2.

We will now describe briefly how the non-abelianization construction works. The rough idea of non-abelianization is that we modify a local system on what is topologically a two dimensional surface by "cutting and regluing" along a series of one dimensional loci, that together form a piece of data called a *spectral network*. The following is a simplified

[2] R_ρ is here the ramificarion divisor of $\overline{X} \to X$

example

Example 1.0.1 (Non-abelianization by "cutting and regluing" on S^1.). Suppose we have a G-local system $\mathcal{E} \to S^1$. Let $[0,1] \xrightarrow{p} S^1$ be the surjective map such that $p(0) = p(1)$, and the preimage of every point other than $p(1)$ consists of a single element. Informally words p identifies $[0,1]$ with the circle S^1 being "cut" at the point $p(0)$. We can then consider $p^*\mathcal{E}$ as being the G-local system $\mathcal{E}$ "cut" at $p(0)$.

If we have an isomorphism of G-torsors $a : p^*\mathcal{E}|_0 \xrightarrow{\cong} p^*\mathcal{E}|_1$, then $p^*\mathcal{E}/\{p^*\mathcal{E}|_0 \xrightarrow{a} p^*\mathcal{E}|_1\}$ is a G-local system on S^1. Informally we refer to this procedure of gaining a new local system on S^1 as regluing. Note that if the automorphism we reglue by is the monodromy around the circle, and the direction we glue by is the opposite to the direction with respect to which we described the monodromy, we will get a G-local system with trivial monodromy, which can hence be extended to a G-local system on the disc D with boundary $\partial D = S^1$.

In the construction of non-abelianization, the types of G-local system we consider will be those that have been induced from N-local systems.

When we do this construction on the topological space underlying a punctured Riemann surface X the locus on which we "cut and reglue" can be more complicated. The locus we use is a *spectral network*, which [16] described how to construct from certain points in the Hitchin base for $G = GL(n)$ or $G = SL(n)$, and [27] described how to construct in the case of groups of type ADE, with the exception of $E8$. The cases of $GL(n)$ and $SL(n)$ spectral networks had already appeared in the setting of exact WKB analysis under the name of Stokes graphs, see e.g. [25, 24, 5, 34, 1, 2, 22, 3]. Unfortunately it is not entirely clear for $G \neq SL_2$, PGL_2, $GL(2)$ for which points in the Hitchin base the construction of a spectral network works. We do not resolve this point. We define spectral networks for arbitrary reductive algebraic groups G, starting from appropriate points in the Hitchin base in chapter 3 following [23].

Our main result follows:

Construction–Theorem 1.0.2 (See Construction–Theorem 5.15 of [23], and Constructions 4.3.9, Equation 4.3.1 and Theorem 4.3.10 of this document.)**.** *To an abstract spectral network (see definition 3.2.1) on a punctured Riemann surface X with cameral cover $\tilde{X}$, the maps s_{WKB} defined in equation 4.3.1, and the reglue map of Construction 4.3.9 provide a map*

$$Loc_N^{\tilde{X}\backslash R,S}(X\backslash P) \to Loc_G(X),$$

where $Loc_N^{\tilde{X},S}(X\backslash P)$ refers to the moduli of N-local systems on $X\backslash P$ corresponding to the W-bundle $\tilde{X}\backslash R \to X\backslash P$, satisfying a restriction on the monodromy called the S-monodromy condition defined in 4.2.2. Note that P and R are the branch and ramification divisors of the map $\tilde{X} \to X$, where $\tilde{X}$ is smooth.

The moduli of N-local systems used in Theorem 1.0.2 can instead be described in terms of T-local systems on the cameral cover $\tilde{X}\backslash R$, as the following theorem shows. We reference chapter 4 for precise definitions of the moduli spaces involved.

Theorem 1.0.3 (See Theorem 5.14 of [23], and Theorem 4.2.7 of this document.)**.** *For $\tilde{X}^\circ \to X^\circ$ a principal W-bundle coming from a smooth cameral cover $\tilde{X} \to X$, there is a commutative diagram:*

$$\begin{tikzcd}
Loc_N^{\tilde{X}^\circ}(X^\circ) \arrow[r, "\cong"] & Loc_T^N(\tilde{X}^\circ) \\
Loc_N^{\tilde{X}^\circ,S}(X^\circ) \arrow[r, "\cong"] \arrow[u] & Loc_T^{N,S}(\tilde{X}^\circ) \arrow[u]
\end{tikzcd}$$

Here $Loc_N^{\tilde{X}^\circ}(X^\circ)$ is the moduli space of N-local systems on X° which correspond to the W-bundle $\tilde{X}^\circ$, $Loc_N^{\tilde{X}^\circ,S}(X^\circ)$ is the moduli space of such N-local systems which also satisfy the S-monodromy condition. The space $Loc_T^N(\tilde{X}^\circ)$ is the moduli space of N-shifted, weakly W-equivariant T-local systems on $\tilde{X}^\circ$, and $Loc_T^{N,S}(\tilde{X}^\circ)$ is the moduli space of T-local systems which also satisfy the S-monodromy condition.

Finally in chapter 5 we consider how to describe these moduli spaces in spectral terms.

One of the results is the compatibility between the non-abelianization construction of Construction–Theorem 1.0.2, and the path detour non-abelianization construction of [16, 27], described here in construction 5.2.2, given by the following commutative diagram.

Theorem 1.0.4 (See Theorem 5.2.3 of this document, and Theorem 6.11 of [23])**.** *In the setting where both maps are defined, there is a commutative diagram:*

$$\begin{array}{ccc} Loc_N^{\tilde{X}^\circ,S}(X^\circ) & \xrightarrow{Sp_\rho} & Loc_{\mathbb{G}_m}(\overline{X}_\rho^{ne,\circ_R}) \\ \Big\downarrow{\scriptstyle nonab} & & \Big\downarrow{\scriptstyle nonab_{PD}} \\ Loc_G(X) & \xrightarrow{\rho} & Loc_{GL(V)}(X^\circ) \end{array} \tag{1.0.1}$$

where nonab is the nonabelianization map of Construction–Theorem 1.0.2, and $nonab_{PD}$ is the path detour nonabelianization map descibed in construction 5.2.2 following [27, 16].

We also give precise descriptions of the moduli spaces of N-local systems we describe in terms of the moduli space of $\mathbb{G}_m$-local systems on spectral covers for $GL(n)$, and $SL(n)$. These are summarized in the following theorem:

Theorem 1.0.5 (Spectral descriptions, see Propositions 5.3.5, 5.3.4, 5.3.7, 5.3.3 of this document and §6.4 of [23].)**.** *For $G = GL(n)$ and $G = SL(n)$, the moduli spaces of N-local systems $Loc_N^{\tilde{X}\backslash R}(X\backslash P)$ and $Loc_N^{\tilde{X}\backslash R,\alpha}(X\backslash P)$ are isomorphic to those in the below table 1.1.*

Group	$Loc_N^{\tilde{X}\backslash R}(X\backslash P)$	$Loc_N^{\tilde{X}\backslash R,\alpha}(X\backslash P)$
$GL(n)$	$Loc_{\mathbb{G}_m}(\overline{X}\backslash\pi_\rho^{-1}(P))$	$Loc_{\mathbb{G}_m}(\overline{X}\backslash R_\rho)\times_{(\mathbb{G}_m/\mathbb{G}_m)^{\# R_\rho}}(\{-1\}/\mathbb{G}_m)^{\# R_\rho}$
$SL(n)$	$Loc_{\mathbb{G}_m}(\overline{X}\backslash\pi_\rho^{-1}(P))\times_{Loc_{\mathbb{G}_m}(X\backslash P)}\{\mathbb{C}_{X\backslash P}\}$	$Loc_{\mathbb{G}_m}(\overline{X}\backslash R_\rho)\times_{Loc_{\mathbb{G}_m}(X\backslash P)}\{\mathbb{C}_{X\backslash P}\}$

Table 1.1: Spectral descriptions of moduli spaces

Here $\overline{X}$ is the associated spectral cover, R_ρ is the ramification locus of the spectral cover. The map $Loc_{\mathbb{G}_m}(\overline{X}\backslash R_\rho)\to(\mathbb{G}_m/\mathbb{G}_m)^{\# R_\rho}$ comes from restricting to loops around points in R_ρ.

We also provide analogous results for the other classical groups in section 5.3.

1.1 Outline

In chapter 2 we provide some background on the moduli of local systems, the Hitchin base, and some conventions and notation involving Lie algebras and algebraic groups. We draw attention here to the fact that we use a non-standard sign convention for Chevalley bases in definition 2.3.1. There is no new material in this section.

In chapter 3 we describe how to construct a spectral network from an appropriate point in the Hitchin base, for an arbitrary reductive algebraic group G. We stress that it is not entirely clear when this construction works, though there are many examples. This generalizes the construction for groups of type ADE in[3] [27], which in turn generalizes the earlier construction of [16] which works for $G=SL(n)$ and $G=GL(n)$. We also give an abstract formulation of the notion of spectral network, which we use in chapter 4.

In chapter 4 we first provide an alternative description of N-shifted, weakly W-

[3]With the exception of $E8$.

equivariant T-local systems on $\tilde{X}\backslash R$ in terms of N-local systems on $X\backslash P$ (where P is the branch locus), which correspond to the given cameral cover $\tilde{X}\backslash R$. We also describe a condition that we call the S-monodromy condition, that we also have to apply. We then describe the construction of the non-abelianization map using the data of an (abstract) spectral network.

In chapter 5 we describe the moduli spaces of N-shifted, weakly W-equivariant T-local systems on $\tilde{X}\backslash R$ satisfying the S-monodromy condition in terms of $\mathbb{G}_m$-local systems on spectral covers for the classical groups. We also describe the relation between the non-abelianization map we construct here and the path detour rule construction of non-abelianization given in [16] for $SL(n)$ and $GL(n)$, and proposed for simply laced groups in [27], in section 5.2. .

Chapter 2

Local Systems, Hitchin base, and Lie Algebras

The purpose of this chapter is to recall some well known theory about local systems and their moduli, the Hitchin base, and Lie algebras that will be used in the sequel.

2.1 Local Systems and Their Moduli

Let $X = X^c \backslash D$ be a punctured Riemann surface, where X^c is a compact Riemann surface, and $D \subset X^c$ is a non-zero reduced divisor.

Definition 2.1.1. A G*-local system* on X is a locally constant sheaf E of sets with right G-action on X, such that for all $x \in X$, E_x is a G-torsor.

Associated to such a local system, a choice of $x \in X$, and a choice of trivialization $x' \in E_x$, we can construct a representation $\rho : \pi_1(X, x) \to G$, defined by $\rho(\gamma) = g$, where $g \in G$ is the unique element such that $x'g$ is the parallel transport of x' along γ.

Changing the choice of $x \in X$, and changing the choice of trivialization of E_x has the affect of conjugating the representation ρ constructed above.

Conversely given a representation $\rho : \pi_1(X) \to G$, we can construct a G-local system on X, by taking the quotient $X^{un} \times G/\pi_1(X)$, where $X^{un} \to X$ is the universal cover of X, and $\pi_1(X)$ acts diagonally by deck transformations and by the representation ρ.

Definition 2.1.2 (Classical Moduli stack of Local systems)**.** We define the lassical[1] moduli space of G-local systems as

$$Loc_G(X)_{cl} := Hom_{Grp}(\pi_1(X), G)/G,$$

where we are taking the stacky quotient.

Remark 2.1.3. We note that $Hom_{Grp}(\pi_1(X), G)$ is an affine scheme. This is because we recall that there is a choice of generators (called the canonical cycles) $\alpha_1, \beta_1, ..., \alpha_g, \beta_g, \gamma_1, ..., \gamma_d$ (g is the genus of X, and $d = deg(D)$) of $\pi_1(X)$, such that we have an isomorphism:

$$\pi_1(X) = \left\langle \alpha_1, \beta_1, ..., \alpha_g, \beta_g, \gamma_1, ..., \gamma_d \,\middle|\, \prod_{i=1}^{g} [\alpha_i, \beta_i] \prod_{i=1}^{d} \gamma_i = Id \right\rangle. \tag{2.1.1}$$

We hence can write

$$Hom_{Grp}(\pi_1(X), G) \hookrightarrow G^{2g+d} \ni (x_{\alpha_1}, x_{\beta_1}, ..., x_{\alpha_g}, x_{\beta_g}, x_{\gamma_1}, ..., x_{\gamma_d})$$

as being cut out by the equation $\prod_{i=1}^{g} [x_{\alpha_i}, x_{\beta_i}] \prod_{j=1}^{d} x_{\gamma_j} = Id$.

We want to consider the (derived) moduli space of G-local systems on X. Note that the role of derived geometry in this thesis is very small, and can be ignored if desired, see remark 2.10 of [23] for a precise statement about when use of derived geometry is necessary.

[1] We use classical here in the sense that we are not referring to the derived moduli space.

Definition 2.1.4 (Betti Stack)**.** The derived stack X_B is the constant functor $dSch^{op} \to Top$, from the opposite category[2] of derived schemes to the category of topological spaces, such that it maps all derived schemes to the topological space underlying the punctured Riemann surface X.

For ℓ a topological space we denote ℓ_B the constant functor $dSch^{op} \to Top$, which maps all derived schemes to the topological space ℓ.

Definition 2.1.5 (Moduli of G-local systems)**.** We define the moduli space of G-local systems as the (derived) mapping stack

$$Loc_G(X) := Hom_{dSt}(X, BG).$$

See e.g. §19.1 of [28] for an overview of derived mapping stacks.

In [36] it is shown that $t_0(Loc_G(X)) = Loc_G(X)_{cl}$, where t_0 is the truncation functor from derived stacks to stacks in the ordinary sense.

2.2 The Hitchin Base

The Hitchin base associated to X^c and the line bundle $K_{X^c}(D)$, for D a non-zero reduced divisor is the space $\mathcal{A} := \Gamma(X^c, \mathfrak{t}_{K_{X^c}(D)}/W)$, where $\mathfrak{t}_{K_{X^c}(D)} := Tot(K_{X^c}(D)) \times_{\mathbb{G}_m} \mathfrak{t}$.

Definition 2.2.1. Given a point $a \in \mathcal{A}$ we define the associated *cameral cover* $\tilde{X}^c \to X^c$ as the pullback:

$$\begin{tikzcd} \tilde{X}^c_a \arrow[r, "\tilde{a}"] \arrow[d, "\pi"] & \mathfrak{t}_{\mathcal{L}} \arrow[d] \\ X^c \arrow[r, "a"] & \mathfrak{t}_{\mathcal{L}}/W \end{tikzcd} \tag{2.2.1}$$

[2]Here category should be understood to mean $(\infty, 1)$-category in the sense of [29].

We will often simply denote a cameral cover as $\tilde{X}^c$ if we do not want to emphasize the point of the Hitchin base it is associated to. We also denote $\tilde{X} := \tilde{X}^c \times_{X^c} X$. We also refer to $\tilde{X}$ as a cameral cover. In the case where $\tilde{X}$ is reduced, we let $R \subset \tilde{X}$ and $P \subset X$ be the ramification and branch divisors of $\tilde{X}$. We let $\tilde{X}^\circ := ReBl_R(\tilde{X})$, and $X^\circ := ReBl_P(X)$ be the real blow ups at these divisors[3]. We will also refer to $\tilde{X}^\circ \to X^\circ$ as a cameral cover.

The cameral cover $\tilde{X}$ has an action of W induced by the action of W on $\mathfrak{t}$. If $\tilde{X}$ is reduced, and the map $\tilde{X} \to X$ is unramified, then $\tilde{X} \to X$ is a principal W-bundle. In particular $\tilde{X}^\circ \to X^\circ$ is a principal W-bundle.

Definition 2.2.2. We define the Zariski open $\mathcal{A}^\diamondsuit \subset \mathcal{A}$ as the subset of $\mathcal{A} = \Gamma(X^c, \mathfrak{t}_{K_{X^c}(D)}/W)$, where X intersects $\left(\cup_{\alpha\in\Phi} (H_\alpha)_{K_{X^c}(D)}\right)/W \subset \mathfrak{t}_{K_{X^c}(D)}/W$ transversely.

Here $(H_\alpha)_{K_{X^c}(D)}$ denotes $Tot(K_{X^c}(D)) \times_{\mathbb{G}_m} H_\alpha$.

Lemma 2.2.3 (See [10, 30])**.** *For $a \in \mathcal{A}^\diamondsuit$, the cameral cover $\tilde{X}_a$ is smooth. Furthermore around the ramification points the map is locally modelled on $\mathbb{A}^1_z \to \mathbb{A}^1_w$, $w \mapsto z^2$.*

Proof. The transversality condition implies that $\tilde{X}$ does not intersect the singular part of $\left(\cup_{\alpha\in\Phi} (H_\alpha)_{K_{X^c}(D)}\right)$, that is to say it does not intersect $(H_\alpha)_{K_{X^c}(D)} \cap (H_\beta)_{K_{X^c}(D)}$ for any pair of roots $\alpha \neq \beta$. Then as each intersection with some $(H_\alpha)_{K_{X^c}(D)}$ is transveral, it is smooth, and the quotient by $s_\alpha \in W$ is locally modelled on $\mathbb{A}^1_z/\{\pm 1\} = \mathbb{A}^1_w$, where $w = z^2$. □

Lemma 2.2.4. *The set $\mathcal{A}^\diamondsuit \subset \mathcal{A}$ is non-empty for the genus of X^c satisfying $g \geq 1$ (recall*

[3]To take the real blow up we consider the complex analytic manifold associated to the Riemann surface, and consider the underlying real analytic space.

we assumed deg$(D) \geq 1$*). For* $X = \mathbb{P}^1$ *nonemptyness of* $\mathcal{A}^{\diamondsuit}$ *holds if deg*$(D) \geq 2$*.*

Proof. We modify the proof of Proposition 4.6.1 of [30]. To adapt this proof to our situation we need only show that $K_{X^c}(D)$ is ample. This is the case under the conditions imposed the lemma. $\square$

2.3 Algebraic Groups

Let G be a reductive algebraic group with maximal torus $T \subset G$. We denote by Φ the root system, and $N = N_G(T)$ the normalizer of T, and $W = N_G(T)/T$ the Weyl group.

We denote by

$$q : N \to W \tag{2.3.1}$$

the quotient map.

We now wish to introduce a version of Chevalley basis with a non-standard choice of signs, which is more convenient for our purposes.

Definition 2.3.1 (Chevalley basis)**.** Let $\mathfrak{g}$ be a semisimple Lie algebra with root space decomposition $\mathfrak{g} = \mathfrak{t} \oplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$. A Chevalley basis is a set of elements:

- $\{h_\alpha\}_{\alpha \text{ a simple root.}}$ forming a basis of $\mathfrak{t}$.
- An element $e_\alpha \in \mathfrak{g}_\alpha$ for each $\alpha \in \Phi$.

Furthermore for $\gamma = \sum_i a_i \alpha_i$, with α_i a simple root, we define $h_\gamma := \sum_i a_i h_i$.

These elements are required to satisfy:

$$[h_\alpha, e_\gamma] = 2\frac{(\alpha,\gamma)}{(\alpha,\alpha)}e_\gamma, \tag{2.3.2}$$

$$[e_\alpha, e_{-\alpha}] = -h_\alpha, \tag{2.3.3}$$

$$[e_\alpha, e_\gamma] = \begin{cases} 0 & \text{if } \alpha+\gamma \text{ not a root,} \\ \pm(p_{\alpha,\gamma}+1)e_{\alpha+\gamma} & \text{if } \alpha+\gamma \text{ is a root.} \end{cases} \tag{2.3.4}$$

The number $p_{\alpha,\gamma}$ in equation 2.3.4, denotes the largest integer with the property that $\alpha - p_{\alpha,\gamma}\gamma$ is a root.

Choose a Chevalley basis (see [7]) of the semisimple part of $\mathfrak{g}$ – the Lie algebra of G. For each root $\alpha \in \Phi$, we denote by

$$I_\alpha : SL_2 \to G \tag{2.3.5}$$

be the map of group corresponding to the map of Lie algebras

$$i_\alpha : \mathfrak{sl}_2 \to \mathfrak{g}$$

specified by the Chevalley basis, with $i_\alpha(\mathfrak{sl}_2) = span(e_\alpha, h_\alpha, e_{-\alpha})$.

We now introduce some subgroups and elements that will be used in the sequel.

Definition 2.3.2. For a root $\alpha \in \Phi$, we define the abelian subgroup: $T_\alpha := I_\alpha(T_{SL_2}) \subset T$.

We denote by $\mathfrak{t}_\alpha$ it's Lie algebra.

Definition 2.3.3. We define $n_\alpha \in N$, for a root $\alpha \in \Phi$, as

$$n_\alpha := I_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Definition 2.3.4. We define $T_{Ker(\alpha)}$ to be the subgroup of T corresponding to the sub Lie algebra

$$\mathfrak{t}_{Ker(\alpha)} \hookrightarrow \mathfrak{t}_{\alpha} \xrightarrow{\alpha} \mathbb{C}$$

which is the kernel of α.

Chapter 3

Spectral and Cameral Networks

We fix a reductive algebraic group G with chosen maximal torus $T \subset G$. We also fix a Riemann surface with punctures $X = X^c \backslash D$ (where $D \neq \emptyset$ is a reduced divisor on the compact Riemann surface X^c). *Spectral networks* are a topological object consisting of a union of oriented lines[1] $\ell \subset X$. These lines are labelled by additional data. The networks and the data are most easily described using *cameral networks*, which are topological objects on a cameral cover $\tilde{X} \to X$, labelled by roots, which are compatible with the action of W on $\tilde{X}$, and the root system. In this chapter we introduce this construction following [23].

We also describe how these abstract topological objects can be constructed using certain points of the Hitchin base associated to the group G, and the line bundle $K_{X^c}(D)$.

In the case where $G = SL(2)$ we essentially recover the trajectories of a quadratic differential. For groups of type ADE, we recover the definitions of [27], and that of [16]

[1]By lines we mean one dimensional submanifolds. We are here considering the two dimensional real analytic manifold underlying the complex analytic manifold/scheme X.

in the case of $G = GL(n)$ or $G = SL(n)$.

3.1 Abstract Cameral Networks

We firstly introduce the notion of 2d scattering diagrams which provide a local description of cameral networks when multiple lines in the cameral network intersect.

We first introduce some additional terminology.

Definition 3.1.1. A set of roots $C \subset \Phi$ is *convex* if the cone $Cone(C) := \{a = \sum_{\alpha_i \in C} a_i \alpha_i | a_i \in \mathbb{R}, a_i \geq 0\}$, does not contain any non-trivial vector subspace of $\mathfrak{t}^\vee$.

Definition 3.1.2. Let $C \subset \Phi$ be a set of roots, we define

$$Conv^{\mathbb{N}}(C) := \{\beta \in \Phi | \beta = \sum_i a_i \alpha_i, a_i \in \mathbb{Z}, a_i \geq 0\}.$$

Definition 3.1.3. To each root α we let U_α denote the unipotent subgroup of G, with Lie algebra being the weight space $\mathfrak{u}_\alpha \hookrightarrow \mathfrak{g}$ of the adjoint representation corresponding to α.

Definition 3.1.4. A *2d scattering diagram* is a set of oriented rays in the plane[2], either starting or ending at the origin, together with:

- A bijection between the set of incoming rays, and a convex set of roots $C \subset \Phi$.
- A bijection between the set of outgoing rays and roots in $Conv^{\mathbb{N}}(C)$.

Definition 3.1.5 (Abstract Cameral Networks, Essentially definition 4.4 of [23])**.** We fix a smooth cameral cover $\tilde{X} \to X$ associated to $a \in \mathcal{A}^\diamondsuit$, and denote its branch and

[2] Considered as an oriented manifold.

ramification divisors by P and R respectively. We denote by $\tilde{X}^\circ \xrightarrow{\pi} X^\circ$ the principal W bundle defined in definition 2.2.1, and also refer to this as the cameral cover.

An *abstract cameral network* $\tilde{\mathcal{W}}$ on $\tilde{X}^\circ$, is the data of

- A subspace $\tilde{X}^{\circ\prime} \hookrightarrow X^\circ$, preserved under the action of W, such that the inclusion is a homotopy equivalence.

- A union of a finite set of smooth oriented lines with single boundary $\{\ell_i\}_i$ with $\ell_i \hookrightarrow \tilde{X}^{\circ\prime} \hookrightarrow \tilde{X}^\circ$, each labelled by a root $\alpha_i \subset \Phi$. By a single boundary we mean that the subset $\ell_i \subset X^{\circ\prime}$ can be described by $i([0,1))$, for some smooth $i : [0,1) \to X^{\circ\prime}$. These are required to have the following property: The closure of ℓ inside the closure $\overline{\left(\tilde{X}^{\circ\prime}\right)} \subset ReBl_{R\cup\pi^{-1}(D)}(\tilde{X}^c)$ of $\tilde{X}^{\circ\prime}$ in the real blow up of $\tilde{X}^c$ along both R and $\pi^{-1}(D)$, has to consist of $\ell \cup pt$, where pt is a single point in $\overline{\left(\tilde{X}^{\circ\prime}\right)} \backslash \tilde{X}^{\circ\prime}$.

These are required to satisfy the following properties:

- *Equivariance*: Let $\ell \subset \tilde{\mathcal{W}}$ be a line in $\tilde{\mathcal{W}}$ labelled by α, then $w(\ell)$ is in $\tilde{\mathcal{W}}$, and has the label $w(\alpha)$, for any $w \in W$.

- *Behaviour at ramification points*: For $p \in P$, and $S^1_r \subset \tilde{X}^\circ$ be a connected component of the preimage of S^1_p, corresponding to a ramification point $r \in \tilde{X}$. There are 6 lines of the cameral network that intersect S^1_r. The labels of these alternate between α and $-\alpha$, where these are the two roots such that at the ramification point r, $\tilde{X}$ intersects the root hyperplane $(H_\alpha)_{K_{X^c(D)}}$. The orientation of these lines is coming out of the circle S^1_r.

- *Joints* We call the intersection points of any pair of lines ℓ_i and ℓ_j, for $i \neq j$, joints. The set $\tilde{\mathcal{J}}$ of joints is finite. Furthermore at each joint the half of the tangent space to incoming and outgoing lines that corresponds to the direction the line is either coming in from or going out towards, together with the labels of their lines form a 2d scattering diagram in the sense of 3.1.4.

- *Acyclicity* Consider the directed graph, with vertices being the joints $\tilde{\mathcal{J}}$ the ramification points R and the points $d \in \pi^{-1}(D)$, and edges being connected components of $\tilde{\mathcal{W}} \backslash \tilde{\mathcal{J}}$, with the orientation of the line providing the direction of the edge. We require that this directed graph is acyclic.

- *Non-denseness* Let $f : [0,1] \to X^{\circ'}$ intersect ℓ_i transversely at $f(1/2)$ for all lines $\ell_i \ni f(1/2)$. Then there is an open set $1/2 \in U \subset [0,1]$, such that $f(U) \cap \tilde{\mathcal{W}} \subset 1/2$.

We often abuse notation and write $\tilde{\mathcal{W}}$ to denote the topological space $\cup_i \ell_i$. We also sometimes refer to the lines ℓ_i as Stokes lines. We hope this does not cause confusion.

Definition 3.1.6. For $\tilde{X}^{\circ'}$ as above we can define $X^{\circ'} := \tilde{X}^{\circ'}/W$, this gives us a principal W-bundle $\tilde{X}^{\circ'} \to X^{\circ'}$. Furthermore it is clear that the inclusion $X^{\circ'} \hookrightarrow X^{\circ}$ is a homotopy equivalece.

3.2 Spectral Networks

Definition 3.2.1 (Abstract Spectral Network)**.** An *abstract spectral network* $\mathcal{W}$ is the set $\pi_* \tilde{\mathcal{W}} \subset X^{\circ'}$, for $\tilde{\mathcal{W}} \subset \tilde{X}^{\circ'} \hookrightarrow \tilde{X}^{\circ}$ an abstract cameral network.

Note that we can still write $\mathcal{W} = \cup_i \ell_i$ for some lines $\ell_i \hookrightarrow X^{\circ'}$. These lines inherit orientations from their preimages on $\tilde{X}$.

We can view these line segments ℓ_i as being labelled by a W-equivariant section of $Hom(\tilde{X}^{\circ\prime}|_{\ell_i}, \Phi)$.

We denote $\mathcal{J} = \pi_* \tilde{\mathcal{J}}$ and call it the set of joints of the spectral network.

Remark 3.2.2. Picking a trivialization of the cameral cover away from a series of branch cuts, gives labels of the lines by roots by precomposing the homomorphism $Hom(\tilde{X}^{\circ\prime}|_{\ell_i}, \Phi)$ with the section trivializing the cameral cover. This recovers the labelling of lines of the spectral network by roots, given such a choice of trivializations found in earlier work, in particular in [27, 16].

Lemma 3.2.3 (Acyclicity)**.** *The directed graph with vertices being points of $\mathcal{J} \cup P \cup D$ and, and edges being connected components of $\mathcal{W}$, with orientations inherited from the orientations of lines of the spectral network. Then this graph is acyclic.*

Proof. Suppose otherwise. Then we can lift a cycle in this graph, which includes a point $x \in (\mathcal{J} \cup P \cup D) \subset X$ to a cycle in the analogous graph (which we will call the cameral graph) defined using the cameral network. This gives a path in the cameral graph between the vertices correpsonding to two preimages $\tilde{x}_1$ and $\tilde{x}_2$ of x. We can commpose this with the lift of the cycle starting at $\tilde{x}_2$. Iterating this process as there are finitely many preimages of x we will obtain a cycle contradicting the acyclicity requirement of definition 3.1.5. □

Lemma 3.2.4. *Consider the directed graph, with vertices being the union of joints $\mathcal{J}$ and branch points P, and edges being connected components of $\mathcal{W} \backslash \mathcal{J}$, with the orientation of the line providing the direction of the edge.*

There is a filtration $F_\bullet(\mathcal{J} \cup P)$ on $\mathcal{J} \cup P$ (the vertices of this graph), with the property that $F_0(\mathcal{J} \cup P) = P$, and the property that any edge ending at a vertex in $F_m(\mathcal{J} \cup P)$ starts at a vertex in $F_{m-1}(\mathcal{J} \cup P)$.

Proof. Lemma 3.2.3 shows that the graph with vertices being the joints and the circles S^1_p ($p \in P$) and edges being the connected components of $\mathcal{W} \backslash \mathcal{J}$ is an acyclic directed graph.

The local behaviour of the cameral network around ramification points shows that there are no incoming edges to the vertices associated to the branch points $p \in P$.

It is well known that in this situation there is a filtration on the edges of the graph which has the desired properties, in particular a proof can be found in lemma 4.5 of [23]. □

Definition 3.2.5. We use the filtration $F_\bullet\mathcal{J}$ on joints to define a filtration $F_\bullet\mathcal{W}$ on connected components of $\mathcal{W} \backslash \mathcal{J}$ as follows:

We define $F_i\mathcal{W}$ to consist of all connected components of $\mathcal{W} \backslash \mathcal{J}$ that start (using the orientation of the line) at a joint in $F_i\mathcal{J}$.

Note that as there are finitely many joints, and finitely many lines, this filtration is eventually constant, that is there is $N >> 0$, such that $\forall m \geq N$, we have $F_m\mathcal{W} = F_N\mathcal{W}$.

3.3 WKB Cameral Networks

In this section an iterative procedure to define lines $\ell_i \subset \tilde{X}^\circ$, starting from a point in the Hitchin base. In good situations, after restricting to $\tilde{X}^{\circ'} \hookrightarrow \tilde{X}^\circ$, this will give an abstract cameral network in the sense of definition 3.1.5.

3.3.1 The Hitchin Base and Differentials

Construction 3.3.1. Let $\alpha \in \Phi$ be a root, we construct a differential form χ_α on $\tilde{X}_a$.

Consider the map:

$$\tilde{X}_a^c \xrightarrow{\tilde{a}} \mathfrak{t}_{K_{X^c(D)}} \xrightarrow{\alpha} Tot(K_X^c(D))$$

This provides a map $\tilde{X}_a^c \to \pi^* K_{X^c(D)} \to K_{\tilde{X}^c(\pi^*(D))}$. We denote this differential, and its restriction to $\tilde{X}$, by χ_a.

Associated to the differential χ_a is a real projective vector field $V_\alpha \in \Gamma(\tilde{X}, T\tilde{X}/\mathbb{R}_+)$, defined as follows:

$$v \in [V_\alpha] \subset T\tilde{X} \tag{3.3.1}$$

if $\chi_\alpha(v) \in \mathbb{R}_+$.

We now provide what we will call the WKB construction of an abstract cameral network, starting at certain points $a \in \mathcal{A}^\diamondsuit$. We stress that this will not work starting at an arbitrary point, and for $G \neq SL(2), PGL(2)$ the precise nature of the locus of points where this construction does not work is unclear. However there is a wealth of examples in e.g. [16, 27] of cases where this construction works for simply laced groups. More examples can be produced by the software [26] which draws spectral networks for simply laced groups.

We break this construction into two parts (constructions 3.3.2 and 3.3.7). The first consists of drawing lines on $\tilde{X}^\circ$, and the second consists of restricting to a specified subset $\tilde{X}^{\circ\prime} \subset \tilde{X}^\circ$.

Construction 3.3.2 (Part 1 of the WKB Construction)**.** We start with a point $a \in \mathcal{A}^\diamondsuit$, and construct the associated smooth cameral cover $\tilde{X}$. We then construct the modified version $\tilde{X}^\circ \to X^\circ$, which is a principal W-bundle.

This part is a two step construction, firstly we draw what we call the "initial" Stokes lines, which start at the preimages of S^1_p, for $p \in P$. We then iteratively draw what we call "new" Stokes lines which start where Stokes lines intersect.

This part of the construction does not work for arbitrary $a \in \mathcal{A}^\diamondsuit$, and we end with a list of points that can stop the iteration, or in which the produced result is unsuccessful.

- Firstly we take the closure in $\tilde{X}^\circ$, of for each $r \in R \subset \tilde{X}$ the trajectories of the projective vector field χ_α (as oriented, unparameterized lines), where α is a root such that $\tilde{a}(r) \in (H_\alpha)_{K_{\tilde{X}^c(D)}}$. We label such a line by the root α.
- We then iterate the following step, assuming that none of the phenomena in the list below occur. For each intersection J (which we again call a joint), where one of the lines produced in the previous step intersects another line, we draw new trajectories of the projective vector field V_α starting at J for each $\alpha \in Conv^{\mathbb{N}}_C$, where C is the set of lines passing through the joint J, assuming that these trajectories have not already been drawn.

The behaviours that can cause the procedure to fail at one of the iterative steps follow:

- A loop forms, that is there is a non-contractible map of oriented manifolds $S^1 \to \cup_i \ell_i$ to the union of all lines so far produced.
- One of the lines drawn intersects one of the preimages of S^1_p, $p \in P$, or equivalently if considering the line on $\tilde{X}$, contains a ramification point.

- A line drawn violates the non-denseness requirement of definition 3.1.5, with $X^{\circ\prime}$ replaced by X°.

- At an intersection point the labels of the incoming roots do not form a convex set in the sense of 3.1.1, or the tangent spaces of multiple incoming or outgoing lines are equal so that the joints condition of definition 3.1.5 is violated.

We take the union of all labelled oriented lines produced in (possibly countably infinitely many) steps of the iterative process above.

Furthermore the construction can fail if after taking the union of the lines produced at all steps of the iteration one of the following phenomena happens:

- The set of joints $\tilde{\mathcal{J}}$ has a limit point in $\tilde{X}^{\circ}$ (we allow it to have a limit point in $\tilde{X}^{c}$).

Lemma 3.3.3. *The set of labelled, oriented curves produced in Construction 3.3.2 (if the construction does not fail) is W-equivariant in the sense of the equivariance condition of definition 3.1.5.*

Proof. Immediate from the W-equivariance of the map $\tilde{X}^{c} \to \mathfrak{t}_{K_{X^c}(D)}$, which means that $w^{*}\chi_{\alpha} = \chi_{w\alpha}$, and hence $w^{*}V_{\alpha} = V_{w\alpha}$. □

Lemma 3.3.4. *The set of labelled, oriented curves produced in Construction 3.3.2 (assuming the construction does not fail) satisfies the "behaviour at ramification points" condition of definition 3.1.5.*

Proof. This is partly due to the requirement of terminating the construction if one of the trajectories on $\tilde{X}$ intersects a ramification point, and partly as a direct computation of the local behaviour of the initial Stokes lines shows that this behaviour holds, as shown in lemma 4.20 of [23]. □

We furthermore note that the acyclicity condition holds by assumption, if the construction is successful. The finiteness condition on both joints, and on lines does not necessarily hold, and if there are infinitely many lines the non-denseness condition does not necessarily hold. Examples of cases where these do not hold are found in §5.5 of [16], and are the reason for restricting to $X^{\circ\prime}$.

The idea of the second step of the WKB construction of an abstract cameral network (construction 3.3.7 see also construction 3.3.2) is to pick a (W-equivariant) subspace $X^{\circ\prime} \hookrightarrow X^{\circ}$, such that the inclusion is a homotopy equivalence, and restrict the lines in construction 3.3.2 to this subset. Before we do this, we prove some more results about the behaviour of trajectories of V_α for some α near $\pi^{-1}(D) \subset \tilde{X}^c$.

3.3.2 Behaviour of Differentials near $D \subset \tilde{X}^c$

Following the case of the Quadratic differential considered in, for example, [33, 6], we find a condition (which we call condition R) under which we can appropriately restrict the lines produced in construction 3.3.2 to a subspace $\tilde{X}^{\circ\prime} \hookrightarrow \tilde{X}^{\circ}$.

Definition 3.3.5 (Condition R)**.** We say that $a \in \mathcal{A} = \Gamma(X^c, \mathfrak{t}_{K_{X^c}(D)}/W)$ satisfies condition R if for each $d \in D$ using a local coordinate x, at d we can write

$$a(x) = \left(W \bullet (\tilde{r_d}\frac{dx}{x})\right)/W,$$

for $\tilde{r_d} \in \mathfrak{t}\backslash \left(\cup_{\alpha\in\Phi}\alpha^{-1}(i\mathbb{R})\right)$.

Note that as the subset $\left(\cup_{\alpha\in\Phi}\alpha^{-1}(i\mathbb{R})\right)$ of $\mathfrak{t}$ is W-equivariant, it does not matter which lift $\tilde{r}_d$ of an element in $\mathfrak{t}/W$ we use.

Proposition 3.3.6. *Let $a \in \mathcal{A}^{\diamondsuit}$ such that a satisfies condition R, and that construction 3.3.2 applied to a is successful. Then for any $d \in D$, there exists an open set $d \in B_d \subset X^c$ such that for any line ℓ produced in construction 3.3.2 entering $\pi^{-1}(B_d)$ does not leave, and furthermore any new Stokes line starting at a joint inside $\pi^{-1}(B_d)$ does not leave $\pi^{-1}(B_d)$.*

Proof. Note that by condition R, the covering $\tilde{X}^c \xrightarrow{\pi} X^c$ is locally trivial around $d \in D$. We will restrict to such a neighbourhood.

We will first outline how the trajectories of the V_α relate to quadratic differentials.

We note that for $\alpha \in \Phi$ we factor π as

$$\tilde{X}^c \to \tilde{X}^c/s_\alpha \xrightarrow{\pi_\alpha} X^c,$$

the differential form χ_α corresponds to a quadratic differential on $\tilde{X}^c/s_\alpha$:

$$\tilde{X}^c/s_\alpha \to \mathfrak{t}/s_\alpha \otimes \pi_\alpha^*(K_{X^c(D)}) \to K_{\tilde{X}^c/s_\alpha}(\pi_\alpha^{-1}(D))/\{\pm 1\}.$$

which corresponds to a section

$$q_\alpha \in \Gamma(\tilde{X}^c/s_\alpha, (K_{\tilde{X}^c/s_\alpha}(\pi^{-1}(D)))^{\otimes 2})$$

To a quadratic differential q on a Riemann surface Y, we can associate a foliation of Y by the curves $\gamma : \mathbb{R}_t \to X$ such that:

$$\frac{d}{ds}\int_{t=t_0}^{s} \pm\sqrt{q}(\gamma(t)) \in \mathbb{R}. \tag{3.3.2}$$

Note that this does not depend on parametrization, and furthermore the lines of this foliation starting at ramification points correspond to lines of the $SL(2)$ spectral network associated to (Y, q).

The trajectories labelled by α on $\tilde{X}$ are then, forgetting the orientation, given by the pullback of sections of the folitation on $\tilde{X}/s_\alpha$ determined by the quadratic differential q_α.

Then by the analysis of the case of quadratic differentials found in [33] or section 3 of [6], for each α there is an open set $B_{d,\alpha}$ for which every line of the cameral network labelled by α in $\pi^{-1}(B_{d,\alpha})$ either has tangent direction pointed towards a point in $\pi^{-1}(d)$ or away from it[3]. Taking $B_d := \cap_\alpha B_{d,\alpha}$the result follows up to the question of orientation – we need to show that the tangent directions are always oriented towards $d \in D$. This statement follows by induction on construction 3.3.2, and using lemma 4.14 of [23]. Firstly note that any line entering the circle will has tangent direction towards d. Secondly lemma 4.14 of [23] says that any line starting at $J \in B_d$ has tangent vector between the tangent vectors of the incoming lines at J which by the induction hypothesis are oriented towards d, and hence the tangent vector of the new Stokes line is oriented towards d.

See example 4.25 and proposition 4.32 in [23] for more details. □

Construction 3.3.7 (Part II of the WKB construction)**.** Starting at a point $a \in \mathcal{A}^\diamondsuit$ satisfying condition R, firstly apply construction 3.3.2. Assuming this construction is successful we then restrict to the lines intersecting $\tilde{X}^{\circ\prime} := \tilde{X}^\circ \setminus \cup_{d \in D} \pi^{-1}(B_d)$, where B_d is determined in proposition 3.3.6. We take the restrictions of the lines we keep to the subset $\tilde{X}^{\circ\prime}$.

[3]The uncertainty in the direction comes from the fact that the trajectories of a quadratic differential are unoriented.

Theorem 3.3.8 (This is Proposition 4.19 in [23])**.** *Apply constructions 3.3.2 and 3.3.7 to a point $a \in \mathcal{A}^{\diamondsuit}$ satisfying condition R. Assume that this is successful, then we have produced an abstract cameral network in the sense of definition 3.1.5.*

Proof. Note that it is clear that $\tilde{X}^{\circ\prime} \hookrightarrow X^{\circ}$ is preserved under the action of W, and is a homotopy equivalence.

Furthermore by the assumption in construction 3.3.2 the joints $\tilde{\mathcal{J}}$ had no accumulation point in $\tilde{X}^{\circ}$, hence there are only finitely many in $\tilde{X}^{\circ\prime}$. Hence there are only finitely many lines.

We now need to prove the five conditions in definition 3.1.5. Note that the equivariance and behaviour at ramification points conditions were proved in lemmas 3.3.3 and 3.3.4. The joints condition follows by construction, and by the assumption about accumulation points noted above. The acyclicity condition was required for success of construction 3.3.2. The non-denseness condition follows from the fact that there are finitely many lines, and the non-denseness condition for the success of construction 3.3.2. The result follows. $\square$

Chapter 4

Nonabelianization

4.1 Moduli Spaces of N and T Local Systems

Consider the short exact sequence

$$1 \to T \to N \to W \to 1$$

The morphism $N \to W$ induces a map $Loc_N(X^\circ) \to Loc_W(X^\circ)$, this allows us to define the moduli of N-local systems that correspond to a given cameral cover as follows:

Definition 4.1.1 (Moduli N local systems corresponding to a cameral cover $Loc_N^{\tilde{X}^\circ}(T)$)**.** Let $\tilde{X}^\circ \to X^\circ$ be a principal W-bundle, we define

$$Loc_N^{\tilde{X}^\circ}(X^\circ) := Loc_N(X^\circ) \times_{Loc_W(X^\circ)} \{\tilde{X}^\circ\}.$$

Consider $\mathcal{E}_N \to X_B^\circ \times Y$ an N-bundle, equipped with an isomorphism $\mathcal{E}_N/T \cong \tilde{X}_B^\circ \times Y$. This gives the map $\mathcal{E}_N \to \mathcal{E}_N/T \times Y \xrightarrow{\cong} \tilde{X}^\circ \times Y$ the structure of a T-local system (using the fact T is abelian). Letting $Y = Loc_N^{\tilde{X}^\circ}(X^\circ)$, and $\mathcal{E}_N$ be the universal object provides, by the universal property, a map

$$Loc_N^{\tilde{X}^\circ}(X^\circ) \to Loc_T(\tilde{X}^\circ). \tag{4.1.1}$$

Clearly this is not an equivalence. We now follow [9] which considers the analogous case in the setting of principal bundles rather than local systems to describe how to upgrade the map in equation 4.1.1 to an isomorphism.

Firstly we make the following definition to be used in the sequel:

Definition 4.1.2. Let $\mathcal{E}_T$ be a T-local system on $\tilde{X}^\circ$, we define the local system

$$Aut_{X^\circ}(\mathcal{E}_T) := \{(w, \rho) | w \in W, \rho : \mathcal{E}_T \xrightarrow{\cong} w^*\mathcal{E}_T\},$$

where we are considering the action of w on $\tilde{X}^\circ$ given by the fact that $\tilde{X}^\circ \to X^\circ$ is a principal W-bundle.

Definition 4.1.3 (N-shifted weakly W-equivariant T-local system)**.** An N-shifted, weakly W equivariant T-local system on a cameral cover $\tilde{X}^\circ$ is a T-local system $\mathcal{E}_T \to \tilde{X}^\circ$ together with the additional data of a morphism $\gamma : N \to Aut_{X^\circ}(\mathcal{E}_T)$, such that the following diagram of algebraic groups commutes:

$$\begin{tikzcd}
1 \arrow[r] \arrow[d] & T \arrow[r] \arrow[d, "\Delta"] & N \arrow[r] \arrow[d, "\gamma"] & W \arrow[r] \arrow[d, "Id"] & 1 \arrow[d] \\
1 \arrow[r] & \mathrm{Hom}_{lc}(\tilde{X}^\circ, T) \arrow[r] & Aut_{X^\circ}(\mathcal{L}) \arrow[r] & W \arrow[r] & 1.
\end{tikzcd} \tag{4.1.2}$$

In the above the map $\Delta : T \to Hom_{lc}(\tilde{X}^\circ, T)$, refers to the map from $t \in T$, to the constant morphism with image $t \in T$, the lc subscript stands for locally constant.

Furthermore this clearly sheafifies over X° in the classical topology, and hence gives a sheaf that we denote $\mathcal{A}ut_{X^\circ}(\mathcal{E}_T)$

Definition 4.1.4 (Moduli of N-shifted weakly W-equivariant T-local systems)**.** We have group objects in the category of (derived) stacks over $Loc_T(\tilde{X}^\circ) \times (X^\circ)_B$: corresponding to $N \times Loc_T(\tilde{X}^\circ) \times (X^\circ)_B$, and to $Aut_{X^\circ}(\mathcal{E}_T)$, where $\mathcal{E}_T \to Loc_T(\tilde{X}^\circ) \times (X^\circ)_B$ denotes the universal object. We now define the moduli stack $Loc_T^N(\tilde{X}^\circ)$ of N-shifted, weakly W-equivariant T-local systems on X° as $\Gamma_{X_B^\circ}(\mathcal{F})$, where $\mathcal{F}$ is the fiber product:

$$\begin{tikzcd}
\mathcal{F} \arrow[r] \arrow[d] & \mathrm{Hom}_{Grp(dSt/Loc_T(\tilde{X}^\circ)\times X_B^\circ)}\big(N, Aut_{X^\circ}(\mathcal{E}_T)\big) \arrow[d] \\
\{\Delta, Id\} \arrow[r] & \mathrm{Hom}_{Grp(dSt/Loc_T(\tilde{X}^\circ)\times X_B^\circ)}\big(T, Hom_{lc}(\tilde{X}^\circ, T)\big) \times \mathrm{Hom}_{Grp(dSt/Loc_T(\tilde{X}^\circ)\times X_B^\circ)}(W, W)
\end{tikzcd}$$

Note that in the above commutative diagragm we sometimes write e.g. W as shorthand for the group object $W \times Loc_T(X^\circ) \times X_B^\circ$ in $dSt_{/Loc_T(X^\circ)\times X_B^\circ}$.

Remark 4.1.5. For group objects G, H of $dSt_{/X}$, the notation $Hom_{Grp(dSt_{/X})}(G, H)$ is defined as the equalizer (limit):

$$Hom_{Grp(dSt/X)}(G, H) \to Hom_{dSt/X}(G, H) \rightrightarrows Hom_{dSt/X}(G \times_X G, H)$$

where the two morphisms $Hom_{dSt/X}(G, H) \rightrightarrows Hom_{dSt/X}(G \times_X G, H)$ are given by $\phi \mapsto (\phi \circ m_G)$, and $\phi \mapsto (m_H \circ (\phi \times_{Id_X} \phi))$, for m_G and m_H the group multiplications of G, and H respectively, and $Hom_{dSt/X}(G, H)$ referring to the derived mapping stack. See §19.1 of [28] and [37] for more about derived mapping stacks.

Proposition 4.1.6. *(Proposition 5.8 of [23], cf. [9]) There is an isomorphism of stacks*

$$Loc_N^{\tilde{X}^\circ}(X^\circ) \to Loc_T^N(\tilde{X}^\circ)$$

Remark 4.1.7. This result is closely related to the unramified case of the result of [9], which in an entirely analogous way gives a version of this result for principal bundles rather than local systems.

Proof. We note that the map of equation 4.1.1 factors as

$$\begin{tikzcd} Loc_N^{\tilde{X}^\circ}(X^\circ) \arrow[rr] \arrow[dr] & & Loc_T^N(\tilde{X}^\circ) \arrow[ll, bend right] \arrow[dl, "forget"] \\ & Loc_T(\tilde{X}^\circ) & \end{tikzcd}$$

with the inverse map $Loc_T^N(\tilde{X}^\circ) \to Loc_N^{\tilde{X}^\circ}(X^\circ)$ also being given by the universal property, taking a T-local system $\mathcal{E}_T \to \tilde{X}_B^\circ \times Y$ to the N-bundle $\mathcal{E}_T \to \tilde{X}_B^\circ \times Y \to X_B^\circ \times Y$, with N-action given by $N \to Aut_{X^\circ}(\mathcal{E}_T) \circlearrowright \mathcal{E}_T$. Applying this to $Y = Loc_T^N(\tilde{X}^\circ)$ gives the desired inverse map. □

4.2 The S-monodromy Condition

In the spectral case of non-abelianization considered in sections 5.2, 5.1 , and in the original paper [16] we can see that the space of all N-local systems is too large. The S-monodromy condition is a restriction on the monodromy of N-local systems around branch points $p \in P \subset X$ (or equivalently $S_p^1 \subset X^\circ$) that rectifies this discrepancy.

Recall definitions 2.3.3 and 2.3.2.

Lemma 4.2.1. *The subset $\bigcup_{\alpha \in \mathcal{O}} n_\alpha T_\alpha \subset N$, where $\mathcal{O} \subset \Phi$ is a W-orbit, is preserved by the action of N by conjugation.*

Proof. See lemma 3.15 in [23]. □

We can now define the S-monodromy condition for N-local systems. Fix a smooth cameral cover $\tilde{X} \to X$, and let $\tilde{X}^\circ \xrightarrow{\pi} X^\circ$ be the unramified cover constructed in definition 2.2.1. For each $p \in P$ let $\mathcal{O}_p \subset \Phi$ be the subset of roots, such that the W-conjugacy class corresponding to the monodromy around S_1^p of $\tilde{X}^\circ$ corresponds to the set $\{s_\alpha | \alpha \in \mathcal{O}_p\}$.

We then define:

Definition 4.2.2. Let $\mathcal{E}_N \to X^\circ$, be an N-local system, equipped with an isomorphism of W-bundles, $\mathcal{E}_N/T \xrightarrow{\cong} \tilde{X}^\circ$. We say that $\mathcal{E}_N$ satisfies the S-monodromy condition if for each $p \in P$, the monodromy of $\mathcal{E}_N$ around S^1_p is contained in $\bigcup_{\alpha \in \mathcal{O}_p} n_\alpha T_\alpha$ with respect to any choice of trivialization at a point.

Definition 4.2.3. We define the moduli space of N-local system corresponding to $\tilde{X}^\circ$ satisfying the S-monodromy condition by:

$$Loc_N^{\tilde{X}^\circ, S}(X^\circ) := Loc_N^{\tilde{X}^\circ}(X^\circ) \times_{\Pi_{p \in P} Loc_N^{\pi^{-1}(S^1_p)}(S^1_p)} \Pi_{p \in P} \left(\left(\bigcup_{\alpha \in \mathcal{O}_p} n_\alpha T_\alpha \right) /N \right)$$

Remark 4.2.4. Let $\mathcal{C}_p \subset W$, be the W-conjugacy class corresponding to the monodromy of $\tilde{X}^\circ$ around S^1_p. Recall we use $q : N \to W$ to denote the quotient map. We then have that

$$Loc_N^{\pi^{-1}(S^1_p)}(S^1_p) \cong q^{-1}(\mathcal{C}_p)/N.$$

We now consider how to describe the S-monodromy condition in terms of N-shifted, weakly W-equivariant T-local systems on the cameral cover $\tilde{X}^\circ$. This is less natural, however we will be able to more or less describe the moduli space of such T-local systems as a component of a moduli space of N-shifted, weakly W-equivariant T-local systems, together with a restriction on the monodromy.

Definition 4.2.5. We say that an N-shifted, weakly W-equivariant T-local system satisfies the S-monodromy condition if the associated N-local system satisfies the S-monodromy condition of definition 4.2.2.

Picking a point on S^1_p, and trivializing $\tilde{X}^\circ|_{S^1_p}$ restricted to this point, for each $p \in P$ means we can interpret the S-monodromy condition, as corresponding to a reduction of structure of the N-local system on S^1_p to SL_2, with respect to the morphism $I_\alpha : SL_2 \to G$, where α is the root, such that s_α is the monodromy of $\tilde{X}^\circ|_{S^1_p}$.

For SL_2, noting that $W \cong \{1, s\}$, and that for any $n \in q^{-1}(s)$, we have $n^2 = -Id$, we have that in the case $G = SL(2)$, an N-local system satisfying the S-monodromy condition is mapped under the map of proposition 4.1.6 to an N-shifted, weakly W-equivariant T-local system with the property that the monodromy around S^1_r is $I_{\alpha(r)}(-Id)$ for each $r \in R$, where $\alpha(r)$ is one of the two roots such that $\tilde{X}$ intersects the hyperplane H_α at r.

The reverse does not hold, for example for $G = GL(3)$ the element

$$g = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

also squares to $I_\alpha(-Id)$.

We now want to describe all elements $n \in q^{-1}(s_\alpha) \subset N$ that square to $I_\alpha(-Id)$, in order to describe the subset of these which correspond to the S-monodromy condition. Recall that we have a map $T_\alpha \times T_{Ker(\alpha)} \to T$ (see definitions 2.3.2, 2.3.3) that is either an isomorphism or has a kernel with two elements (see Lemma 3.11 of [23]). We can hence write any element in the N such that $q(N) = s_\alpha$ in at most two different ways as $n = n_\alpha t_\alpha t_{Ker(\alpha)}$ with $t_\alpha \in T_\alpha$, and $t_{Ker(\alpha)} \in T_{Ker(\alpha)}$. We have that

$$\begin{aligned} (n_\alpha t_\alpha t_{Ker(\alpha)})^2 &= (n_\alpha t_\alpha)^2 t^2_{Ker(\alpha)} \\ &= I_\alpha(-Id) t^2_{Ker(\alpha)}. \end{aligned}$$

This tells us that the monodromies (in N/N) which correspond to satisfying the S-monodromy condition form a connected component of the monodromies (in N/N) which upon choosing a trivialization of the cameral cover to reduce from N/N to N/T square to an element $I_\alpha(-Id)$ for $\alpha \in \mathcal{O}_p$.

Note that for $deg(D) > 0$ we have that $Loc_N^{\tilde{X}}(X)$ is connected (as the character variety is connected because D is non-zero). Furthermore it is clear that $Loc_N^{\tilde{X},S}(X)$ is connected.

We can hence make the following definition:

Definition 4.2.6. We define the moduli stack of N-shifted, weakly W-equivariant T-bundles on $\tilde{X}^\circ$ as the connected component of

$$Loc_T^N(\tilde{X}^\circ) \times_{Loc_T(\coprod_{r \in R}(S^1_r))} \Pi_{r \in R}\{-Id\},$$

containing the image of $Loc_N^{\tilde{X},S}(X)$.

Because $n_\alpha T_\alpha$ is a connected component of the stack $q^{-1}(s_\alpha) \times_T \{-Id\}$, where the map $q^{-1}(s_\alpha) \to T$ is $n \mapsto n^2$, we gain the following theorem.

Theorem 4.2.7 (Theorem 5.14 in [23].)**.** *For $\tilde{X}^\circ \to X^\circ$ a principal W-bundle coming from a smooth cameral cover $\tilde{X} \to X$, there is a commutative diagram:*

$$\begin{tikzcd}
Loc_N^{\tilde{X}^\circ}(X^\circ) \arrow[r, "\cong"] & Loc_T^N(\tilde{X}^\circ) \\
Loc_N^{\tilde{X}^\circ,S}(X^\circ) \arrow[r, "\cong"] \arrow[u] & Loc_T^{N,S}(\tilde{X}^\circ) \arrow[u]
\end{tikzcd}$$

where the top arrow is that of proposition 4.1.6.

4.3 The Nonabelianization Map

In this section we give describe the construction of the non-abelianization map from [23].

By definition 3.2.5 we have a filtration $F_i\mathcal{W}$ of the connected components of $\mathcal{W}\backslash\mathcal{J}$, with the property that $F_0\mathcal{W}$ is the union of the lines emerging from branch points $p \in P \subset X$ (or strictly speaking from the associated boundary circles S^1_p) as shown in lemma 3.2.4.

The large scale structure of the non-abelianization construction can be seen as follows. We are going to "cut and reglue" a G local system induced from any given N-local system, corresponding to $\tilde{X}^\circ$ satisfying the S-monodromy condition, along the locus $\mathcal{W}$. To do this we need to produce automorphisms of the G local system restricted to each connected component of $\mathcal{W}\backslash\mathcal{J}$. We will call these automorphisms Stokes factors. We do this inductively using the filtration $F_\bullet\mathcal{W}$. We then define the reglue map (for specified automorphisms) in construction 4.3.9, which describes "cutting" along $\mathcal{W}$ and then "regluing" by specified automorphisms.

We first define some moduli stacks that parameterize appropriate N-local systems $\mathcal{E}_N$, together with certain automorphisms of the induced G-local system $\mathcal{E}_G := \mathcal{E}_N \times_N G$. Note that inducing a G-local system provides a map $Loc_N(X^\circ) \to Loc_G(X^\circ)$, which given the mapping stack description $Loc_N(X^\circ) = Map(X^\circ, BN)$, corresponds to postcomposing a map with the map $BN \to BG$ induced from the inclusion $N \hookrightarrow G$.

We first give a definition of the moduli of G-local systems on X° equipped with automorphisms of the local system restricted to a set of lines[1]. Firstly note that the universal bundle $pt \to BG$ has the sheaf of automorphisms $G/G \to BG$. Hence we make the following definition:

[1]By lines we mean one dimensional topological submanifolds of the topological manifold X°.

Definition 4.3.1. For $F_i\mathcal{W}\backslash\mathcal{J} = \coprod_j \ell_j$ for disjoint line segments ℓ_j we define

$$Aut_G^{F_i\mathcal{W}}(X^{\circ\prime}) := Loc_G(X^{\circ\prime}) \times_{Loc_G(\coprod_j \ell_j)} Hom(\coprod_j (\ell_j)_B, G/G).$$

Similarly for $\mathcal{W}\backslash\mathcal{J} = \coprod_j \ell_j$ for disjoint line segments ℓ_j we define

$$Aut_G^{\mathcal{W}}(X^{\circ\prime}) := Loc_G(X^{\circ\prime}) \times_{Loc_G(\coprod_j \ell_j)} Hom(\coprod_j (\ell_j)_B, G/G).$$

Definition 4.3.2. We define

$$Aut^{F_i\mathcal{W},\tilde{X}^{\circ\prime}}(X^{\circ\prime}) := Loc_N^{X^{\circ\prime},S}(X^{\circ\prime}) \times_{Loc_G(X^{\circ\prime})} Aut_G^{F_i\mathcal{W}}(X^{\circ\prime}),$$

where the map $Loc_N^{X^{\circ\prime},S}(X^{\circ\prime}) \to Loc_G(X^{\circ\prime})$ is the composition of the forgetful map, with the map coming from inducing from N-local systems to G-local systems;

$$Loc_N^{X^{\circ\prime},S}(X^{\circ\prime}) \xrightarrow{forget} Loc_N(X^{\circ\prime}) \to Loc_G(X^{\circ\prime}).$$

Definition 4.3.3. Similarly we define

$$Aut^{\mathcal{W},\tilde{X}^{\circ\prime}}(X^{\circ\prime}) := Loc_N^{X^{\circ\prime},S}(X^{\circ\prime}) \times_{Loc_G(X^{\circ\prime})} Aut_G^{\mathcal{W}}(X^{\circ\prime}).$$

Note that for N sufficiently large we have $Aut^{F_N\mathcal{W},\tilde{X}^{\circ\prime}}(X^{\circ\prime}) = Aut^{\mathcal{W},\tilde{X}^{\circ\prime}}(X^{\circ\prime})$.

We now provide a section of the forgetful map

$$Aut^{\mathcal{W},\tilde{X}^{\circ\prime}}(X^{\circ\prime}) \to Loc_N^{\tilde{X}^{\circ\prime},S}(X^{\circ\prime}).$$

We do this in two steps; firstly we provide a section s_0 of $Aut^{F_0\mathcal{W},\tilde{X}^{\circ\prime}}(X^{\circ\prime}) \to Loc_N^{\tilde{X},S}(X^{\circ\prime})$, and secondly for each non-negative integer i we provide a lifting s_i of the section s_{i-1}:

$$\begin{tikzcd}
 & Aut^{F_i\mathcal{W},\tilde{X}^{\circ\prime}}(X^{\circ\prime}) \arrow[d, "forget"] \\
Loc_N^{\tilde{X}^{\circ\prime},S}(X^{\circ\prime}) \arrow[r, "s_{i-1}"] \arrow[ur, "s_i"] & Aut^{F_{i-1}\mathcal{W},\tilde{X}^{\circ\prime}}(X^{\circ\prime})
\end{tikzcd}$$

Noting that for N sufficiently large we have $Aut^{F_N\mathcal{W},\tilde{X}^{\circ\prime}}(X^{\circ\prime}) = Aut^{\mathcal{W},\tilde{X}^{\circ\prime}}(X^{\circ\prime})$, composing the sections obtained above gives a section we denote s_{WKB} of the forgetful map

$$\begin{tikzcd} Aut^{\mathcal{W},\tilde{X}^{\circ\prime}}(X^{\circ\prime}) \arrow[r, "forget"'] & Loc_N^{\tilde{X}^{\circ\prime},S}(X^{\circ\prime}). \arrow[l, bend right, "s_{WKB}"'] \end{tikzcd} \tag{4.3.1}$$

4.3.1 Stokes Factors for Initial Stokes Lines

We will being by giving a very explicit calculation for $G = SL_2$. We note that for simply laced groups with a faithful miniscule representation, the path detour rules explained in section 5.2 provide an alternative construction.

Example 4.3.4 (Stokes factors for SL_2). We first note that the monodromy of an N-local system corresponding to a cameral cover $\tilde{X}^\circ \to X^\circ$, will have monodromy around S^1_p (for some $p \in P$ upon choosing a trivialization at a point) of the form

$$\begin{pmatrix} 0 & -a \\ 1/a & 0 \end{pmatrix}.$$

There are three lines in the spectral network emerging from S^1_p. We associate to these lines the automorphisms[2]

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1/a & 1 \end{pmatrix}, \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \tag{4.3.2}$$

[2] We have to say how we which automorphism to associate with a Stokes line. The choice of trivialization of the N-bundle induces a trivialization of the Cameral cover, so parallel transporting around the circle S^1_p in the direction specified by the orientation allows us to label the lines of the spectral network by roots. We then assign to a Stokes line labelled by α the parallel transport of the automorphism in the unipotent subgroup U_α with Lie algebra being the root space $\mathfrak{g}_\alpha \hookrightarrow \mathfrak{g}$.

which are considered as automorphisms of the local system restricted to this point, by parallel transport in the clockwise direction around S^1_p.

These have the property that

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/a & 1 \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -a \\ 1/a & 0 \end{pmatrix}^{-1},$$

which means that after cutting and gluing by these automorphisms we will get a local system with no monodromy around S^1_p.

If we were always looking at groups such that all maps $I_\alpha : SL_2 \to G$ did not factor through PGL_2 we could use the image of the above relations under I_α to define the Stokes factors for the segments of Stokes lines emerging from S^1_p, $p \in P$. However as we need to work in more generality we do the following:

We note that we can write any $n \in N_{SL_2}$, as $n = n_\alpha t$, $t \in T_{SL(2)}$. The map to the automorphisms in equation 4.3.2 then corresponds to mapping $n_\alpha t = Ad_{t^{-1/2}} n_\alpha$ (see Lemma 5.22 in [23]) to

$$Ad_{t^{-1/2}} exp(-e_\alpha), Ad_{t^{-1/2}} exp(-e_{-\alpha}), Ad_{t^{-1/2}} exp(-e_\alpha),$$

where the choice of the square root of t is irrelevant, and we note that we can exponentiate the Chevalley bases elements e_α because these are nilpotent elements of the Lie algebra.

Our map is then the manifestly T_{SL_2}-equivariant map

$$Ad_{t^{-1/2}} n_\alpha \mapsto (Ad_{t^{-1/2}} exp(-e_\alpha), Ad_{t^{-1/2}} exp(-e_{-\alpha}), Ad_{t^{-1/2}} exp(-e_\alpha))$$

Applying this map to SL_2, and taking its image under I_α for each $\alpha \in \mathcal{O}_p$ provides a map of schemes

$$\coprod_{\alpha \in \mathcal{O}_p} n_\alpha T_\alpha \to \coprod_{\alpha \in \mathcal{O}_p} U_\alpha \times U_{-\alpha} \times U_\alpha.$$

Furthermore this is N-equivariant (see Lemma 5.24 of [23]), and hence descends to a map of schemes

$$\left(\coprod_{\alpha \in \mathcal{O}_p} n_\alpha T_\alpha \right) /N \to \left(\coprod_{\alpha \in \mathcal{O}_p} U_\alpha \times U_{-\alpha} \times U_\alpha \right) /N, \tag{4.3.3}$$

where N acts on the factor on the right by conjugating each factor.

This hence (by applying this to each $p \in P$ and translating the automorphisms on the right hand side in the clockwise direction around S^1_p) provides the desired map to Stokes factors for initial Stokes lines, and the section s_0:

$$\begin{tikzcd} Aut^{F_0 \mathcal{W}, \tilde{X}^{\circ\prime}}(X^{\circ\prime}) \arrow[r] & Loc_N^{\tilde{X}, S}(X^{\circ\prime}) \arrow[l, bend right, "s_0"'] \end{tikzcd}$$

4.3.2 Stokes Factors for New Stokes Lines

Definition 4.3.5. A *decorated* 2d scattering diagram is a 2d scattering diagram, such that for each ray r, labelled by α_r we have an element $u_r \in U_{\alpha(r)}$.

These elements u_r must have the property that

$$\prod_{\text{Rays } r} u_r^{\pm 1} = Id,$$

where the order in which we take this product is starting at one ray and then moving around the origin in a clockwise direction[3]. The orientation on rays, and on the plane specify a preferred normal direction to each ray. The exponent in the above product is $+1$ if we cross the ray in this direction, and -1 if we cross the ray in this opposite direction, as we move clockwise around the origin.

[3]The direction and the starting point are irrelevant.

Definition 4.3.6. A *solution* to a 2d scattering diagram, is a morphism of schemes

$$\prod_{\text{Incoming rays } r} U_{\alpha(r)} \xrightarrow{p} \prod_{\text{Outgoing rays } r} U_{\alpha(r)}$$

with the property that for any

$$(u_1, ..., u_i) \in \prod_{\text{Incoming rays } r} U_{\alpha(r)},$$

the labels $(u_1, ..., u_i), p(u_1, ..., u_i)$ make the 2d scattering diagram a decorated 2d scattering diagram.

Theorem 4.3.7 (Theorem 3.35 of [23].)**.** *Every 2d scattering diagram has a solution p. Furthermore, for any $(u_1, ..., u_i)$, the set $(u_1, ..., u_i), p(u_1, ..., u_i)$ is the only decorated 2d scattering diagram, corresponding to the give 2d scattering diagram, and with the incoming rays decorated by $(u_1, ..., u_i)$.*

We omit the proof of the above, and refer to [23].

Corollary 4.3.8. *For a given 2d scattering diagram, and the scattering diagrams produced from it by acting on the roots by the Weyl group W, the solutions gives a map:*

$$\left(\coprod_C \left(\prod_{\alpha \in C} U_\alpha\right)\right)/N \to \left(\coprod_C \left(\prod_{\beta \in Conv^{\mathbb{N}}(C)} U_\beta\right)\right)/N, \tag{4.3.4}$$

where the disjoint union is taken under the sets C of labels of incoming lines, over the different scattering diagrams (related by action of W on labels).

Proof. Firstly note that we have a map

$$\left(\coprod_C \left(\prod_{\alpha \in C} U_\alpha\right)\right) \xrightarrow{\coprod p} \left(\coprod_C \left(\prod_{\beta \in Conv^{\mathbb{N}}(C)} U_\beta\right)\right), \tag{4.3.5}$$

given by the solutions to the 2d scattering diagrams. The N-equivariance follows from the uniqueness statement of Theorem 4.3.7, because if $n \in N$ with $q(n) = w \in W$, and $(u_1, ..., u_i, u_{i+1}, ..., u_{i+j})$ is a decoration of a scattering diagram, then

$$(nu_1n^{-1}, ..., nu_in^{-1}, nu_{i+1}n^{-1}, ..., nu_{i+j}n^{-1})$$

is a decoration of the scattering diagram obtained by acting on all labels of rays by w.

Hence if we denote p_1 the solution to the first scattering diagram, and p_2 the solution to the second, we have that

$$np_1(u_1, ..., u_i)n^{-1} = p_2(nu_1n^{-1}, ..., nu_in^{-1}),$$

where by $np_1(u_1, ..., u_i)n^{-1}$ we mean acting by conjugation on each of the factors U_β, $\beta \in Conv^{\mathbb{N}}(C)$.

Hence the map of equation 4.3.5 is N-equivariant, and descends to the desired map

$$\left(\coprod_C \left(\prod_{\alpha \in C} U_\alpha\right)\right)/N \to \left(\coprod_C \left(\prod_{\beta \in Conv^{\mathbb{N}}(C)} U_\beta\right)\right)/N.$$

$\square$

Applying the map 4.3.4 to the automorphisms of $\mathcal{E}_N|_J \times_N G$ given by the Stokes lines coming into J for each J such that outgoing line segments are in $F_i\mathcal{W}$ provides the desired lift s_i:

$$\begin{tikzcd} & Aut^{F_i\mathcal{W},\tilde{X}^{\circ\prime}}(X^{\circ\prime}) \arrow[d, "forget"] \\ Loc_N^{\tilde{X}^{\circ\prime},S}(X^{\circ\prime}) \arrow[r, "s_{i-1}"] \arrow[ur, "s_i"] & Aut^{F_{i-1}\mathcal{W},\tilde{X}^{\circ\prime}}(X^{\circ\prime}) \end{tikzcd}$$

4.3.3 Reglue Map

We now describe the regluing map, which corresponds to "cutting and regluing" a given G-local system along certain lines, using specified automorphisms of the G-local system.

Construction 4.3.9 (Reglue Map)**.** We construct here a map

$$Aut_G^{\mathcal{W}}(X^{\circ'}) \xrightarrow{reglue} Loc_G(X^{\circ'} \backslash \mathcal{J}).$$

Let $X^{\circ'} \backslash \mathcal{W} = \cup_i U_i$, be the decomposition into connected components. Let $\overline{U_i}$ denote the closure of U_i in $X^{\circ'} \backslash \mathcal{J}$.

Consider the map $\coprod_i \overline{U_i} \xrightarrow{cut} X^{\circ'} \backslash \mathcal{J}$.

Let the universal object over $Aut_G^{\mathcal{W}}(X^{\circ'})$ be $(\mathcal{E}, \{f_i \in Aut(\mathcal{E})|_{Aut_G^{\mathcal{W}}(X^{\circ'}) \times (\ell_i)_B}\})$. Then $cut^*(\mathcal{E})$ is the pullback of $\mathcal{E}$ to the disjoint union of partially compactified connected components of $X^{\circ'} \backslash \mathcal{W}$.

Along each line $l_i \subset \mathcal{W}$, we identify $cut^*(\mathcal{E})$ restricted to the two preimages of $\ell_i \backslash (\ell_i \cap \mathcal{J})$ by $f_i|_{\ell_i \backslash (\ell_i \cap \mathcal{J})}$ (when we move in the preferred normal direction to ℓ_i specified by the orientations on X° and on ℓ_i). We denote this by $cut^*(\mathcal{E})/\{f_i\}$, it is a G-local system on $X_B^{\circ'} \times Aut_G^{\mathcal{W}}(X^{\circ'})$, and hence by the universal property provides a map

$$Aut_G^{\mathcal{W}}(X^{\circ'}) \xrightarrow{reglue} Loc_G(X^{\circ'} \backslash \mathcal{J}),$$

which we call the reglue map.

Theorem 4.3.10 (See Construction–Theorem 5.15 in [23])**.** *The map*

$$Loc_N^{\tilde{X}^\circ, S}(X^{\circ'}) \xrightarrow{\cong} Loc_N^{\tilde{X}^{\circ'}, S}(X^{\circ'}) \xrightarrow{s_{WKB}} Aut^{\mathcal{W}, \tilde{X}^{\circ'}}(X^{\circ'}) \to Aut_G^{\mathcal{W}}(X^{\circ'}) \xrightarrow{reglue} Loc_G(X^{\circ'} \backslash \mathcal{J}),$$

factors as

$$\begin{tikzcd}
Loc_N^{\tilde{X}^\circ,S}(X^\circ) \arrow[rr, "nonab"] \arrow[dr] & & Loc_G(X) \arrow[dl, "restriction"] \\
& Loc_G(X^{\circ\prime}\backslash\mathcal{J}) &
\end{tikzcd}$$

This diagram provides the definition of the map $nonab : Loc_N^{\tilde{X}^\circ,S}(X^\circ) \to Loc_G(X)$*.*

Proof. To prove this claim we need to show that the monodromy around S_p^1 for each $p \in P$, and around each $j \in \mathcal{J}$ is the identity.

For joints $j \in \mathcal{J}$ this follows from the definition of a decorated scattering diagram (definition 4.3.5).

For $p \in P$ the monodromy around S_p^1 of the bundle produced by the reglue map is given by

$$\begin{aligned}
&Ad_{t^{-1/2}} n_\alpha Ad_{t^{-1/2}} exp(-e_\alpha) Ad_{t^{-1/2}} exp(-e_{-\alpha}) Ad_{t^{-1/2}} exp(-e_\alpha) \\
&= I_\alpha \left(Ad_{t^{-1/2}} \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) \right) \\
&= I_\alpha(Id) \\
&= Id.
\end{aligned}$$

The result follows. $\square$

Chapter 5

Spectral Descriptions of Nonabelianization

5.1 Spectral and Cameral Covers

Let $\tilde{X} \to X$ be a cameral cover. We will give alternative descriptions of the data provided by these, and of the N-bundles corresponding to these in terms of *spectral covers*, and local systems of vector spaces on these. This is the ahistorical direction: Spectral covers were introduced in [20] to describe fibers of the Hitchin moduli space for classical groups. See [4, 31] for further results in this direction for $G = GL(n)$. Cameral covers were introduced in [10, 32, 8] as a replacement of spectral covers for arbitrary reductive algebraic groups G. In [9] the Hitchin fiber for a smooth cameral cover is described in terms of N-shifted, weakly W-equivariant T-bundles on the cameral cover. Some results on the relation between descriptions of Hitchin fibers using cameral and spectral covers are found in [8], and for $GL(n)$ the precise relation for the regular part of Hitchin fibers is found in [9]. In this section we are using the *unramified* cameral cover $\tilde{X}^\circ \to X^\circ$, and as such don't need to engage with the difficult question of what happens at ramification points, and instead only need to provide a description of the S-monodromy condition in terms of the local systems on the spectral covers.

5.1.1 Constructing Spectral Covers from Cameral Covers

Construction 5.1.1 (Spectral covers from unramified Cameral covers.)**.** We work with a fixed reductive algebraic group G with maximal torus T, and with a cameral cover $\tilde{X} \to X$.

Let $\rho : G \to GL(V)$ be a representation, with weights Ω_ρ.

We define the *non-embedded* spectral cover associated to ρ as

$$\overline{X}_\rho^{ne} := \tilde{X} \times_W \Omega_\rho. \tag{5.1.1}$$

Note that this is equipped with a map $\pi : \overline{X}_\rho^{ne} \to X$

We denote by $R_\rho \subset \overline{X}_\rho^{ne}$ the ramification divisor. We retain P to denote the branch locus of $\tilde{X} \to X$. Note that the branch locus of $\overline{X}_\rho^{ne}$ is a subdivisor of P.

We also define the real blow up

$$\overline{X}_\rho^{ne,\circ_P} := ReBl_{\pi^{-1}(P)}(\overline{X}_\rho^{ne}). \tag{5.1.2}$$

Note that $\overline{X}_\rho^{ne,\circ_P} \cong \tilde{X} \times_W \Omega_\rho$.

Finally we define

$$\overline{X}_\rho^{ne,\circ_R} := ReBl_{\pi^{-1}(R_\rho)}(\overline{X}_\rho^{ne}). \tag{5.1.3}$$

We also introduce the following variant:

Pick decomposition of the induced representation of the normalizer $N \hookrightarrow T$, as $\rho|_N = \oplus_i \rho_i$ where the weight spaces of each ρ_i are one dimensional. Let Ω_{ρ_i} be the weights of ρ_i. Note that there always exists such a decomposition; one can be constructed by taking the weight space decomposition of ρ, and then for one weight in each W-orbit of the weights, we pick a basis of the weight space. We then take the ρ_i to be the representation of N corresponding to the vector subspace of V spanned by the N-orbit of one of the basis elements.

Definition 5.1.2. Let G, T, $\tilde{X} \to X$, ρ, $\rho|_N = \oplus_i \rho_i$ be as above.

We then define:

$$\overline{X}_\rho^{ne,r} := \coprod_i \tilde{X} \times_W \Omega_{\rho_i}.$$

We again use $R_\rho \subset \overline{X}_\rho^{ne,r}$ to denote the ramification divisor, we hope this does not cause confusion.

As in the preceding case we also define the real blow ups:

$$\overline{X}_\rho^{ne,r,\circ_P} := ReBl_{\pi^{-1}(P)}(\overline{X}_\rho^{ne,r}), \tag{5.1.4}$$

and

$$\overline{X}_\rho^{ne,r,\circ_R} := ReBl_{\pi^{-1}(R_\rho)}(\overline{X}_\rho^{ne,r}). \tag{5.1.5}$$

5.1.2 Local systems on Spectral covers from N-local systems

In this section we describe how to construct local systems of vector spaces on spectral covers from N-local systems corresponding to a given cameral cover. We also consider how imposing the S-monodromy condition affects which local systems we produce.

Construction 5.1.3. We fix T, G, $\tilde{X} \to X$, and ρ as before.

Let $\mathcal{E}_N \to X^\circ$ be an N-bundle on X°, equipped with an isomorphism $\mathcal{E}_N/T \xrightarrow{\cong} \tilde{X}^\circ$ as W-bundles.

Let $V = \oplus_{\omega \in \Omega_\rho} V_\omega$ be the decomposition of V as a direct sum of weight spaces V_ω of T.

Define

$$\mathcal{L} := \mathcal{E}_N \times_N \left(\coprod_{\omega \in \Omega_\rho} V_\omega \right).$$

Note that the map $\left(\coprod_{\omega \in \Omega_\rho} V_\omega\right) \to \Omega_\rho$, which maps the weight space V_ω to the point ω provides a map

$$\mathcal{L} \to \overline{X}_\rho^{ne,\circ_P}.$$

This realizes $\mathcal{L}$ as a local system of vector spaces on $\overline{X}_\rho^{ne,\circ_P}$.

Applying this to the universal bundle, and applying the universal property this provides a map:

$$Loc_N^{\tilde{X}^\circ}(X^\circ) \to LocVect(\overline{X}_\rho^{ne,\circ_P}),$$

where $LocVect(\overline{X}_\rho^{ne,\circ_P})$ is the indstack of local systems of vector bundles on $\overline{X}_\rho^{ne,\circ_P}$.

We provide also the following variant:

Construction 5.1.4. We fix T, G, $\tilde{X} \to X$, and ρ as before, together with a decomposition $\rho|_N = \oplus_i \rho_i$ as in section 5.1.1.

Applying construction 5.1.3 to each representation ρ_i, noting that we actually only require an N-representation rather than a G representation in this construction, provides a map which we denote Sp_ρ:

$$Loc_N^{\tilde{X}^\circ}(X^\circ) \xrightarrow{Sp_\rho} Loc_{\mathbb{G}_m}(X_\rho^{ne,r,\circ_P}),$$

where we are using the equivalence between local systems of one dimensional vector spaces and $\mathbb{G}_m$ local systems.

5.2 Path Detour Rules and Miniscule Representations

For $G = GL(n)$ [16] interpret the automorphisms that we "cut and reglue" by in the non-abelianization construction of section 4.3 in terms of sums of monodromies on paths produced from the spectral network by the "path-detour rules." In the case of a simply laced group, together with a miniscule and faithful representation, [27] suggest extending path detour rules to this setting, but leave open the question of the precise maps of moduli spaces of local systems that one could construct in this way.

In this section we define a version of this which we call path detour non-abelianization in the case of a simply laced group G, and a miniscule representation $\rho : G \to GL(V)$. We also show the compatibility of path detour non-abelianization with the non-abelianization of section 4.3.

Recall:

Definition 5.2.1. A *miniscule* representation $\rho : G \to GL(V)$ is an irreducible representation where the Weyl group W acts transitively on the weights of the representation.

We now describe the path detour nonabelianization construction following [16] for $SL(n)$ and $GL(n)$, and the proposal of [27] to extend this.

Construction 5.2.2 (Path Detour Nonabelianization, following [16, 27])**.** Pick a simply laced reductive algebraic group G, an abstract spectral network $\mathcal{W}$ where at all joints only two Stokes lines intersect, and a miniscule representation $\rho : G \to GL(V)$.

We will construct the *path detour non-abelianization* map $nonab_{PD}$ as the factoring of a map $reglue \circ s_{PD}$, for a map s_{PD} we will define:

$$\begin{tikzcd} Loc_{\mathbb{G}_m}(\overline{X}_\rho^{ne,\circ_R}) \arrow[r, dashed, "nonab_{PD}"] \arrow[d, "s_{PD}"] & Loc_{GL(V)}(X^\circ) \arrow[d] \\ Aut_{GL(V),\mathcal{W}}(X^{\circ\prime}) \arrow[r, "reglue"] & Loc_{GL(V)}(X^\circ \backslash \mathcal{J}) \end{tikzcd}$$

To define S_{PD} we will apply the iterative process as in section 4.3, providing first a map

$$s_{PD,0} : Loc_{\mathbb{G}_m}(\overline{X}_\rho^{ne,\circ_R}) \to Aut_{GL(V)}^{F_0\mathcal{W}}(X^{\circ\prime}),$$

and then for each $i \in \mathbb{N}$ we provide a lift

$$\begin{tikzcd} & Aut_{GL(V)}^{F_i\mathcal{W}}(X^{\circ\prime}) \arrow[d, "forget"] \\ Loc_N^{\tilde{X}^{\circ\prime},S}(X^{\circ\prime}) \arrow[ur, "s_{PD,i}"] \arrow[r, "s_{PD,i-1}"] & Aut_{GL(V)}^{F_{i-1}\mathcal{W}}(X^{\circ\prime}) \end{tikzcd}$$

To define $s_{PD,i}$ for $i \neq 0$ we note that by assumption that there is at most one new Stokes line coming out of each joint. For Stokes lines entering a joint, we continue the automorphism (as a locally constant section of $Aut(\mathcal{E})$) to the segment of this Stokes line emerging from this joint. For the new Stokes line we assign the unique automorphism, such that after regluing the monodromy around the joint will be the identity.

To define $s_{PD,0}$ we do the following: Let $x_p \in S_p^1$. Choosing a trivialization of $\tilde{X}^\circ|_{x_p}$ gives a labelling of $\overline{X}_\rho^{ne} \times_X \{x_p\}$ by weights of the representation ρ. For a root α, (note that $-\alpha$ is also such a root) such that the monodromy of $\tilde{X}^\circ$ around S_p^1 is s_α with respect to this trivialization we consider the set of pairs (x_i, x_j), $x_i, x_j \in \overline{X}_\rho^{ne} \times_X \{x_p\}$

with the property that $l(x_i) - l(x_j) = \alpha$, where $l(x_m)$ refers to the weight labelling the branch x_m is on, with respect to the chosen trivialization. Note that the set of pairs of $\overline{X}_\rho^{ne} \times_X \{x_p\}$ is independent of the choice of trivialization because if $l(x_i) - l(x_j) = \alpha$, then $w(l(x_i)) - w(l(x_j)) = w(\alpha)$.

To a line bundle $\mathcal{L}$ on $\overline{X}_\rho^{ne,\circ_R}$, we define $\mathcal{E} := \pi_*\mathcal{L}$. We define d_{x_i,x_j} to be the parallel transport in $\mathcal{L}$ attached to the lift starting at x_i of the path γ_p going around p in the opposition orientation to that of S^1_p.

To a Stokes line emitted from S^1_p, which is labelled by α with respect to the choice of trivialization of $\tilde{X}^\circ$, we let

$$S_{detour} := \exp\left(\sum d_{x_i,x_j}\right) \in GL(\mathcal{E}|_{x_s}),$$

(using d_{x_i,x_j} for the choice of root α).

Doing this in families, for each initial Stokes line provides the section

$$s_{PD,0} : Loc_{\mathbb{G}_m}(\overline{X}_\rho^{ne,\circ_R}) \to Aut^{F_0\mathcal{W}}_{GL(V)}(X^{\circ\prime}),$$

together with the specified lifts $s_{PD,i}$ this specifies the map s_{PD} as the composition:

$$\begin{tikzcd} Loc_{\mathbb{G}_m}(\overline{X}_\rho^{ne,\circ_R}) \arrow[r, "s_{PD,N}"] \arrow[rr, bend left, "s_{PD}"] & Aut^{F_N\mathcal{W}}_{GL(V)}(X^{\circ\prime}) \arrow[r, "\cong"] & Aut^{\mathcal{W}}_{GL(V)}(X^{\circ\prime}). \end{tikzcd}$$

Theorem 5.2.3 (See Theorem 6.11 in [23])**.** *In the setting where both maps are defined, there is a commutative diagram:*

$$\begin{tikzcd} Loc_N^{\tilde{X}^\circ,S}(X^\circ) \arrow[r, "Sp_\rho"] \arrow[d, "nonab"] & Loc_{\mathbb{G}_m}(\overline{X}_\rho^{ne,\circ_R}) \arrow[d, "nonab_{PD}"] \\ Loc_G(X) \arrow[r, "\rho"] & Loc_{GL(V)}(X^\circ) \end{tikzcd} \tag{5.2.1}$$

We will use the following lemma:

Lemma 5.2.4. *Let G be a simply laced reductive algebraic group.*

Let $c : SL(2) \xrightarrow{I_\alpha} G \xrightarrow{\rho} GL(V)$ be the composition of I_α with a minuscule representation ρ.

Then the representation c is a direct sum of one and two dimensional irreducible representations of $SL(2)$.

Proof. If the group G is simple Corollary 6.6.6 of [19] says that the weights λ of the Lie algebra representation $Lie(\rho)$ have the property that $s_\alpha(\lambda) \in \{\lambda, \lambda+\alpha, \lambda-\alpha\}$.

The identification of $\Lambda_{char} \hookrightarrow \mathfrak{t}_{SL(2)}$ with $\mathbb{Z} \hookrightarrow \mathbb{C}$ identifies the maps $\mathfrak{t}_G^\vee \to \mathfrak{t}_{SL(2)}^\vee$ with the map $\mathfrak{t}_G^\vee \xrightarrow{\alpha^\vee} \mathfrak{t}_{SL(2)}^\vee$. The reason for this is that the dual maps are $\mathfrak{t}_{SL(2)} \to \mathfrak{t}_G$, $h_{\alpha,SL(2)} \mapsto h_\alpha$, and $\mathbb{C} \to \mathfrak{t}_G$, $1 \mapsto \alpha^\vee$, and $h_\alpha = \alpha^\vee$. As $\alpha^\vee(\alpha) = 2$, corollary 6.6.6 of [19] implies that the weights $\lambda_{\mathfrak{sl}(2)}$ of the $\mathfrak{sl}(2)$ representation c satisfy $s_\alpha(\lambda_{\mathfrak{sl}(2)}) \in \{\lambda, \lambda+2, \lambda-2\}$. The result follows for G simple.

For a general reductive algebraic group, noting that the result only depends on the Lie algebras we can assume without loss of generality that G is simply connected. In this case $G = T_1 \times \Pi_i G_i$, where T_1 is abelian, and G_i is simple for each i. A miniscule representation of G is of the form $V = V_{T_1} \otimes \bigotimes_i V_i$ of $(T_1 \times \Pi_i G_i)$, where V_i is miniscule representation of G_i for each i, and V_{T_1} is a one dimensional representation of T_1. The result then follows from the simple case. $\square$

Proof of Theorem 5.2.3. It is clear that we need only show that the following diagram commutes:

$$\begin{tikzcd} Loc_N^{\tilde{X}^\circ}(X^\circ) \arrow[r, "s_0"] \arrow[d, "Sp_\rho"] & Aut^{F_{i-1}\mathcal{W}, \tilde{X}^{\circ\prime}}(X^{\circ\prime}) \arrow[d] \\ Loc_{\mathbb{G}_m}(\overline{X}_\rho^{ne,\circ_R}) \arrow[r, "s_{PD,0}"] & Aut_{GL(V)}^{F_0\mathcal{W}}(X^{\circ\prime}) \end{tikzcd}$$

where the map on the right hand side of the above commutative square is the composition

$$Aut^{F_{i-1}\mathcal{W},\tilde{X}^{\circ\prime}}(X^{\circ\prime}) \to Aut_G^{F_0\mathcal{W}}(X^{\circ\prime}) \to Aut_{GL(V)}^{F_0\mathcal{W}}(X^{\circ\prime}).$$

We pick a trivialization of the an N-bundle corresponding to $\tilde{X}^{\circ\prime}$ at $x_p \in S_p^1 \subset X^{\circ\prime}$, such that the monodromy around S_p^1 (in the direction specified by the orientation of S_p^1) is n_α. We then have that the monodromy around the path γ_p in construction 5.2.2 is $\rho(n_\alpha^{-1})$. By lemma 5.2.4 we have that

$$\rho(n_\alpha^{-1}) = \rho(-e_\alpha) + \rho(-e_{-\alpha}) + Id_{\oplus_{\omega|\omega(\alpha)=0} V_\omega},$$

which gives the identification $\rho(-e_\alpha) = \sum d_{i,j}$.

We hence have that the Stokes factor assigned by $s_{PD,0}$ to a line labelled by α with respect to the chosen trivialization at x_p is given by $s_{PD} = exp(\sum d_{i,j}) = exp(\rho(-e_\alpha)) = \rho(exp(-e_\alpha))$. The case of lines labelled by $-\alpha$ is analogous. □

5.3 Spectral Description of N-Local Systems For Classical Groups

In this section we describe the relation between the moduli of N-local systems corresponding to a given cameral cover (satisfying the S-monodromy condition) and certain moduli spaces of $\mathbb{G}_m$-local systems on spectral covers. These results appear in §6.4 of [23].

5.3.1 The Case of $GL(n)$, and $SL(n)$

Firstly we note that for $GL(n)$ we can describe the Hitchin base, associated to X^c and the line bundle $K_{X^c}(D)$ as

$$\oplus_{i=1}^n \Gamma(X, (K_{X^c}(D))^{\otimes i}) \cong \Gamma(X^c, \mathfrak{t}_{GL(n), K_{X^c}(D)}/W),$$

by using the basis of $\mathbb{C}[\mathfrak{t}]^W$ given by the elementary symmetric polynomials. Specifically the i^{th} elementary symmetric polynomial gives a map $\mathfrak{t}_{K_{X^c}(D)}/W \to K_{X^c(D)}^{\otimes i}$.

For $G = SL(n)$ we have a basis of $\mathbb{C}[\mathfrak{t}]^W$, corresponding under the inclusion $\mathfrak{t}_{SL(n)} \to \mathfrak{t}_{GL(n)}$ to the elementary symmetric polynomials with the exception of the trace.

We hence have an isomorphism

$$\oplus_{i=2}^n \Gamma(X, (K_{X^c}(D))^{\otimes i}) \cong \Gamma(X^c, \mathfrak{t}_{SL(n), K_{X^c}(D)}/W).$$

Given a point in the $GL(n)$ Hitchin base, potentially one corresponding to a point in the $SL(n)$ Hitchin base we can define the spectral curve as follows:

Definition 5.3.1. The *spectral cover* $\overline{X}^c$ associated to $(a_1, ..., a_n) \in \oplus_{i=1}^n \Gamma(X, (K_{X^c}(D))^{\otimes i})$ is the locus in $Tot(K_{X^c}(D))$ where the section

$$\Sigma_{i=0}^n (-1)^{n-i} \lambda^{n-i} a_i$$

(which we call the characteristic polynomial) of $p^*(K_{X^c}(D))^{\otimes n}$ intersects the zero section. Here p denotes the map $p : Tot(K_{X^c}(D)) \to X^c$, and λ is the tautological section of $p^*(K_{X^c}(D))$.

This is equipped with the map $\pi = p|_{\overline{X}^c} : \overline{X}^c \to X^c$.

We denote $\overline{X} := \overline{X}^c \times_{X^c} X$. We denote $\overline{X}^{\circ_R} := ReBl_R(X)$ the real blow up along the ramification locus $R \subset \overline{X}$ of $\overline{X} \to X$. We denote by $\overline{X}^{\circ_P} := ReBl_P(X)$ the real blow of $\overline{X}$ up along the preimage $\pi^{-1}(P)$ of the branch locus P of $\tilde{X} \to X$ along the map $\pi : \overline{X} \to X$.

Proposition 5.3.2. *For $G = GL(n)$ or $G = SL(n)$, and $a \in \mathcal{A}^{\diamondsuit}$ there is an isomorphism of schemes $\overline{X}^{ne}_{\rho} \cong \overline{X}$ between the non-embedded spectral cover for ρ the defining representation, and the spectral cover.*

Proof. Choose an identification $\mathfrak{t} \cong \mathbb{C}^n$, that identifies W with S_n acting by permuting the copies of $\mathbb{C}$. A projection $\mathfrak{t} \cong \mathbb{C}^n \to \mathbb{C}$ to one of these copies of $\mathbb{C}$ provides a map $\mathfrak{t}_{K_{X^c}(D)} \to Tot(K_{X^c}(D))$ that maps $\tilde{X} \to \overline{X}$. This factors through the map $\overline{X}^{ne}_{\rho}$, giving an isomorphism $\overline{X}^{ne}_{\rho} \cong \overline{X}$. $\square$

Proposition 5.3.3 (Spectral and Cameral descriptions of local systems for $Gl(n)$ without S-monodromy condition, cf. [9])**.** *For $G = GL(n)$ there is an isomorphism of stacks*

$$Loc_{\mathbb{G}_m}(\overline{X}^{\circ_P}) \to Loc_N^{\tilde{X}^{\circ}}(X^{\circ_P})$$

Proof. Note that for each $i \in \{1, ..., n\}$ we have an inclusion $S_{n-1} \hookrightarrow S_n$. Let $\pi_i : \tilde{X} \to \overline{X}$ be the map coming from quotienting by S_{n-1}, or equivalently in the proof of 5.3.2 comes from projecting onto the i^{th} copy of $\mathbb{C}$.

Let $\mathcal{L} \in Loc_{\mathbb{G}_m}(\overline{X}^{\circ_P})$, and $\mathcal{E}_T \in Loc_N^{\tilde{X}^{\circ}}(X^{\circ_P})$. We have inverse maps given by

$$\mathcal{L} \mapsto \oplus_{i=1}^{n} \pi_i^* \mathcal{L},$$

and construction 5.1.4.

Note that to realize $\oplus \pi_i^* \mathcal{L}$ as an N-local system we are using proposition 4.1.6, and that as for $GL(n)$ the sequence $1 \to T \to N \to W \to 1$ splits, an N-shifted weakly

W-equivariant T-bundle on $\tilde{X}^\circ$ is equivalent to a W-equivariant T bundle on $\tilde{X}^\circ$, which is what $\oplus\pi_i^*\mathcal{L}$ is. □

Proposition 5.3.4 (Spectral and Cameral descriptions of local systems for $SL(n)$, without S-monodromy condition. cf. [8])**.** *We have an isomorphism of stacks:*

$$Loc^{\tilde{X}^\circ}_{N_{SL(n)}}(X^\circ) \xrightarrow{\cong} Loc_{\mathbb{G}_m}(\overline{X}^{\circ P}) \times_{Loc_{\mathbb{G}_m}(X^\circ)} \{\underline{\mathbb{C}}_X|_{X^\circ}\}. \tag{5.3.1}$$

The map $Loc_{\mathbb{G}_m}(\overline{X}^{\circ P}) \to Loc_{\mathbb{G}_m}(X^\circ)$ is given by $\mathcal{L} \mapsto det(\pi_*\mathcal{L}) = (\pi_*\mathcal{L})^{\wedge n}$. The local system $\underline{\mathbb{C}}_X$ is the constant local system on X.

Proof. Firstly note that it is clear that the map of construction 5.1.4 factors through the map in the above proposition.

We apply the map $\mathcal{L} \mapsto \oplus_{i=1}^n \pi_i^*\mathcal{L}$ to a $\mathbb{G}_m$-local system on the spectral curve. We have the additional data of an isomorphism $\otimes_{i=1}^n \pi_i^*\mathcal{L} \cong \mathbb{C}_{\tilde{X}^\circ}$.

Note that $T_{SL(n)}$ can be described as the kernel in the short exact sequence

$$1 \to T_{SL(n)} \to T_{GL(n)} \xrightarrow{det} \mathbb{G}_m \to 1.$$

Hence the data of this isomorphism is a reduction of structure of the $T_{GL(n)}$-local system to a $T_{SL(n)}$-local system $\mathcal{E}_{T_{SL(n)}}$.

This inclusion $N_{SL(n)} \to N_{GL(n)} \xrightarrow{\gamma} Aut_X(\oplus_{i=1}^n \mathcal{L}_i)$ factors through $Aut_X(\mathcal{E}_T)$ as shown in the diagram:

$$\begin{tikzcd} N_{SL(n)} \arrow[r] \arrow[d] & N_{GL(n)} \arrow[d] \\ Aut_{X^\circ}(\mathcal{E}_{T_{SL(n)}}) \arrow[r] & Aut_{X^\circ}(\oplus_{i=1}^n \mathcal{L}_i). \end{tikzcd}$$

This provides an inverse to the map of equation 5.3.1. □

We now consider the modification to the above statements to include the S-monodromy condition:

Proposition 5.3.5 (Spectral and Cameral descriptions of local systems for $SL(n)$ with S-monodromy condition.)**.** *Let $\tilde{X} \to X$ be a smooth cameral cover.*

There is an isomorphism of stacks

$$Loc^{\tilde{X}^\circ, S}_{N_{SL(n)}}(X^\circ) \xrightarrow{\cong} Loc_{\mathbb{G}_m}(\overline{X}^{\circ R}) \times_{Loc_{\mathbb{G}_m}(X^\circ)} \{\, \underline{\mathbb{C}}_{X^\circ}\}$$

Proof. We will show that the maps in proposition 5.3.4 factors as

$$\begin{tikzcd}
Loc^{\tilde{X}^\circ}_{N_{SL(n)}}(X^\circ) \arrow[r, bend right] & Loc_{\mathbb{G}_m}(\overline{X}^{\circ P}) \times_{Loc_{\mathbb{G}_m}(X^\circ)} \{\, \underline{\mathbb{C}}_{X^\circ}\} \arrow[l, bend right] \\
Loc^{\tilde{X}^\circ, S}_{N_{SL(n)}}(X^\circ) \arrow[u] \arrow[r, bend right, dashed] & Loc_{\mathbb{G}_m}(\overline{X}^{\circ R}) \times_{Loc_{\mathbb{G}_m}(X^\circ)} \{\, \underline{\mathbb{C}}_{X^\circ}\} \arrow[u] \arrow[l, bend right, dashed]
\end{tikzcd}$$

Consider a boundary circle $S^1_p \subset \tilde{X}^\circ$ for $p \in P$. The restriction on the right hand side, means that the N-local systems mapped to by the inverse of 5.3.1 have monodromy of the form:

$$\left(\begin{array}{cc|c} 0 & a & 0 \\ -1/a & 0 & 0 \\ \hline 0 & 0 & Id_{n-2} \end{array}\right)$$

where we are using a basis for the fiber at p corresponding to the branches of the spectral cover, with the first two being the ramified branches. Note that there are precisely two such branches, by the assumption that the cameral cover is smooth, and is branched at p.

This restriction is precisely the S-monodromy condition. □

Remark 5.3.6. Proposition 5.3.5 is the motivation for the S-monodromy condition.

Proposition 5.3.7 (Spectral and Cameral descriptions of local systems for $GL(n)$ with S-monodromy condition)**.** *There is an isomorphism of stacks:*

$$Loc^{\tilde{X}^\circ,S}_{N_{GL(n)}}(X^\circ) \xrightarrow{\cong} Loc_{\mathbb{G}_m}(\overline{X}^{\circ R}) \times_{(\mathbb{G}_m/\mathbb{G}_m)^{\#R_\rho}} (\{-1\}/\mathbb{G}_m)^{\#R_\rho},$$

where the map

$$Loc_{\mathbb{G}_m}(\overline{X}^{\circ R}) \to (\mathbb{G}_m/\mathbb{G}_m)^{\#R_\rho}$$

is given by restricting to the preimage of the ramification divisor R_ρ under the map $\overline{X}^{\circ R} \to \overline{X}$.

Proof. We will show that the maps in proposition 5.3.3 factors as

$$\begin{tikzcd}
Loc^{\tilde{X}^\circ}_{N_{GL(n)}}(X^\circ) \arrow[rr, bend right] & & Loc_{\mathbb{G}_m}(\overline{X}^{\circ P})\{\, \underline{\mathbb{C}}_{X^\circ}\} \arrow[ll, bend right] \\
Loc^{\tilde{X}^\circ,S}_{N_{GL(n)}}(X^\circ) \arrow[u] \arrow[rr, dashed, bend right] & & Loc_{\mathbb{G}_m}(\overline{X}^{\circ R}) \times_{(\mathbb{G}_m/\mathbb{G}_m)^{\#R_\rho}} (\{-1\}/\mathbb{G}_m)^{\#R_\rho}\} \arrow[u] \arrow[ll, dashed, bend right]
\end{tikzcd}$$

As in the $SL(n)$ case the S-monodromy condition on an N-local system means that the monodromy of the associated $\mathbb{G}_m$-local system considered as a sheaf on X° $\mathcal{L} \to \overline{X}^{\circ P} \to X^\circ$ must have monodromy in

$$\left(\begin{array}{cc|c} 0 & a & 0 \\ -1/a & 0 & 0 \\ \hline 0 & 0 & Id_{n-2} \end{array}\right)$$

where the basis corresponds to the branches of the spectral cover. This corresponds to a $\mathbb{G}_m$ local system that extends to a local system on $X^{\circ R}$ with monodromy -1 around each ramification point R.

Furthermore all such $\mathbb{G}_m$-local systems correspond to an N-local system satisfying the S-monodromy condition. $\square$

5.3.2 The Case of $Sp(2n)$

We have that $T \cong \mathbb{G}_m^n$, and $W \cong S_n \ltimes \{\pm 1\}^n$.

The defining representation $Sp(2n) \xrightarrow{\rho} GL(2n)$ induces a map $\mathfrak{t}_{Sp(2n)} \to \mathfrak{t}_{GL(2n)}$, which induces a map of the Hitchin bases. By the spectral curve associated to a point in the $Sp(2n)$ Hitchin base we mean the spectral curve associated to the image of this point in the $GL(2n)$ Hitchin base. The weights of the defining representation form a single W-orbit, and we can identify $\overline{X} \cong \overline{X}_\rho^{ne}$ the non-embedded spectral cover and the spectral cover in this case.

Proposition 5.3.8 (Non-embedded and embedded spectral covers for $Sp(2n)$, [8])**.** *There is an isomorphism* $\overline{X}_\rho^{ne} \cong \overline{X}$*.*

Proof. Identify $\mathfrak{t} \cong \mathbb{C}^n$ in a way that intertwines $W \circlearrowright \mathfrak{t}$ with $S_n \ltimes \{\pm 1\}^n \circlearrowright \mathbb{C}^n$. Projecting onto one copy of $\mathbb{C}$ gives a map $\mathfrak{t}_{K_{X^c(D)}} \to Tot(K_{X^c(D)})$ that gives rise to the desired identification. $\square$

The involution $i : Tot(K_{X^c(D)}) \to Tot(K_{X^c(D)})$ coming from the action of -1 preserves the spectral curve, and hence descends to an involution of $\overline{X}$. We also denote the induced involutions of $\overline{X}^{\circ_P}$ and $\overline{X}^{\circ_R}$ by i.

We denote the quotient map by p_i: $\overline{X}^{\circ_P} \xrightarrow{p_i} \overline{X}^{\circ_P}/i$.

Proposition 5.3.9 (Spectral and Cameral Descriptions for $Sp(2n)$ without S-monodromy condition. Cf. [20, 8])**.** *There is an isomorphism of stacks*

$$\begin{array}{c} Loc_N^{\tilde{X}^\circ}(X^\circ) \\ \big\downarrow \cong \\ Loc_{\mathbb{G}_m}(\overline{X}^{\circ_P}) \times_{Loc_{\mathbb{G}_m}(\overline{X}^{\circ_P}/i)} \{\mathbb{C}_{\overline{X}^{\circ_P}/i}\}, \end{array} \tag{5.3.2}$$

where the map

$$Loc_{\mathbb{G}_m}(\overline{X}^{\circ P}) \to Loc_{\mathbb{G}_m}(\overline{X}^{\circ P}/i)$$

corresponds to the map $\mathcal{L} \mapsto \det((p_i)_*\mathcal{L})$ *(on one dimensional local systems, which are equivalent to* $\mathbb{G}_m$ *local systems).*

Proof. Consider $\tilde{X}^\circ \to \overline{X}^{\circ P}/i$ as an $Stab(\{\alpha, -\alpha\})$-bundle, where $Stab(\{\alpha, -\alpha\}) \subset W$ is the stabilizer of the unordered set of the pair of weights $\{\alpha, -\alpha\}$.

There is an isomorphism $\tilde{X}^\circ \times_{Stab(\{\alpha,-\alpha\})} (V_\alpha \wedge V_{-\alpha}) \cong \mathbb{C}_X|_{X^\circ}$ because $Stab(\{\alpha, -\alpha\})$, acts trivially on $V_\alpha \wedge V_{-\alpha}$.

Construction 5.1.4 factors through a map as in equation 5.3.2. We will construct an inverse.

Let $\mathcal{L} \to \overline{X}^{\circ P}$ be a locally constant line bundle on $\overline{X}^{\circ P}$ together with a trivialization of $det((p_i)_*\mathcal{L})$.

Let $A := \coprod_{-n \leq i \leq n, i \neq 0} \mathbb{C}_i$ be the disjoint union of $2n$ copies of $\mathbb{C}$. Consider invertible homomorphisms from $A \to A$ that consist of:

- A permutation σ of $\{-n, ..., -1, 1, ..., n\}$,
- A linear map $\mathbb{C}_i \to \mathbb{C}_{\sigma(i)}$ for each $i \in \{-n, ..., -1, 1, ..., n\}$.

These can be identified with $N_{GL(2n)}$. We will now describe a subset that can be identified with $N_{Sp(2n)} \hookrightarrow N_{GL(2n)}$.

For each $i \in \{1, ..., n\}$ define the isomorphism $\mathbb{C}_i \wedge \mathbb{C}_{-i} \xrightarrow{\cong} \mathbb{C}$ that sends $1 \wedge 1$ to 1 for $i > 0$ (and hence $1 \wedge 1 \mapsto -1$ for $i < 0$).

Define the subset of invertible homomorphisms $A \to A$ such that:

- $\sigma \in S_n \ltimes \{\pm 1\}^n \cong W_{Sp(2n)} \hookrightarrow W_{GL(n)} \cong S_{2n}$.

- For $1 \leq i \leq n$ the following diagram commutes:

$$\begin{tikzcd}
\mathbb{C}_i \wedge \mathbb{C}_{-i} \arrow[d] \arrow[dr, "\cong"] & \\
& \mathbb{C} \\
\mathbb{C}_{\sigma(i)} \wedge \mathbb{C}_{\sigma(-i)} \arrow[ur, "\cong"] &
\end{tikzcd}$$

The group of these homomorphisms is $N_{Sp(2n)}$.

To $\mathcal{L} \in Loc_{\mathbb{G}_m}(\overline{X}^{\circ P}) \times_{Loc_{\mathbb{G}_m}(\ \overline{X}^{\circ P}/i)} \{\underline{\mathbb{C}}_{\overline{X}^{\circ P}/i}\}$ we define a locally constant sheaf $Hom^{Sp}(\mathcal{L}, A \times X^\circ)$ on X°, where over an open set $U \subset X^\circ$ an element $b \in Hom^{Sp}(\mathcal{L}, A \times X^\circ)$ consists of the data of:

- An invertible morphism $b_1 : \overline{X}^{\circ P}|_U \to \{-n, ..., -1, 1, ..., n\} \times U$ of locally constant sheaves of sets over U. We require that b intertwines i and the involution sending $j \mapsto -j$. We denote by $b_{1,K}^{-1}$ is the map $(X^{\circ,P}/i)|_U \to \{1, ..., n\} \times U$ induced by b_1

- Isomorphisms $\mathcal{L}|_{w_1^{-1}(j \times X)} \xrightarrow{b_{2,j}} \underline{\mathbb{C}_j}$, of locally constant sheaves of sets such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{L}|_{b_1^{-1}(\{j\} \times U)} \wedge \mathcal{L}|_{b_1^{-1}(\{-j\} \times U)} \arrow[r] \arrow[d, "b_{2,j} \wedge b_{2,-j}"] & \underline{\mathbb{C}}_{\overline{X}^{\circ P}/i}|_{b_{1,K}^{-1}\{j\} \times U} \arrow[d, "\cong"] \\
\underline{\mathbb{C}_j}_U \wedge \underline{\mathbb{C}_{-j}}_U \arrow[r] & \underline{\mathbb{C}}_U
\end{tikzcd}$$

Postcomposing these morphisms with the morphisms $A \to A$ that were identified with $N_{Sp(2n)}$ identifies $\mathrm{Hom}^{Sp}(\mathcal{L}, A \times X^\circ)$ as an $N_{Sp(2n)}$ bundle. The associated W bundle is $\tilde{X}^\circ$. The map from $\mathcal{L}$ to $\mathrm{Hom}^{Sp}(\mathcal{L}, A \times X^\circ)$ is an inverse to that of equation 5.3.2.

□

We now consider the case where we impose the S-monodromy condition.

The spectral cover has two different types of branch point. The first is where an eigenvalue, and hence two eigenvalues are equal to zero. We let $R_{\rho,0} \subset \overline{X}$ be the subdivisor of the ramification divisor corresponding to these ramification points.

The second type corresponds to $R \backslash R_{\rho,0}$, This is stable under the involution i, and hence at the associated branch point, there are two pairs of sheets of the spectral cover which coalesce. We let $R_{\rho,1}$ be a subdivisor of the ramification divisor, such that $R_{\rho,1} \cap i_{R_{\rho,1}} = \emptyset$, $R_{\rho,0} \cap R_{\rho,1} = \emptyset$ and $R_{\rho,1} \cup i_{R_{\rho,1}} \cup R_{\rho,0} = R$.

Proposition 5.3.10 (Spectral and Cameral Descriptions for $Sp(2n)$ with S-monodromy condition.)**.** *Construction 5.1.4 applied to $G = Sp(2n)$ induces an isomorphism of stacks:*

$$\begin{gathered} Loc^{\tilde{X},S}_{N_{Sp(2n)}}(X^\circ) \\ \downarrow \cong \\ Loc_{\mathbb{G}_m}(\overline{X}^{\circ R}) \times_{Loc_{\mathbb{G}_m}(\overline{X}^{\circ R}/i) \times (\mathbb{G}_m/\mathbb{G}_m)^{\# R_{\rho,1}}} \left(\{ \underline{\mathbb{C}}_{\overline{X}^{\circ R}/i} \} \times (\{-1\}/\mathbb{G}_m)^{\# R_{\rho,1}} \right). \end{gathered}$$

The map to $(\mathbb{G}_m/\mathbb{G}_m)^{R_{\rho,1}}$ comes from restricting a local system to the preimage of the divisor $R_{\rho,1}$ under the map $\overline{X}^{\circ R} \to \overline{X}$.

Proof. The case of $R_{\rho,0}$ is identical to the $SL(2)$ case by just considering the two branches that are ramified.

For $R_{\rho,1}$ if we consider just one pair of ramified branches the case is identical to the $GL(2)$ case. After we impose the requirement that the monodromy is -1 around one of the ramification points, the monodromy around the other is automatically -1. $\square$

5.3.3 The case of $SO(2n)$

For $G = SO(2n)$ we have that the maximal torus is $T = \mathbb{G}_m^n$ and the Weyl group is $W \cong S_n \ltimes H_n$, for H_n the kernel of the short exact sequence

$$1 \to H_n \to \{\pm 1\}^n \xrightarrow{p} \{\pm 1\} \to 1.$$

The map $p : \{\pm 1\}^n \to \{\pm 1\}$ corresponds to taking the product.

We again associate a point in the Hitchin base to a spectral cover, here by taking the map $\mathfrak{t}_{SO(2n)}/W_{SO(2n)} \to \mathfrak{t}_{GL(2n)}/W_{GL(n)}$ induced by the map $SO(2n) \to GL(2n)$. Here the coefficients of the characteristic polynomial do not form a basis of $\mathbb{C}[\mathfrak{t}_{SO(2n)}]^{W_{SO(2n)}}$, though one can upgrrade this to a basis by replacing the determinant with the Pfaffian. This is the underlying reason that the spectral cover and the embedded spectral cover are here different [8] (cf. [20]).

Proposition 5.3.11 (Non embedded spectral covers and spectral covers for $SO(2n)$ [8], [20])**.** *For $G = SO(2n)$ and $a \in \mathcal{A}^{\diamondsuit}$, the spectral cover $\overline{X}$ associated to a has singularities on the intersection of the zero section of $Tot(K_{X^c}(D))$ and the vanishing locus of the Pfaffian. These are nodal* [1] *singularities.*

There is a map $\overline{X}^{ne} \to \overline{X}$ from the associated non-embedded spectral cover. This map corresponds to the blow up along the intersection of $\overline{X}$ with zero section of $Tot(K_{X^c}(D))$.

Proof. Pick a map $\mathfrak{t} \xrightarrow{\cong} \mathbb{C}^n$ intertwining the actions of W and of $S_n \ltimes H_n$. The action of $S_n \ltimes H_n$ is that induced as a subgroup of $S_n \ltimes \{\pm 1\}^1$. The projection to one copy of $\mathbb{C}$

[1] If we removed the assumption of smoothness of the cameral cover there would be other possible singularities.

induces a map $\mathfrak{t}_{\mathcal{L}} \to \mathcal{L}$ which restricts to a map $\tilde{X} \to \overline{X}$, which factors through a map $\overline{X}^{ne} \to \overline{X}$.

The characteristic polynomial is given by

$$p(\lambda) = \lambda^{2n} + a_2\lambda^{2n-2} + ... + a_{2n-2}\lambda^2 + p_f^2$$

where $p(f)$ is the Pfaffian in $\mathbb{C}[\mathfrak{t}]^W$. Hence the spectral curve is singular at the intersection of the Pfaffian with the zero section of $Tot(K_{X^c}(D))$ [20]. These are accidental singularities in the sense of [8].

As a smooth cameral cover intersects the locus $Pf^{-1}(0) \subset \mathfrak{t}_{K_{X^c}(D)}$ (here 0 denotes the zero section of $K_{X^c}(D)$, and Pf denotes the Pfaffian) transversely, and hence $\tilde{X}/S_{n-1} \ltimes H_{n-1}$ is smooth. Hence as the map $\overline{X}^{ne} \to \overline{X}$ is an isomorphism away from this locus, we have that the map $\overline{X}^{ne} \to \overline{X}$ is the blow up along the intersection of $\overline{X}$ with zero section of $Tot(K_{X^c}(D))$. $\square$

The involution given by the action of -1 on $Tot(K_{X^c}(D))$ induces an involution $i : \overline{X} \to \overline{X}$ as in the case of $Sp(n)$. We also denote by i the induced involutions of $\overline{X}^{ne}$, $\overline{X}^{ne,\circ_R}$, and $\overline{X}^{ne,\circ_P}$. We again have a map $p_i : \overline{X}^{ne,\circ_P} \to \overline{X}^{ne,\circ_P}/i$.

Proposition 5.3.12 (Spectral and Cameral Descriptions for $SO(2n)$ without S-monodromy condition)**.** *Construction 5.1.4 can be enhanced to an isomorphism of stacks*

$$\begin{array}{c} Loc_N^{\tilde{X}^\circ}(X^\circ) \\ \downarrow \cong \\ Loc_{\mathbb{G}_m}(\overline{X}^{ne,\circ_P}) \times_{Loc_{\mathbb{G}_m}(\overline{X}^{ne,\circ_P}/i)} \{\mathcal{M}_{\overline{X}^{ne,\circ_P}/i}\}, \end{array} \tag{5.3.3}$$

where

- *The map*

$$Loc_{\mathbb{G}_m}(\overline{X}^{ne,\circ_P}) \to Loc_{\mathbb{G}_m}(\overline{X}^{ne,\circ_P}/i))$$

is given by $\mathcal{L} \mapsto \det\big((p_i)_*\mathcal{L}\big)$ *(applied to the universal object).*

- *The local system* $\mathcal{M}_{\overline{X}^{ne,\circ_P}/i}$ *on* $\overline{X}^{ne,\circ_P}/i$ *is defined by* $\mathcal{M}_{\overline{X}^{ne,\circ_P}/i} := \tilde{X}^\circ \times_{Stab(\{\alpha,-\alpha\})} (\mathbb{C})$. *Here the action of* $Stab(\{\alpha,-\alpha\})$ *on* $\mathbb{C}$ *is via the map* $Stab(\{\alpha,-\alpha\}) \to \{\pm 1\}$ *with kernel* $Stab(\alpha)$.

Proof. As locally constant sheaves on $\overline{X}^{ne,\circ_P}$ we have $\tilde{X}^\circ \times_{Stab(\{\alpha,-\alpha\})} (V_\alpha \wedge V_{-\alpha}) \cong \mathcal{M}_{\overline{X}^{\circ_P}/i}$. Here α is one of the weights of the defining representation of $SO(2n)$, and we note that $\tilde{X}^\circ \to \overline{X}^{ne,\circ_P}$ is a $Stab(\{\alpha,-\alpha\})$-bundle. The isomorphism is because $Stab(\{\alpha,-\alpha\})$ acts by ± 1 on $V_\alpha \wedge V_{-\alpha}$ (with the subgrouop $Stab(\alpha)$ acting trivially).

Consider A as defined in the proof of proposition 5.3.9. Recall the definition of homomorphisms $A \to A$, and recall that invertible homomorphisms corresponded to $N_{GL(2n)}$. We will now define a subset of such homomorphisms which correspond to $N_{SO(2n)} \hookrightarrow N_{GL(2n)}$.

Consider the isomorphisms $\mathbb{C}_i \wedge \mathbb{C}_{-i} \xrightarrow{\rho} \mathbb{C}$ that sends $1 \wedge 1$ to 1 (for $i > 0$). Consider the subset of homomorphisms $A \to A$ such that:

- $\sigma \in S_n \ltimes H_n$ (identified as a subgroup of S_{2n} via $W_{SO(2n)} \hookrightarrow W_{GL(2n)}$).

- The following diagram commutes for all $1 \le i \le n$;

$$\begin{tikzcd}
\mathbb{C}_i \wedge \mathbb{C}_{-i} \arrow[dr, "\rho"] \arrow[dd] & \\
& \mathbb{C} \\
\mathbb{C}_{\sigma(i)} \wedge \mathbb{C}_{\sigma(-i)} \arrow[ur, "\pm\rho"] &
\end{tikzcd}$$

where we use $+1\rho$ if $\sigma(i) > 0$, and $-\rho$ otherwise.

These homomorphisms correspond to $N_{SO(2n)} \hookrightarrow N_{GL(2n)}$.

We define a locally constant sheaf $Hom^{SO(2n)}(\mathcal{L}, A \times X^\circ)$ on X° associated to each $\mathcal{L} \in Loc_{\mathbb{G}_m}(\overline{X}^{ne,\circ_P}) \times_{Loc_{\mathbb{G}_m}(\overline{X}^{ne,\circ_P}/i)} \{\mathcal{M}_{\overline{X}^{ne,\circ_P}/i}\}$.

Over an open set $U \subset X^\circ$ an element $b \in Hom^{SO(2n)}(\mathcal{L}, A \times X^\circ)$ consists of the data of:

- An isomorphism $a_1 : \overline{X}^{ne,\circ_P}|_U \to \{-n, ..., -1, 1, ..., n\} \times U$ of locally constant sheaves of sets over U. We require that a_1 intertwines i and the involution of $\{-n, ..., -1, 1, ..., n\}$ given by multiplication by -1. We denote by $a_{1,K}^{-1}$ the induced map from $(X^{\circ_P}/i)|_U \to \{1, ..., n\} \times U$ induced by a_1
- Locally constant isomorphisms $\mathcal{L}|_{a_1^{-1}(j \times X)} \xrightarrow{a_{2,j}} \underline{\mathbb{C}_j}$ for each $j \in \{-n, ..., -1, 1, ..., n\}$.
- A morphism of principal $\mathbb{Z}/2$-bundles on $a_{1,K}^{-1}(\{j\} \times U) \subset \overline{X}^{ne,\circ_P}/i$:

$$\overline{X}^{ne,\circ_P}|_{a_{1,K}^{-1}(\{j\}\times U)} \xrightarrow{\cong} (a_{1,K}^{-1}(\{j\} \times U)) \times \mathbb{Z}/2$$

  We denote the induced map $\mathcal{M}_{\overline{X}^{ne,\circ_P}/i}|_{a_{1,K}^{-1}(\{j\}\times U)} \to \mathbb{C}_U$ by $a_{3,j}$,

We require these satisfy the conditions:

- The following diagram commutes:

$$\begin{tikzcd}
\mathcal{L}|_{a_1^{-1}(\{j\}\times U)} \wedge \mathcal{L}|_{a_1^{-1}(\{-j\}\times U)} \arrow[r] \arrow[d, "a_{2,j}\wedge a_{2,-j}"] & \mathcal{M}_{\overline{X}^{ne,\circ_P}/i}|_{a_{1,K}^{-1}\{j\}\times U} \arrow[d, "a_{3,j}"] \\
\underline{\mathbb{C}_j}_U \wedge \underline{\mathbb{C}_{-j}}_U \arrow[r] & \underline{\mathbb{C}}_U
\end{tikzcd}$$

•
$$\bigwedge^{2n}(\oplus_{-n<j<n,j\neq 0}b_{2,j}) = Id.$$

Postcomposing with the homomorphisms $A \to A$ specified above makes $Hom^{SO(2n)}(\mathcal{L}, A\times X^{\circ})$ a $N_{SO(2n)}$ bundle. The map $\mathcal{L} \mapsto Hom^{SO}(\mathcal{L}, A \times X)$ is the inverse of the map in of equation 5.3.3.

□

We now need to consider the S-monodromy condition. The situation is closely related to that of $Sp(2n)$.

We define the divisor $R_{\rho,1} \subset \overline{X}^{ne}$ as in section 5.3.2. In contrast to that section we do not define an $R_{\rho,0}$, the non-embedded spectral cover is the blow up of the locus given by the intersection of the zero section and the vanishing locus of the pfaffian.

Proposition 5.3.13 (Spectral and Cameral Descriptions for $SO(2n)$ with S-monodromy condition.)**.** *For $G = SO(2n)$ there is an isomorphism of stacks, induced by the isomorphism of proposition 5.3.12*

$$\begin{array}{c} Loc^{\tilde{X}^{\circ},S}_{N_{SO(2n)}}(X^{\circ}) \\ \Big\downarrow\cong \\ Loc_{\mathbb{G}_m}(\overline{X}^{ne,\circ_R}) \times_{Loc_{\mathbb{G}_m}(\overline{X}^{ne,\circ_R}/i))\times(\mathbb{G}_m/\mathbb{G}_m)^{\#R_{\rho,1}}} \left(\left\{\mathcal{M}_{\overline{X}^{ne,\circ_R}/i}\right\} \times (-1/\mathbb{G}_m)^{\#R_{\rho,1}}\right) \end{array}$$

where the map to $(\mathbb{G}_m/\mathbb{G}_m)^{R_{\rho,1}}$ corresponds to the restricting the local systems to the preimage of $R_{\rho,1}$ under the map $\overline{X}^{ne,\circ_R} \to \overline{X}^{ne}$, and $\mathcal{M}_{\overline{X}^{ne,\circ_R}/i} := \overline{X}^{ne,\circ_R} \times_{\{\pm 1\}} \mathbb{C}$, where we consider $\overline{X}^{ne,\circ_R} \to \overline{X}^{ne,\circ_R}/i$ as a $\{\pm 1\}$-bundle.

Remark 5.3.14. Note that $\mathcal{M}_{\overline{X}^{ne,\circ_R}/i}|_{\overline{X}^{ne,\circ_P}/i} \cong \mathcal{M}_{\overline{X}^{ne,\circ_R}/i}$ (defined in proposition 5.3.12).

Proof. The analysis is identical to that of Proposition 5.3.10, with the exception of the fact we do not need to consider $R_{\rho,0}$ as mentioned above. $\square$

5.3.4 The case of $SO(2n+1)$

For $G = SO(2n+1)$, the maximal torus is $T \cong \mathbb{G}_m^n$, and the Weyl group is $W \cong S_n \ltimes \{\pm 1\}^n$.

The weights of the defining representations consist of two W-orbits, one consisting of the single weight 0, and one of all the other weights of this representation

We again associate a spectral cover $\overline{X}$ to a point in the Hitchin base, by considering the induced map $\mathfrak{t}_{SO(2n+1)}/W_{SO(2n+1)} \to \mathfrak{t}_{GL(2n+1)}/W_{GL(2n+1)}$ from the map of groups $SO(2n+1) \to GL(2n+1)$. For $a \in \mathcal{A}^{\diamondsuit}$ the associated spectral cover has a partial normalization $\overline{X}_0 \coprod \overline{X}_1 \to \overline{X}$, where $\overline{X}_0$ corresponds to the zero weight, and $\overline{X}_1$ corresponds to the other W-orbit. Note that this map is an isomorphism when restricted away from the locus where $\overline{X}_1 \to Tot(K_{X^c}(D))$ intersects the zero section, because $\overline{X}_0$ corresponds to the zero section of $Tot(K_{X^c}(D))$.

We can identify this disjoint union with the non-embedded spectral cover:

Proposition 5.3.15. *There is an isomorphism*

$$\overline{X}_1 \coprod \overline{X}_0 \cong \overline{X}_\rho^{ne}.$$

Proof. We use the decomposition of the non-embedded spectral cover coming from decomposing Ω_ρ into W-orbits in equation 5.1.1. Denote the orbits by 0, and 1 and write $\overline{X}^{ne,0}$ and $\overline{X}^{ne,1}$ to denote the corresponding subschemes of the non-embedded spectral cover $\overline{X}_\rho^{ne}$. Then $\overline{X}^{ne,0} \cong X \cong \overline{X}_0$. The identification of $\overline{X}^{ne,1} \cong \overline{X}_1$ is identical to the identification in proposition 5.3.8. $\square$

As in the case of $Sp(2n)$ and $SO(2n)$ we have an involution $i : \overline{X}_\rho \to \overline{X}_\rho$ coming from the action of -1 on $Tot(K_{X^c}(D))$.

This involution fixes $\overline{X}_0$ pointwise. This involution maps $\overline{X}_1$ to $\overline{X}_1$.

We decompose the ramification locus as $R_\rho = R_a \coprod R_b$ where $R_a \subset \overline{X}_1$, and $R_b \subset \overline{X}_0$ be the ramification locus of the map $\overline{X}^{ne}_\rho \to X$. We denote by $\overline{X}_1^{\circ_R}$ and $\overline{X}_0^{\circ_R}$ the oriented real blow ups of $\overline{X}_1$ and $\overline{X}_0$ at R_a and R_b respectively. The analysis of the ramification points is closely related to that of $Sp(2n)$; we have two types of branch points, the first is where $\overline{X}_1$ intersects the zero section. We note that by the symmetry of $\overline{X}$ at such a point $\overline{X}_1 \to X$ is ramified. The second is points in $R\backslash(R \cap X_0)$. As in the $Sp(2n)$ these come in pairs, and so as in the $Sp(2n)$ case we decompose $R_a = R_{\rho,0} \coprod R_{\rho,1} \coprod i(R_{\rho,1})$, where $R_{\rho,0}$, $R_{\rho,1}$, $i(R_{\rho,1})$ are effective divisors corresponding to disjoint sets of points.

Proposition 5.3.16 (Spectral and Cameral Descriptions for $SO(2n+1)$ with S-monodromy condition. Cf. [20, 8])**.** *For $G = SO(2n+1)$ construction 5.1.4 induces an isomorphism:*

$$\begin{array}{c} Loc^{\tilde{X}^\circ, S}_{N_{SO(2n+1)}}(X^\circ) \\ \downarrow\cong \\ Loc_{\mathbb{G}_m}(\overline{X}^{ne,\circ_R}) \times_{Loc_{\mathbb{G}_m}(\overline{X}^{ne,\circ_R}/i)\times(\mathbb{G}_m/\mathbb{G}_m)^{\#R_{\rho,1}}} \left(\{\mathcal{M}\} \times (-1/\mathbb{G}_m)^{\#R_{\rho,1}}\right), \end{array}$$

where the map

$$Loc_{\mathbb{G}_m}(\overline{X}^{ne,\circ_R}) \to Loc_{\mathbb{G}_m}(\overline{X}^{ne,\circ_R}/i)$$

corresponds to the map $\mathcal{L} \mapsto \det((p_i)_*\mathcal{L})$, and the map to $(\mathbb{G}_m/\mathbb{G}_m)^{\#R_{\rho,1}}$ corresponds to taking the monodromy around each point in $R_{\rho,1}$.

The $\mathbb{G}_m$-local system $\mathcal{M}$ is defined on $\overline{X}_0^{\circ_R}/i$ by $\mathcal{M}|_{\overline{X}_0^{\circ_R}} := \tilde{X}^\circ \times_W V_0$, where V_0 is the representation of W coming from the N-action on the zero weight space of the defining *representation of* $SO(2n+1)$*. On* $\overline{X}_1^{\circ R}/i$ *it is defined by* $\mathcal{M}|_{\overline{X}_1^{\circ R}/i} := \tilde{X}^\circ \times_{Stab(\{\alpha,-\alpha\})} (\mathbb{C})$*, where* $Stab(\{\alpha,-\alpha\})$ *acts on* $\mathbb{C}$ *via the map* $Stab(\{\alpha,-\alpha\}) \to \{\pm 1\}$ *with kernel* $Stab(\alpha)$*.*

Proof. We first note that the map of 5.1.4 does induce such a map. The argument the induced line bundle on $\overline{X}_1^{\circ R}/i$ is $\mathcal{M}|_{\overline{X}_1^{\circ R}/i}$ is identical to that in the $SO(2n)$ case of propositions 5.3.12, 5.3.13.

The local system induced on $\overline{X}_0^{\circ R} \cong X^\circ$ is then given by:

$$\tilde{X}^\circ \times_W V_0,$$

where V_0 is the representation of W on the zero-weight space of the defining representation of $SO(2n+1)$. Note that this is the representation

$$S_n \ltimes \{\pm 1\}^n \to \{\pm 1\} \circlearrowright \mathbb{C},$$

where the map corresponds to taking the product of the ± 1 factors, and the action on $\mathbb{C}$ is by multiplication.

Consider A as defined in the proof of proposition 5.3.9. Recall the definition of homomorphisms $A \to A$. We will define a subset of the invertible homomorphisms that we identify with $N_{SO(2n+1}$.

Consider the isomorphisms $\mathbb{C}_i \wedge \mathbb{C}_{-i} \xrightarrow{\rho} \mathbb{C}$ sending $1 \wedge 1$ to 1 (for $i > 0$) that were also defined previously.

Consider the subset of invertible homommorphisms $A \to A$ such that.

- $\sigma \in S_n \ltimes \{\pm 1\}^n$ (identified as a subgroup of S_{2n} via the map $W_{SO(2n+1)} \hookrightarrow W_{GL(2n+1)}$ factoring through $W_{GL(2n)}$).

- The following diagram commutes for all $1 \leq i \leq n$;

$$\begin{tikzcd} \mathbb{C}_i \wedge \mathbb{C}_{-i} \arrow[d] \arrow[dr, "\rho"] & \\ & \mathbb{C} \\ \mathbb{C}_{\sigma(i)} \wedge \mathbb{C}_{\sigma(-i)} \arrow[ur, "\pm\rho"] & \end{tikzcd}$$

  where we use $+1\rho$ if $\sigma(i) > 0$, and $-\rho$ otherwise.

We can identify the homomorphisms satisfying these conditions with $N_{SO(2n+1)}$.

Consider the locally constant sheaf on X° associated to

$$\mathcal{L} \in Loc_{\mathbb{G}_m}(\overline{X}_1^{\circ_P}) \times_{Loc_{\mathbb{G}_m}(\overline{X}^{ne,\circ_P}/i)} \{\mathcal{M}\}$$

denoted by $Hom^{SO(2n+1)}(\mathcal{L}|_{\overline{X}_1^{ne,\circ_P}}, A \times X^\circ)$, where over an open set $U \subset X^\circ$ an element $a \in Hom^{SO(2n)}(\mathcal{L}, C \times X^\circ)$ consists of the data of;

- An isomorphism $a_1 : \overline{X}^{ne,\circ_P}|_U \to \{-n, ..., -1, 1, ..., n\} \times U$ of locally constant sheaves of sets over U. We require that a_1 intertwines i and the involution of $\{-n, ..., -1, 1, ..., n\}$ given by multiplication by -1. We denote by $a_{1,K}^{-1}$ the induced map from $(X^{\circ_P}/i)|_U \to \{1, ..., n\} \times U$ induced by a_1

- Locally constant isomorphisms $\mathcal{L}|_{a_1^{-1}(j \times X)} \xrightarrow{a_{2,j}} \underline{\mathbb{C}_j}$ for each $j \in \{-n, ..., -1, 1, ..., n\}$.

- Morphisms of principal $\mathbb{Z}/2$-bundles on $b_{1,K}^{-1}(\{j\} \times U) \subset \overline{X}^{\circ_P}/i$ from $\overline{X}^{\circ_P}|_{b_{1,K}^{-1}(\{j\} \times U)} \xrightarrow{\cong}$ $(b_{1,K}^{-1}(\{j\} \times U)) \times \mathbb{Z}/2$. We denote by $a_{3,j} : \mathcal{M}_{\overline{X}^{\circ_P}/i}|_{b_{1,K}^{-1}(\{j\} \times U)} \to \mathbb{C}_U$ the induced map of locally constant sheaves of vector bundles.

We require that;

- The following diagram commutes:

$$\begin{tikzcd}
\mathcal{L}|_{a_1^{-1}(\{j\}\times U)} \wedge \mathcal{L}|_{a_1^{-1}(\{-j\}\times U)} \arrow[r] \arrow[d, "a_{2,j}\wedge a_{2,-j}"] & \mathcal{M}_{\overline{X}^{\circ_P}/i}|_{a_{1,K}^{-1}\{j\}\times U} \arrow[d, "a_{3,j}"] \\
\underline{\mathbb{C}_j}_U \wedge \underline{\mathbb{C}_{-j}}_U \arrow[r] & \underline{\mathbb{C}}_U
\end{tikzcd}$$

Postcomposing with $A \to A$ corresponding to $n \in N_{SO(2n+1)}$ gives $Hom^{SO(2n+1)}(\mathcal{L}|_{\overline{X}_1^{ne,\circ_R}}, A\times X^\circ)$ the structure of a $N_{SO(2n+1)}$ bundle. The map $\mathcal{L} \mapsto Hom^{SO(2n+1)}(\mathcal{L}, A \times X)$ is then the inverse to the map in proposition 5.3.16.

We now consider the S-monodromy. Considering the effect on the monodromy around $R_{\rho,1}$ is analogous to the $SO(2n)$ and $Sp(2n)$ cases in the proof of propositions 5.3.10 and 5.3.13. The effect at $R_{\rho,0}$ is analogous to that of $Sp(2n)$ in proposition 5.3.10, As the monodromy of the local system on $\overline{X}_0$ then restricts the monodromy to be -1, we have to restrict the monodromy of the system on $\overline{X}_1$ around $r \in R_{\rho,0}$ to be -1. The result follows. $\square$

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