DIFFERENTIAL ESSENTIAL DIMENSION

Man Cheung Tsui

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Supervisor of Dissertation

____

Julia Hartmann, Professor of Mathematics

Graduate Group Chairperson

____

Julia Hartmann, Professor of Mathematics

Dissertation Committee

David Harbater, Christopher H. Browne Distinguished Professor in the School of Arts and Sciences
Julia Hartmann, Professor of Mathematics
Robin Pemantle, Merriam Term Professor of Mathematics

ABSTRACT

DIFFERENTIAL ESSENTIAL DIMENSION

Man Cheung Tsui

Julia Hartmann

We define an analogue of essential dimension in differential Galois theory. As application, we show that the number of coefficients in a general homogeneous linear differential equation over a field cannot be reduced via gauge transformations over the given field. We also give lower bounds on the number of parameters needed to write down certain generic Picard-Vessiot extensions.

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CHAPTER 1

Introduction

Roughly speaking, the essential dimension of an algebraic object is the minimum number of parameters needed to specify the object. It was first introduced by J. Buhler and Z. Reichstein in Buhler and Reichstein 1997 in the context of simplifying polynomials by means of Tschirnhaus transformations, which we now discuss following their presentation. In this section, all fields will be of characteristic zero.

1.1. Simplifying polynomials via Tschirnhaus transformations.

Consider a quadratic polynomial $p(x) = x^2 + ax + b$ in two parameters a and b. The change of variables $y = x - a/2$ simplifies $p(x)$ to a polynomial of the form $q(y) = y^2 + c$ in one parameter c. Similarly, a cubic polynomial may be simplified by a linear transformation to a polynomial $y^3 + cy + c$ in one parameter.

More generally, we allow simplifying polynomials by means of nondegenerate Tschirnhaus transformations. A monic polynomial $q(y)$ is a *nondegenerate Tschirnhaus transformation* of a monic polynomial $p(x)$ over a field K if there exists an isomorphism of K-algebras $K[x]/(p(x)) \cong K[y]/(q(y))$.

Let F be a field and let $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a *general polynomial* of degree n over F, that is, $a_0, ..., a_{n-1}$ are algebraic indeterminates over F. Then $p(x)$ has coefficients in the field $K = F(a_0, ..., a_{n-1})$. We are interested in the minimal number $d(n)$ of algebraically independent coefficients of $q(y)$, as $q(y)$ ranges over the nondegenerate Tschirnhaus transformations of $p(x)$ over K. Above, we saw that $d(2) = d(3) = 1$. Classical results by Klein and Hermite imply that $d(5) = 2$. The determination of $d(n)$ is therefore a classical problem, and the exact values of $d(n)$ remain unknown for $n \geq 8$.

We can reformulate $d(n)$ in the following way which will be more amenable to further study. Let $F \subset K_0 \subset K$ be an inclusion of fields, and M a K-algebra. We say that K_0 is a *field of definition* of M if there exist a K_0-algebra M_0 and an isomorphism of K-algebras $M_0 \otimes_{K_0} K \cong M$. We define the *essential dimension* of M to be the minimal transcendence degree taken over the fields of definition of M. In this language, $d(n)$ is the essential dimension of the algebra $K[x]/(p(x))$ where $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ is the general polynomial of degree n and $K = F(a_0, ..., a_{n-1})$.

To understand the new language, consider a polynomial $p(x) = x^2 + ax + b$ over a field K containing a base field F. The change of variables $y = x - a/2$ simplifies $p(x) = x^2 + ax + b$ to a polynomial $q(y) = y^2 + c$ over K and gives an isomorphism of K-algebras $K[x]/(p(x)) \cong K[y]/(q(y))$. Initially, the algebra $K[x]/(p(x))$ only appears to be defined over the subfield $F(a, b)$ of K. Rewriting it as $K[y]/(q(y))$ makes clear that it is also defined over the subfield $F(c)$ of $F(a, b)$. The essential dimension of $K[x]/(p(x))$ is equal to $\operatorname{trdeg}_F F(c) \leq 1$. If a and b are algebraic indeterminates over F, the essential dimension of $K[x]/(p(x))$ is equal to $\operatorname{trdeg}_F F(c) = 1$.

In this thesis, we define an analogue of essential dimension in differential Galois theory, motivated by a similar problem.

1.2. Simplifying differential equations

Let F be a differential field. Consider the *general differential equation* $p(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y$ of order n over F, that is, $a_0, ..., a_{n-1}$ are indeterminates over F. Then $p(y)$ has coefficients in $K = F\langle a_0, ..., a_{n-1}\rangle$, the field generated by the indeterminates $a_0, ..., a_{n-1}$ and their formal derivatives. We are interested in the minimal number $e(n)$ of differentially algebraically independent coefficients in $q(z)$, as $q(z)$ ranges over the gauge transformations of $p(y)$ over K.

As before, we can formalize this as follows. Let $F \subset K_0 \subset K$ be an inclusion of differential fields, and M a differential K-module. We say that K_0 is a *differential field of definition* of M if there exist a differential K_0-module M_0 and an isomorphism of differential K-modules $M_0 \otimes_{K_0} K \cong M$. We define the *differential essential dimension* of M to be the minimal differential transcendence degree across the differential fields of definition of M. In the setting of the previous paragraph, the number $e(n)$ is simply the differential essential dimension of $M = K[\partial]/K[\partial] \cdot p$. In Chapter 6, we will show that $e(n) = n$ for all $n \geq 1$.

The differential essential dimension also arises in the study of generic Picard-Vessiot extensions. We review the Picard-Vessiot theory in Chapter 2.

1.3. Generic Picard-Vessiot extensions

In Galois theory, given a finite group G, one would like to parametrize the family of all Galois extensions over a field F with Galois group G. For example, quadratic extensions K/F are of the form $K = F(\sqrt{a})$ for some $a \in K$. Therefore one may view the "generic" quadratic field extension as $F(\sqrt{a})/F(a)$ in one parameter a, since any other quadratic extension is obtained by specifying a value of a. The constructions of these "generic" Galois extensions is an active topic of interest.

In differential Galois theory, starting from a homogeneous linear differential equation $p(y)$ over a field of differentiable functions F, one can construct a minimal field extension K/F containing a full set of solutions to $p(y) = 0$. We call K/F a *Picard-Vessiot extension* of $p(y)$. From K/F we can construct the differential Galois group of $p(y)$. One expects Picard-Vessiot extensions with the same differential Galois group to have similar properties. Therefore in this setting, one would like to again parametrize the various Picard-Vessiot extension with the same differential Galois group. We call such a parametrizing extension a "generic" Picard-Vessiot extensions for G.

The generic Picard-Vessiot extensions defined here include the generic Picard-Vessiot extensions in the sense of Juan and Ledet 2016, Section 6 when F is a constant field. The papers of L. Juan, A. Ledet, and A. Magid give explicit constructions of generic G-Picard-Vessiot extensions for certain groups G. Others including L. Goldman and A. Pillay have studied generic Picard-Vessiot extensions (see Goldman 1957).

Since we would like to be as efficient as possible when constructing such generic Picard-

Vessiot extensions, we would like the number of parameters to be small. In Chapter 6, we will show that in certain cases, the number of parameters needed to write down a generic Picard-Vessiot extension is bounded below by a number.

1.4. Outline of thesis

This thesis proceeds as follows. In Chapter 2, we review differential algebra. In Chapter 3, we introduce the general notion of essential dimension, as well as the notion of a versal pair following Berhuy and Favi 2003. In Chapter 4, we introduce twisted forms and cohomology in the differential algebraic setting. In Chapter 5, we construct classifying differential torsors and generic differential torsors. Chapter 6 gives the main results of this thesis.

The general strategy we follow is that of Berhuy and Favi 2003. The applications we give are the differential analogues of those in Buhler and Reichstein 1997.

CHAPTER 2

Background

In this chapter, we review background material and set the notations and conventions used in the thesis. For content on differential algebra, excellent sources include the article Van der Put and Singer 2012 and the textbook Van der Put and Singer 2012.

2.1. Algebra

We let $\mathbb{N}$ denote the set $\{0, 1, 2, ...\}$ of natural numbers.

By an *algebraic group*, we mean an affine group scheme defined over a field that is reduced and of finite type over the field. By Milne 2017, Chapter 4(d), any algebraic group is a closed subgroup of GL_n for some value of n.

A *linear algebraic group* is a smooth algebraic group. By Cartier's theorem Chapter 3(g), an (affine) algebraic group is a linear algebraic group in characteristic zero. Note that Milne 2017 defines a linear algebraic group to not require smoothness; see Remark 4.11.

Let G be an algebraic group over a field C, and let F/C be a field extension. We let G_F denote its base change $G \times_{\mathrm{Spec}(C)} \mathrm{Spec}(F)$ and let $F[G]$ denote the coordinate ring of G_F (in place of the more correct notation $F[G_F]$).

2.2. Differential algebra

A *derivation* on a ring R is an additive map $\partial : R \to R$ that satisfies the Leibniz rule $\partial(rs) = \partial(r)s + r\partial(s)$ for all $r, s \in R$. A *differential ring* is a ring equipped with a derivation. For example, the field $\mathbb{C}(x)$ of complex rational functions and the field $\mathbb{C}((x))$ of Laurent series in x are both differential rings with the derivation d/dx. When the context is clear, derivations are often denoted by the symbol ∂. We often refer to a differential ring (R, ∂) as R. We write $r^{(n)} = (\partial \circ \cdots \circ \partial)(r)$ for the n-fold application of the derivation ∂ on an element $r \in R$. We will not use the notation r' to mean $\partial(r)$.

Let (R, ∂_R) and (S, ∂_S) be differential rings. A *homomorphism* of differential rings from R to S is a ring homomorphism $\varphi : R \to S$ satisfying $\partial_S \circ \varphi = \varphi \circ \partial_R$.

Example 1. Let $(F, \partial) = (\mathbb{C}(x), d/dx)$ and let $S = F[y, 1/y]$ be a differential ring extension of F with derivation determined by $\partial(y) = y$. Since the element $e^x = 1 + x + x^2/2! + x^3/3! + \cdots$ in $\mathbb{C}((x))$ also satisfies $\partial(e^x) = e^x$, we can define an injective differential F-algebra homomorphism $S \to \mathbb{C}((x))$ which takes y to e^x. By identifying S with its image in $\mathbb{C}((x))$, we may think of y as e^x.

Let R be a differential ring. An ideal I of R is a *differential ideal* if $\partial(I) \subset I$. A *differential field* is a differential ring that is a field. The *constant ring* of R is the subring

$$C_R = \{r \in R \mid \partial(r) = 0\}$$

of R. We call C_R the *constant field* of R if C_R is a field. Given a multiplicative subset S of R, the derivation on $S^{-1}R$ defined by

$$\partial(r/s) = (\partial(r)s - r\partial(s))/s^2$$

for all $r \in R$ and $s \in S$ is well-defined and makes the localization map

$$R \to S^{-1}R : r \mapsto r/1$$

a differential ring homomorphism. In particular, if R is an integral domain, the derivation ∂_R extends to a derivation on the field of fractions $\mathrm{Frac}(R)$.

One similarly defines notions like *differential algebras*, *differential coalgebras*, and *differential Hopf algebras* to be algebras, coalgebras, and Hopf algebras with compatible derivations. We leave the definitions to the reader.

Let F be a differential field. A *differential module* over F is a vector space M over F equipped with an additive map $\partial : M \to M$ such that $\partial(rm) = \partial(r)m + r \cdot \partial(m)$ holds for all $r \in F$ and $m \in M$. An element m of M is a *generator* of M if the set $\{m, \partial(m), \partial^2(m), ...\}$ is a spanning set of the vector space M over F. The minimal differential equation of m is the least degree differential equation with leading coefficient one, that is satisfied by m.

Example 2. Let F be a differential field. We define $F[\partial]$ be the (non-commutative) differential ring over F as follows. As a F-vector space, it has the basis $\{\partial^n\}_{n\in\mathbb{N}}$ where we write $\partial^0 = 1$. Multiplication is defined on generators by $\partial^n \circ \partial^m = \partial^{n+m}$ and $\partial \circ a = a^{(1)} + a \circ \partial$.

Let $p(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y$ be a homogeneous linear differential equation over F and let $p = \partial^n + a_{n-1}\partial^{n-1} + \cdots + a_0\partial$. Then $F[\partial] \circ p$ is a left ideal of the (non-commutative) differential ring and the quotient $M = F[\partial]/(F[\partial] \circ p)$ is a differential module over F.

Let M and N be differential modules over a differential ring R. A *homomorphism* of differential modules is a homomorphism of modules M to N that commutes with the derivations. Sometimes one calls a map a *differential homomorphism* of differential modules to distinguish it from a homomorphism of the underlying modules.

Example 3. Let $p(y)$ be a homogeneous linear differential equation over a differential field F and let R/F be a differential algebra containing a solution y_0 for $p(y) = 0$. Then we have a homomorphism of differential F-modules

$$F[\partial] \to R$$

taking q to $q(y_0)$, whose image is the differential module $F[\partial] \cdot y_0$ and whose kernel is $F[\partial] \circ p$.

Remark 1. Let $n \geq 1$. There is a correspondence between the equivalence classes of homogeneous linear differential equations of degree n over a differential field F up to gauge transformation and (differential) isomorphism classes of differential modules of dimension n over F. This proceeds as follows. Let p and q be gauge equivalent

We define a derivation on the tensor product $M \otimes_R N$ by $\partial(m \otimes n) = \partial(m) \otimes n + m \otimes \partial(n)$ for all $m \in M$ and $n \in N$, and extend linearly. One similarly defines the tensor product of two differential algebras.

Example 4. Let F be a differential field with constant field C. Let D be a C-algebra. We regard D as a differential ring with the trivial derivation $\partial = 0$. We view $D \otimes_C F$ as a differential ring. We will take this viewpoint for algebraic groups G defined over C, viewing $C[G]$ as a C-algebra with trivial derivation.

Let S be a differential coalgebra over R with comultiplication Δ and counit ϵ. A *differential S-comodule* over R is a differential module M over R together with a differential R-linear map $\rho : M \to M \otimes S$ such that $(1 \otimes \Delta) \circ \rho = (\rho \otimes 1) \circ \rho$ and $(1 \otimes \epsilon) \circ \rho = 1$.

Let I be a set and let $x_i^{(j)}$ be indeterminates where $i \in I$ and $j \in \mathbb{N}$. A *differential polynomial ring* over R in the indeterminates x_i for $i \in I$ is the polynomial ring $R\{x_i\}_{i\in I} := R[x_i^{(j)}]_{i\in I, j\in\mathbb{N}}$ equipped with a derivation ∂ that extends the derivation on R by letting $\partial x_i^{(j)} = x_i^{(j+1)}$ for all $i \in I$ and $j \in \mathbb{N}$. Given a differential R-algebra S and a subset $\{s_i\}_{i\in I}$ of S, the image of the differential ring homomorphism $\varphi : R\{x_i\}_{i\in I} \to S : x_i \mapsto s_i$ is denoted by $R\{s_i\}_{i\in I}$. If φ is injective, we say that $\{s_i\}_{i\in I}$ is *differentially algebraically independent* over R. Otherwise $\{s_i\}_{i\in I}$ is said to be *differentially algebraically dependent* over R. Finally, if F is a differential field, we let $F\langle s_i\rangle_{i\in I}$ denote the field of fractions of $F\{s_i\}_{i\in I}$.

Let K/F be a differential field extension. A *differential transcendence basis* for K over F is a differentially algebraically independent subset of K over F that is maximal with respect to inclusion. By Ellis Robert Kolchin 1973, Chapter 2, any two differential transcendence bases of K over F have the same cardinality. Thus we define the *differential transcendence degree* of K/F, denoted by $\operatorname{trdeg}_F^{\partial} K$, to be the cardinality of any differential transcendence basis for K/F.

Finally we give a standard lemma on the differential transcendence degrees of the residue fields.

Proposition 1. *Let $\mathcal{O}$ be an integral differential F-algebra and let $\mathfrak{p}$ be a differential prime* *ideal of* $\mathcal{O}$. *Then* $\operatorname{trdeg}_F^\partial \kappa(\mathcal{O}) \leq \operatorname{trdeg}_F^\partial \operatorname{Frac}\mathcal{O}$.

Proof. It suffices to prove the inequality $\operatorname{trdeg}_F^\partial \kappa(\mathfrak{p}) \leq \operatorname{trdeg}_F^\partial K$. Let $\{\overline{x_i}\}_{i\in I}$ be a differential transcendence basis of the field $\kappa(\mathfrak{p})$ over F, so $\{\overline{x_i}^{(j)}\}_{i\in I, j\geq 0}$ is algebraically independent over F.

Let x_i be lifts of $\overline{x_i}$ to $\mathcal{O}$. We claim that $\{x_i^{(j)}\}_{i\in I,j\geq 0}$ is algebraically independent over F. Suppose that $f \in F[Y_1, ..., Y_n]$ is a nonzero polynomial satisfying $f(y_1, ..., y_n) = 0$ for some nonempty subset $\{y_1, ..., y_n\}$ of $\{\overline{x_i}^{(j)}\}_{i\in I,j\geq 0}$. Now the local ring $\mathcal{O}_\mathfrak{p}$ is dominated by some valuation ring $\mathcal{O}'$. We may multiply f by a scalar in $\mathcal{O}'$ to assume that f lies in $\mathcal{O}'[Y_1, ..., Y_n]$ and that f has at least one coefficient in $(\mathcal{O}')^\times$. Therefore $\bar{f} \neq 0$ over $\kappa(\mathcal{O}')$ and $\bar{f}(\overline{y_1}, ..., \overline{y_n}) = 0$, a contradiction. We conclude that $\{x_i^{(j)}\}_{i\in I,j\geq 0}$ is algebraically independent over F. That is, $\{x_i\}_{i\in I}$ is differentially algebraically independent over F. Thus $\operatorname{trdeg}_F^\partial \kappa(\mathfrak{p}) \leq \operatorname{trdeg}_F^\partial K$. □

2.3. Picard-Vessiot theory

In Galois theory, we start with a polynomial over a field and construct a splitting field and Galois group for the polynomial. Similarly in the Picard-Vessiot theory, we start with a homogeneous linear differential equation over a differential field and construct its Picard-Vessiot ring extension and differential Galois group. We summarize this theory following the presentation in Dyckerhoff 2008 and Van der Put and Singer 2012.

Recall that if R is a differential ring and $y_1, ..., y_n \in R$ are elements, the Wronskian determinant of $y_1, ..., y_n$ is the element

$$\operatorname{wr}(y_1, ..., y_n) = \det \begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1^{(1)} & y_2^{(1)} & \cdots & y_n^{(1)} \\ \vdots & & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix}$$

in R.

Definition 1. Let

$$p(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y \tag{2.1}$$

be a homogeneous linear differential equation over a differential field F. A differential algebra R/F is a *Picard-Vessiot extension* associated to (2.1) if there exist elements $y_1, ..., y_n \in R$ satisfying $p(y_1) = \cdots = p(y_n) = 0$ such that the following conditions hold.

1. The element $\mathrm{wr}(y_1, ..., y_n)$ is nonzero in R and the ring R is generated by $y_1, ..., y_n$ and the multiplicative inverse of the Wronskian determinant $\mathrm{wr}(y_1, ..., y_n)$ over F, i.e.,

$$R = F[y_1, ..., y_n, \mathrm{wr}(y_1, ..., y_n)^{-1}].$$

2. The ring of constants of R is C, i.e., $C_R = C$.

3. The ring R has no nontrivial proper differential ideal.

We call $\mathrm{Frac}(R)$ a *Picard-Vessiot field extension* associated to (2.1).

Remark 2. If C is an algebraically closed field, it is a standard fact that a Picard-Vessiot extension exists for (2.1) and is unique up to differential isomorphism. See Van der Put and Singer 2012, Proposition 1.20.

Consider a differential algebra R over F. Following Dyckerhoff 2008, page 9, we define the differential Galois group of R/F to be the group-valued functor

$$\begin{aligned} \underline{\mathrm{Gal}}^{\partial}(R/F) : \mathrm{Algs}_C \to \quad & \mathrm{Grps} \\ D \mapsto \quad & \mathrm{Aut}^{\partial}_{F_D}(R_D, R_D). \end{aligned}$$

We let $\mathrm{Gal}^{\partial}(R/F) = \underline{\mathrm{Gal}}^{\partial}(R/F)(C)$ and also call this the differential Galois group of R/F.

By Corollary 2.12, this functor $\underline{\mathrm{Gal}}^{\partial}(R/F)$ is representable by the linear algebraic group $G = \mathrm{Spec}(C_{R\otimes R})$.

There is a Galois correspondence for Picard-Vessiot field extensions.

Proposition 2. *Let F be a differential field with a constant field C that is algebraically closed field and of characteristic zero. Let K/F be a Picard-Vessiot field extension with differential Galois group G. Consider the set $\mathcal{S}$ of closed subgroups of G and the set $\mathcal{L}$ of differential subfields E of K containing F. Then there is a bijection $\mathcal{S} \to \mathcal{L}$ given by sending H in $\mathcal{S}$ to the fixed differential subfield $K^{H(C)}$. The inverse map is given by sending E in $\mathcal{L}$ to the closed subgroup of G whose (algebraic) C-points are* $\mathrm{Gal}^{\partial}(K/E)$.

Proof. See Van der Put and Singer 2012, Proposition 1.34. □

Finally we need an analogue of the separable closure of fields for the Picard-Vessiot theory.

Definition 2. Let F be a differential field. We define F^{PV} to be the direct limit of Picard-Vessiot field extensions K over F, filtered by inclusion.

Note that F^{PV} exists and is unique up to (differential) isomorphism; see Magid 2011, Introduction.

2.4. Generic Picard-Vessiot extensions

Let F be a differential field and let $K = F\langle y_1, ..., y_n\rangle$ where $y_1, ..., y_n$ are indeterminates over F. Let $p_{y_1,...,y_n}(y)$ be a homogeneous linear differential equation over K. Suppose that $p_{y_1,...,y_n}(y)$ determines a Picard-Vessiot extension L/K with differential Galois group G. Then L/K is said to be a *generic G-Picard-Vessiot extension* if for every G-Picard-Vessiot extension L'/K', there exist elements $a_1, ..., a_n \in K'$ such that L'/K' is the Picard-Vessiot extension for the differential equation $p_{a_1,...,a_n}(y)$.

One of our goals will be to prove that generic Picard-Vessiot extension for the groups GL_n and $\mathbb{G}_m^n$ requires at least n parameters.

2.5. Differential Hopf-Galois extension

We give definitions analogous to those in Nuss and Wambst 2007, §0. Let H be a differential Hopf algebra over R with comultiplication Δ_H, counit ϵ_H, and antipode σ_H. Suppose that S is a differential algebra over R equipped with a map $\Delta_S : S \to S \otimes_R H$ such that S is a differential H-comodule via the coaction map Δ_S. If M is a differential S-module and a differential H-comodule with a differential R-linear map $\Delta_M : M \to M \otimes_R H$ satisfying $\Delta_M(ms) = \Delta_M(m)\Delta_S(s)$ for all $m \in M$ and all $s \in S$, we say that M is an *differential (H, S)-Hopf module* over R. The *H-coinvariants* of M is the differential R-submodule

$$M^{\mathrm{co}\,H} := \{m \in M \mid \Delta_M(m) = m \otimes 1\}$$

of M. If $S = R$ and $\Delta_S(s) = s \otimes 1$ for all $s \in R$, then we simply say that M is a *differential H-Hopf module* over R.

A *differential H-Hopf-Galois extension* is a faithfully flat, differential ring extension S/R such that S is a differential H-Hopf module over R, and such that the map

$$\begin{aligned} \mathrm{can}_S : S \otimes_R S &\to S \otimes_R H \\ x \otimes y &\mapsto (x \otimes 1)\Delta_S(y) \end{aligned}$$

is a differential isomorphism.

Example 5. Let R/F be a Picard-Vessiot ring extension with differential Galois group G. Then R/F is a differential $F[G]$-Hopf-Galois extension.

Let S/R be a differential H-Hopf-Galois extension. By Knus 2012, Chapter III, Proposition 1.1.1, the faithful flatness of S/R, gives an exact sequence:

$$0 \longrightarrow R \longrightarrow S \underset{\iota_2}{\overset{\iota_1}{\rightrightarrows}} S \otimes_R S.$$

Combined with the differential isomorphism can_S, we get that $R = S^{\mathrm{co}\,H}$.

2.6. Differential schemes

We now define affine differential schemes. We define the *spectrum* of R to be the set $X = \mathrm{Spec}_D(R)$ consisting of differential prime ideals of R. For a subset S of X, we let $Z(S) = \{\mathfrak{p} \in X \mid S \subset \mathfrak{p}\}$. The $Z(S)$ define the closed sets of a topology on X, called the *Kolchin topology* on X. For $f \in R$, let $D(f) = \{\mathfrak{p} \in X \mid f \notin \mathfrak{p}\}$. The collection $\{D(f)\}_{f\in R}$ defines a basis of open sets for the Kolchin topology on X. We can further define a sheaf $\mathcal{O}_X$ of differential rings on X: for each $f \in R$, we let $\mathcal{O}_X(D(f)) = R_f$. Then $(X, \mathcal{O}_X)$ becomes a (differentially) ringed space. We call such an X an *affine differential scheme*. Since we will only consider affine differential schemes, we leave the definition of a differential scheme to the reader.

A *morphism* of affine differential schemes $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ is a continuous map $f : X \to Y$. As in algebraic geometry, there is a bijection between morphisms $X \to Y$ and differential ring homomorphisms $\mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$: a morphism $\varphi : X \to Y$ induces a differential ring homomorphism $\varphi^\# : \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$, and a differential ring homomorphism $\mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$ gives a morphism $X \to Y$ by taking Spec_D.

Let F be a differential field. An *affine differential variety* over F is an affine differential scheme over F whose coordinate ring is reduced and is a differentially finitely generated ring over F.

Let X and Y be affine differential schemes. A *differential Y-point* of X is a morphism $Y \to X$. The set of differential Y-points of X is denoted by $X(Y)$. When $Y = \mathrm{Spec}_D R$ is affine we also call differential Y-point a *differential R-point* and let $X(R) = X(Y)$.

Example 6. We call $\mathbb{A}^n_R := \mathrm{Spec}_D R\{x_1, ..., x_n\}$ the *differential affine n-space* over R. Each R-point of $\mathbb{A}^n_R$ can be identified with a tuple $(x_1, ..., x_n) \in R^n$.

A morphism of affine differential schemes is *dominant* if it has dense image. As in the case of algebraic geometry, if A and B are differential rings which are integral domains, then

$\mathrm{Spec}_D\, B \to \mathrm{Spec}_D\, A$ is a dominant morphism if and only if the differential ring homomorphism $A \to B$ is injective.

Let X and Y be affine differential varieties. A *rational morphism* $f : X \to Y$ is an equivalence class of pairs (f_U, U) where f_U is a morphism from an affine open differential subscheme of X to Y, and two pairs (f_U, U) and (f_V, V) are considered equivalent if f_U and f_V coincide on the intersection $U \cap V$.

2.7. Differential torsors

One can discuss Galois theory in the language of torsors. Similarly, one can discuss the Picard-Vessiot theory in the language of differential torsors.

Let F be a differential field with constant field C. Let G be an algebraic group over C. Let $X \to Y$ be a morphsim of affine differential schemes and let $X \times_Y G_Y \to X$ define a G_Y-action on X in the category of affine differential schemes over Y. Then we say that $X \to Y$ is a differential G-torsor if the map $X \times_Y \to X \times_Y X : (x, g) \mapsto (x, xg)$ is an isomorphism in the category.

In this language, if R/F is a G-Picard Vessiot ring extension, then $\mathrm{Spec}_D\, R \to \mathrm{Spec}_D\, F$ is a differential G_F-torsor.

CHAPTER 3

Essential Dimension

In this section we formalize the notions of the classes of objects we will consider and their essential dimensions.

3.1. Classes of objects as functors

The classes of objects encountered so far are defined over either fields or differential fields, which in turn can be collected into the following definition.

Definition 3. Consider a category $\mathcal{C}$ together with a function $d : \mathrm{Ob}(\mathcal{C}) \to \mathbb{N} \cup \{\infty\}$. We call the pair $(\mathcal{C}, d)$ a *field-like category* if for every morphism $K \to L$ in $\mathcal{C}$, we have $d(K) \leq d(L)$.

For a field F and a differential field K with constant field $C = C_K$, the pairs $(\mathrm{Fields}_F, \mathrm{trdeg}_F)$, $(\mathrm{Fields}^{\partial}_K, \mathrm{trdeg}^{\partial}_K)$, and $(\mathrm{Fields}^{\partial}_{K,C}, \mathrm{trdeg}^{\partial}_K)$ are field-like categories.

Next we organize the classes of objects we encountered in the introduction by the field each object is defined over. Formally this means writing down a functor from the appropriate field-like category to Sets, as the following examples illustrate.

Example 7.

1. Let F be a field and $n \geq 1$. Given a field K/F, we define $\mathbf{Poly}_n(K)$ to be the set of equivalence classes of separable, monic degree n polynomials over K up to Tschirnhaus transformations over K. For a morphism of fields $i : K \hookrightarrow L$, we define $\mathbf{Poly}_n(i) : \mathbf{Poly}_n(K) \to \mathbf{Poly}_n(L)$ to be the map taking a polynomial $p(x)$ over K to $i(p(x))$ where i is applied to the coefficients of p. This defines a functor $\mathbf{Poly}_n$: $\mathrm{Fields}_F \to \mathrm{Sets}$.

2. Let F be a field and $n \geq 1$. Recall that an algebra is *étale* over a field K if it is a finite product of finite separable field extensions over K. Given a field K, we define $\mathbf{\acute{E}t}_n(K)$ to be the set of the isomorphism classes of n-dimensional étale algebras over K. For

any morphism of fields $i : K \hookrightarrow L$, we define $\mathbf{\acute{E}t}_n(i) : \mathbf{\acute{E}t}_n(K) \to \mathbf{\acute{E}t}_n(L)$ to take an étale algebra E over K to the étale algebra $E \otimes_K L$ over L (extension of scalars). This defines a functor $\mathbf{\acute{E}t}_n : \mathrm{Fields}_F \to \mathrm{Sets}$.

3. Let F be a field and G an algebraic group over F. Given a field K, we define $G\textbf{-tors}(K)$ to be the set of isomorphism classes of G-torsors over K. For a morphism of fields $i : K \to L$, we define $G\textbf{-tors}(i) : G\textbf{-tors}(K) \to G\textbf{-tors}(L)$ to be the extension of scalars map. This defines a functor $G\textbf{-tors} : \mathrm{Fields}_F \to \mathrm{Sets}$.

The association of a polynomial $p(x)$ over a field K to an algebra $K[X]/(p(x))$ in Chapter 1 defines a natural isomorphism $\mathbf{Poly}_n \cong \mathbf{\acute{E}t}_n$ whose pseudo-inverse is given by taking an étale algebra E/K to the minimal polynomial of an element $x \in E$ over K, where x is a generator for the algebra E over K. To study $\mathbf{\acute{E}t}_n$, one further uses an isomorphism $\mathbf{\acute{E}t}_n \cong S_n\textbf{-tors}$ (see Berhuy and Favi 2003).

Example 8.

1. Let F be a differential field and $n \geq 1$. Given a differential field K, we define $\mathbf{DiffEq}_n(K)$ to be the set of equivalence classes of homogeneous linear differential equations over K up to gauge transformations over K. For a morphism of differential fields $i : K \to L$, we define $\mathbf{DiffEq}_n(i) : \mathbf{DiffEq}_n(K) \to \mathbf{DiffEq}_n(L)$ by $p \mapsto i(p)$. This defines a functor $\mathbf{DiffEq}_n : \mathrm{Fields}^{\partial}_{F,C} \to \mathrm{Sets}$.

2. Let F be a differential field and let $n \geq 1$. Given a differential field K, we define $\mathbf{Diff}_n(K)$ to be the set of differential isomorphism classes of differential modules of dimension n over K. For a morphism of differential fields $i : K \to L$, we define $\mathbf{DiffEq}_n(i)$ by extension of scalars. This defines a functor $\mathbf{Diff}_n : \mathrm{Fields}^{\partial}_{F,C} \to \mathrm{Sets}$.

3. Let F be a differential field and G an algebraic group over the constant field C of F. Given a differential field K, we define $G\textbf{-tors}^{\partial}(K)$ to be the set of differential isomorphism classes of differential G_K-torsors over K. For an inclusion of differential

fields $i : K \to L$, we define $G\text{-}\mathbf{tors}(i)$ by extension of scalars. This defines a functor $G\text{-}\mathbf{tors}^\partial : \mathrm{Fields}_F^\partial \to \mathrm{Sets}$.

4. Let F be a differential field and A a differential algebra over F. Then $X = \mathrm{Spec}_D(A)$ can be viewed as a functor

$$X : \mathrm{Fields}_F^\partial \to \mathrm{Sets}$$

taking a differential field K to $X(K)$.

Like in Example 7, we have an isomorphism $\mathbf{DiffEq}_n \cong \mathbf{Diff}_n$. In Chapter 4, we will show $\mathbf{Diff}_n \cong \mathrm{GL}_n\text{-}\mathbf{tors}^\partial$.

3.2. Essential dimension

We now formalize the notion of counting parameters for our classes of objects.

Definition 4. Let $(\mathcal{C}, d)$ be a field-like category and $\mathcal{F} : \mathcal{C} \to \mathrm{Sets}$ a functor. Let L be an object of $\mathcal{C}$ and $a \in \mathcal{F}(L)$. The *essential dimension* of an element a is defined to be the number

$$\mathrm{ed}^{\mathcal{C}}(a) = \min d(K)$$

where the minimum ranges over all morphisms $i : K \to L$ in $\mathcal{C}$ such that a lies in the image of $\mathcal{F}(i) : \mathcal{F}(K) \to \mathcal{F}(L)$. The *essential dimension* of the functor $\mathcal{F}$ is defined to be the number

$$\mathrm{ed}^{\mathcal{C}}(\mathcal{F}) = \sup \mathrm{ed}^{\mathcal{C}}(a)$$

where the supremum ranges over all objects L of $\mathcal{C}$ and $a \in \mathcal{F}(L)$.

If $(\mathcal{C}, d)$ is the field-like category $(\mathrm{Fields}_F, \mathrm{trdeg})$, $\mathrm{ed}^{\mathcal{C}}$ is the usual *essential dimension* ed_F given in Berhuy and Favi 2003. In the introduction, we saw that $\mathrm{ed}_F(\mathbf{Poly_2}) = \mathrm{ed}_F(\mathbf{\acute{E}t}_2) = 1$.

Definition 5. Let F be a differential field with constant field C, $(\mathcal{C}, d) = (\mathrm{Fields}_{F,C}^\partial, \mathrm{trdeg}^\partial)$, and $\mathcal{F} : \mathcal{C} \to \mathrm{Sets}$ a functor. Let L be an object of $\mathcal{C}$ and let $a \in \mathcal{F}(L)$. We define the *differential essential dimension* of an element a to be the number $\mathrm{ed}_F^{\partial}(a) := \mathrm{ed}^{\mathcal{C}}(a)$ and the *differential essential dimension* of the functor $\mathcal{F}$ to be the number $\mathrm{ed}_F^{\partial}(\mathcal{F}) := \mathrm{ed}^{\mathcal{C}}(\mathcal{F})$.

The essential dimension of a functor measures the size of the class of objects corresponding to the functor. If the functor is represented by a differential scheme, then the size should be the “dimension" of the differential scheme (see Berhuy and Favi 2003, Proposition 1.17):

Proposition 3. *Let $X = \mathrm{Spec}_D A$ be an affine differential scheme over a differential field F. Viewing X as a functor $\mathrm{Fields}_F^{\partial} \to \mathrm{Sets}$, we have*

$$\mathrm{ed}_F^{\partial}(X) = \sup \mathrm{trdeg}_F^{\partial} \kappa(\mathfrak{p})$$

where the supremum is taken over all differential prime ideals $\mathfrak{p}$ of A. In particular, if A is an integral domain, then $\mathrm{ed}_F^{\partial}(X) = \mathrm{trdeg}_F^{\partial} F(X)$.

Proof. Let K be in $\mathrm{Fields}_F^{\partial}$ and let $x \in X(K)$. Then x corresponds to a differential homomorphism $x : A \to K$ which factors through $\kappa(\mathfrak{p})$ where $\mathfrak{p} = \ker(x)$ so $x \in X(K)$ is defined over $\kappa(\mathfrak{p})$. However x is not defined over a subfield of $\kappa(p)$. Therefore $\mathrm{ed}_F^{\partial}(x) = \mathrm{trdeg}_F^{\partial} \kappa(\mathfrak{p})$ and we have $\mathrm{ed}_F^{\partial}(X) = \sup \mathrm{trdeg}_F^{\partial} \kappa(\mathfrak{p})$.

If A is an integral domain, by Proposition 1, we have $\kappa((0)) = F(X)$ has $\mathrm{trdeg}_F^{\partial} F(X) \geq \mathrm{trdeg}_F^{\partial} \kappa(\mathfrak{p})$ for all $\mathfrak{p} \in X$. □

The basic properties concerning the usual essential dimension (see Section 1) adapt easily to our general definition, with the exception of Proposition 1.13 which requires the existence of composita in the field-like category. Of these basic properties, we only need the following (see Lemma 1.9).

Proposition 4. *Let $(\mathcal{C}, d)$ be a field-like category and $\eta : \mathcal{F} \Rightarrow \mathcal{G}$ a natural transformation of functors $\mathcal{F} : \mathcal{C} \to \mathrm{Sets}$ and $\mathcal{G} : \mathcal{C} \to \mathrm{Sets}$. If η is surjective (on objects), then $\mathrm{ed}^{\mathcal{C}}(\mathcal{F}) \geq \mathrm{ed}^{\mathcal{C}}(\mathcal{G})$.*

Proof. Let $\mathrm{ed}^{\mathcal{C}}(\mathcal{F}) = n$, possibly infinite. Let K in $\mathcal{C}$ and $b \in \mathcal{G}(K)$ be arbitrary. By surjectivity of η, there exists $a \in \mathcal{F}(K)$ such that $\eta_K(a) = b$. Since $\mathrm{ed}^{\mathcal{C}}(\mathcal{F}) = n$, there exist E in $\mathcal{C}$ with $d(E) \leq n$ and a morphism $i : E \to K$ that induces a morphism $\mathcal{F}(i) : \mathcal{F}(E) \to \mathcal{F}(K)$ taking some object $a' \in \mathcal{F}(E)$ to a. By the commutativity of the diagram

$$\begin{tikzcd}
\mathcal{F}(K) \arrow[r, "\eta_K"] & \mathcal{G}(K) \\
\mathcal{F}(E) \arrow[u, "\mathcal{F}(i)"] \arrow[r, "\eta_E"'] & \mathcal{G}(E), \arrow[u, "\mathcal{G}(i)"']
\end{tikzcd}$$

$\mathcal{G}(i)$ takes $b' := \eta_E(a')$ to b and thus $\mathrm{ed}^{\mathcal{C}}(b) \leq d(b') \leq n$. Since $b \in \mathcal{G}(K)$ and K in $\mathcal{C}$ are arbitrary, we have $\mathrm{ed}^{\mathcal{C}}(\mathcal{G}) \leq n$, as desired. $\square$

3.3. Differential essential dimension of $\mathbb{G}_m^n$

Let F be a differential field with constant field C, with C algebraically closed and of characteristic zero.

Remark 3. Suppose that a G-Picard-Vessiot extension L/K descends to a G–Picard-Vessiot extension L_0/K_0 over a differential subfield K_0 of K. This means that there exists a G-equivariant differential isomorphism $L_0 \otimes_{K_0} K \cong L$ of differential K-algebras. Here, the G-action on $L_0 \otimes_{K_0} K$ is given by $\sigma(x \otimes y) := \sigma(x) \otimes y$ for all $\sigma \in G$, $x \in L_0$, and $y \in K$. Let N be a normal subgroup of G. Taking N-invariants of both sides of the isomorphism gives $L_0^N \otimes_{K_0} K \cong L^N$. Therefore the G/N-Picard-Vessiot extension L^N/K descends to L_0^N/K_0.

Proposition 5. *For $n \geq 0$, we have $\mathrm{ed}_F^{\partial}(\mathbb{G}_m^n\textbf{-tors}^{\partial}) \geq n$.*

Proof. Let $n \geq 0$, $y_1, ..., y_n$ be differential indeterminates over F, $L = F\langle y_1, ..., y_n\rangle$, and $K = F\langle \partial y_1/y_1, ..., \partial y_n/y_n\rangle$. To prove the assertion, it suffices to show that the $\mathbb{G}_m^n$-Picard-Vessiot extension L/K satisfies $\mathrm{ed}_F^{\partial}(L/K) = n$.

For the sake of contradiction, suppose the proposition is false. Let n be the smallest number for which it fails. Since the case of $n = 0$ trivially holds, we have $n \geq 1$. The extension L/K

is then induced by some extension L_0/K_0 for some differential subfield K_0 of K satisfying $\operatorname{trdeg}_F^{\partial} K_0 < n$.

For $i = 1, ..., n$, the $\mathbb{G}_m$-subextension $K\langle y_i\rangle/K$ of L/K is induced by some $\mathbb{G}_m$-subextension of L_0/K_0, which is necessarily of the form $K_0\langle z_i\rangle/K_0$ for some $z_i \in L_0$ satisfying $\partial z_i/z_i \in K_0$ by Van der Put and Singer 2012, Exercise 1.41, page 32. Thus we can write L_0 as $F\langle z_1, ..., z_n\rangle$ and K_0 as $F\langle \partial z_1/z_1, ..., \partial z_n/z_n\rangle$.

$$\begin{tikzcd}[row sep=small]
F\langle y_1, ..., y_n\rangle = & L \arrow[d, dash] \arrow[dr, dash] & & \\
F\langle \partial y_1/y_1, ..., \partial y_n/y_n\rangle = & K \arrow[dr, dash] & L_0 \arrow[d, dash] & = F\langle z_1, ..., z_n\rangle \\
& & K_0 & = F\langle \partial z_1/z_1, ..., \partial z_n/z_n\rangle
\end{tikzcd}$$

Let $L' = F\langle y_1, ..., y_{n-1}\rangle$, $K' = F\langle \partial y_1/y_1, ..., \partial y_{n-1}/y_{n-1}\rangle$, $L'_0 = F\langle z_1, ..., z_{n-1}\rangle$, $K'_0 = F\langle \partial z_1/z_1, ..., \partial z_{n-1}/z_{n-1}\rangle$, and $L'' = L'\langle z_n\rangle$. Note that the extension L'_0/K'_0 induces the $\mathbb{G}_m^{n-1}$-extension L'/K'.

$$\begin{tikzcd}[row sep=small]
L'\langle z_n\rangle = & L'' \arrow[d, dash] & & \\
F\langle y_1, ..., y_{n-1}\rangle = & L' \arrow[d, dash] \arrow[dr, dash] & & \\
F\langle \partial y_1/y_1, ..., \partial y_{n-1}/y_{n-1}\rangle = & K' \arrow[dr, dash] & L'_0 \arrow[d, dash] & = F\langle z_1, ..., z_{n-1}\rangle \\
& & K'_0 & = F\langle \partial z_1/z_1, ..., \partial z_{n-1}/z_{n-1}\rangle
\end{tikzcd}$$

Since n is the minimal value for which the proposition fails, we have $\operatorname{trdeg}_F^{\partial} L'_0 = \operatorname{trdeg}_F^{\partial} K'_0 = n-1$. Noting that $L_0 = L'_0\langle z_n\rangle$, the inequalities

$$n - 1 \geq \operatorname{trdeg}_F^{\partial} K_0 = \operatorname{trdeg}_F^{\partial} L_0 \geq \operatorname{trdeg}_F^{\partial} L'_0 = n - 1$$

now force z_n to be differentially algebraic over L'_0 and hence over L'. Therefore $\operatorname{trdeg}^{\partial}_F L'' = \operatorname{trdeg}^{\partial}_F L' = n - 1$.

To finish proving the proposition, it suffices to show that y_n is differentially algebraic over L'', since we would then get $n = \operatorname{trdeg}^{\partial}_F L = \operatorname{trdeg}^{\partial}_F L'' = n - 1$, resulting in the desired contradiction. Recall that the extension $K\langle y_n\rangle/K$ is induced by the extension $K_0\langle z_n\rangle/K_0$. By the Kolchin-Ostrowski theorem Ellis R Kolchin 1968, pages 1155-1156, there exist nonzero integers r, s and a nonzero element $d \in K$ such that $y_n^r z_n^s = d$ holds. Since $K = F\langle \partial y_1/y_1, ..., \partial y_n/y_n\rangle = K'\langle \partial y_n/y_n\rangle$, we may view d as $f(\partial y_n/y_n)$ where f is a differential rational function in one variable T over K'. Therefore $y_n^r z_n^s = f(\partial y_n/y_n)$. Since d is nonzero, $f(T)$ is nonzero. Furthermore, the differential rational function $g(T) := f(\partial T/T) - T^r z_n^s$ over L'' is nonzero as it is not fixed by $T \mapsto 2T$. Therefore y_n satisfies the nonzero differential rational function $g(T)$ over L'' and so is differentially algebraic over L''. $\square$

A similar argument to the above proves $\operatorname{ed}^{\partial}_F(\mathbb{G}_a^n\text{-}\mathbf{tors}^{\partial}) = \operatorname{ed}^{\partial}_F(C_r^n\text{-}\mathbf{tors}^{\partial}) = n$ where C_r denotes the cyclic group of order r. We omit these cases as we will not use them.

CHAPTER 4

Twisted Forms and Cohomology

Certain classes of objects in differential algebra can be interpreted as twisted forms of a particular object. In this section, we will define the notions of such twisted forms as well as a cohomology set relevant to differential algebra. Finally we will show that the cohomology set is in bijection with the set of such twisted forms. This is analogous to the situation in algebra.

In this chapter, we fix a base differential field F and assume that its constant field C is algebraically closed and of characteristic zero. Moreover R will denote a differential F-algebra, and all unadorned tensor products in this section are taken over R, i.e., $\otimes = \otimes_R$.

4.1. $\mathbf{\Phi}$-structures and descent along differential Hopf-Galois extensions

We loosely follow the formalism in Nardin 2012, Section 1.3. Let I be a set. We define a *tensor-type* to be a subset of $\mathbb{N}^4$ indexed by I. Let $\mathbf{\Phi} = \{(r_{1i}, r_{2i}, r_{3i}, r_{4i})\}_{i \in I}$ be a tensor-type, M a differential module over R, and H a differential Hopf algebra over R. We say that a tuple $(M, \{\Phi_i\}_{i\in I})$ is a $\mathbf{\Phi}$-*structure* over R if the Φ_i are differential R-module homomorphisms of the form

$$\Phi_i : M^{\otimes r_{1i}} \otimes H^{\otimes r_{2i}} \to M^{\otimes r_{3i}} \otimes H^{\otimes r_{4i}}.$$

Example 9.

1. A differential module M over R is a $\mathbf{\Phi}$-structure by taking $\mathbf{\Phi}$ to be the empty set.

2. A differential algebra A over R with multiplication map $m : A^{\otimes 2} \to A$ is a $\mathbf{\Phi}$-structure with $\mathbf{\Phi} = \{(2, 0, 1, 0)\}$.

3. A differential H-Hopf-Galois extension S/R consists of a multiplication map $m : S^{\otimes 2} \to S$ and a coaction map $\Delta_S : S \to S \otimes H$. Therefore S/R defines a $\mathbf{\Phi}$-structure

with $\mathbf{\Phi} = \{(2,0,1,0),(1,0,1,1)\}$.

We will view the above classes of objects (differential modules, differential algebras, differential Hopf-Galois extensions) as $\mathbf{\Phi}$-structures with the $\mathbf{\Phi}$ given in the examples.

Remark 4. Note that the definition of a $\mathbf{\Phi}$-structure relies upon an implicit choice of differential Hopf algebra and indexing set, henceforth denoted by H and I, respectively.

Let $(M, \{\Phi_i\}_{i\in I})$ and $(N, \{\Psi_i\}_{i\in I})$ be $\mathbf{\Phi}$-structures over R. A *morphism of* $\mathbf{\Phi}$*-structures* over R is a differential R-module homomorphism $\varphi : M \to N$ such that $\varphi \circ \Phi_i = \Psi_i$ holds for all $i \in I$. Together the $\mathbf{\Phi}$-structures over R and the morphisms of $\mathbf{\Phi}$-structures over R form a category which we denote by $\mathbf{\Phi}\text{-Struc}_R$.

Let $\varphi : R \to S$ be a differential ring homomorphism. Then a $\mathbf{\Phi}$-structure $(M, \{\Phi_i\}_{i\in I})$ over R induces the $\mathbf{\Phi}$-structure $(M \otimes S, \{\Phi_i \otimes 1_S\}_{i\in I})$ over S by extension of scalars. This map gives a functor

$$\mathbf{\Phi}\text{-Struc}_R \to \mathbf{\Phi}\text{-Struc}_S\,. \tag{4.1}$$

We may also write down an equivariant version of such structures. Let H' be a differential Hopf algebra over R. An *H'-equivariant* $\mathbf{\Phi}$*-structure* is a $\mathbf{\Phi}$-structure $(M, \{\Phi_i\}_{i\in I})$ over R such that M is a differential H'-comodule with coaction map $\Delta_M : M \to M \otimes H'$, and such that Δ_M commutes with the Φ_i, i.e., the following diagram commutes for all $i \in I$:

$$\begin{tikzcd}
M^{\otimes r_{1i}} \otimes H^{\otimes r_{2i}} \arrow[r, "\Phi_i"] \arrow[d, "\Delta_M^{\otimes r_{1i}}\otimes 1"'] & M^{\otimes r_{3i}} \otimes H^{\otimes r_{4i}} \arrow[d, "\Delta_M^{\otimes r_{3i}}\otimes 1"] \\
M^{\otimes r_{1i}} \otimes H^{\otimes r_{2i}} \otimes H' \arrow[r, "\Phi_i\otimes 1_{H'}"'] & M^{\otimes r_{3i}} \otimes H^{\otimes r_{4i}} \otimes H'.
\end{tikzcd}$$

A *morphism of H'-equivariant* $\mathbf{\Phi}$*-structures* $(M, \{\Phi_i\}_{i\in I})$ and $(N, \{\Psi_i\}_{i\in I})$ is a morphism $\varphi : M \to N$ of $\mathbf{\Phi}$-structures that commutes with the H'-coactions on M and N, i.e. the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & N \\ {\scriptstyle\Delta_M}\downarrow & & \downarrow{\scriptstyle\Delta_N} \\ M \otimes H' & \xrightarrow[\varphi\otimes 1_{H'}]{} & N \otimes H'. \end{array}$$

Together the H'-equivariant $\mathbf{\Phi}$-structures and their morphisms form a category which we denote by $\mathbf{\Phi}\text{-Struc}_R^{H'}$.

Let S/R be a differential H'-Hopf-Galois extension. By extension of scalars, a $\mathbf{\Phi}$-structure $(M, \{\Phi_i\}_{i\in I})$ over R extends to $(M \otimes S, \{\Phi_i \otimes 1_S\}_{i\in I})$ over S. Since all the $\Phi_i \otimes 1_S$ commute with Δ_S, the new structure is H'-equivariant over S. Therefore (4.1) restricts to a functor

$$\mathbf{\Phi}\text{-Struc}_R \to \mathbf{\Phi}\text{-Struc}_S^{H'} . \tag{4.2}$$

Conversely, given a H'-equivariant $\mathbf{\Phi}$-structure $(N, \{\Phi_i\})$ over S, we may consider its coinvariant module $N^{\mathrm{co}\,H'} = \{n \in N \mid \Delta_N(n) = n \otimes 1\}$. Since each Φ_i commutes with Δ_N, the $\mathbf{\Phi}$-structure on N restricts to one on $N^{\mathrm{co}\,H'}$. This gives a functor

$$\mathbf{\Phi}\text{-Struc}_S^{H'} \to \mathbf{\Phi}\text{-Struc}_R .$$

We now show that the two functors just considered define an equivalence of categories. We follow the proof of Schneider 1990, Theorem 7.3.1 (1) $\Rightarrow$ (2).

Theorem 6 (Descent along differential Hopf-Galois extensions)**.** *Let H' be a differential Hopf algebra over R, S/R a differential H'-Hopf-Galois extension, and $\mathbf{\Phi}$ a tensor-type. Suppose that S/R is faithfully flat. Then extension of scalars defines an equivalence of categories*

$$\mathbf{\Phi}\text{-Struc}_R \to \mathbf{\Phi}\text{-Struc}_S^{H'} .$$

The pseudo-inverse is given by taking an object N to the coinvariant module $N^{\operatorname{co} H'}$.

Proof. Naturality is clear. It suffices now to check that for all M in $\mathbf{\Phi}$-Struc$_R$ and N in $\mathbf{\Phi}$-Struc$_S^{H'}$, the two maps

$$\mu_N : N^{\operatorname{co} H'} \otimes_R S \to N : n \otimes s \mapsto ns$$

and

$$\iota_M : M \to (M \otimes_R S)^{\operatorname{co} H'} : m \mapsto m \otimes 1$$

are bijections.

Since S/R is a differential H'-Hopf-Galois extension, we have the differential isomorphism

$$\begin{aligned} \operatorname{can}_S : S \otimes_R S &\to S \otimes_R H' \\ x \otimes y &\mapsto (x \otimes 1)\Delta_S(y) \end{aligned}$$

Therefore for any P in $\operatorname{Mod}_S^{\partial}$, the map can_P given by

$$\begin{aligned} \operatorname{can}_P : P \otimes_R S &\to P \otimes_R H' \\ p \otimes s &\mapsto (p \otimes 1)\Delta_S(s) \end{aligned}$$

is also a differential isomorphism.

Consider the following two commutative diagrams.

$$\begin{tikzcd} N^{\operatorname{co} H'} \otimes_R S \arrow[r] \arrow[d, "\mu_N"'] & N \otimes_R S \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\operatorname{can}_N"] & (N \otimes_S H') \otimes_R S \arrow[d, "\operatorname{can}_{N \otimes H'}"] \\ N \arrow[r, "\Delta_N"'] & N \otimes_S H' \arrow[r, shift left, "\Delta_{N \otimes H'}"] \arrow[r, shift right, "1_N \otimes \Delta_{H'}"'] & N \otimes_S H' \otimes_S H' \end{tikzcd} \tag{4.3}$$

$$
\begin{tikzcd}
(M \otimes_R S)^{\operatorname{co} H'} \arrow[r, "\iota"] & M \otimes_R S \arrow[r, shift left] \arrow[r, shift right] & M \otimes_R S \otimes_R H' \\
M \arrow[u, "\iota_M"] \arrow[r, "\iota"'] & M \otimes_R S \arrow[u, equal] \arrow[r, shift left, "\iota_1"] \arrow[r, shift right, "\iota_2"'] & M \otimes_R S \otimes_R S \arrow[u, "\operatorname{can}_{M\otimes S}"']
\end{tikzcd}
\tag{4.4}
$$

The top row of (4.4) is exact by the definition of $\operatorname{co} H'$. The top row of (4.3) is exact by the definition of $\operatorname{co} H'$ and the flatness of S/R. The bottom row of (4.3) is exact by coassociativity of Δ_N. Since S/R is faithfully flat, by Knus 2012, Chapter III, Proposition 1.1.1, the bottom row of (4.4) is exact. The vertical arrows can_N, $\operatorname{can}_{N\otimes H'}$, and $\operatorname{can}_{M\otimes S}$ are differential isomorphisms by our above discussion. Thus μ_N and ι_M are differential isomorphisms. □

4.2. Twisted forms.

Let $\mathbf{\Phi}$ be a tensor-type. Let M and N be $\mathbf{\Phi}$-structures over R, and let S/R be a differential ring extension. We say that M is a (S/R)-*twisted form* of N if there exists a differential isomorphism $\varphi : M \otimes_R S \cong N \otimes_R S$ of $\mathbf{\Phi}$-structures over S. We let $\mathrm{TF}(S/R, M)$ denote the set of differential isomorphism classes of (S/R)-twisted forms of M.

Example 10.

1. Any differential module over F is a (F^{PV}/F)-twisted form of the trivial differential module M of the same rank, giving a bijection between $\mathbf{Diff}_n(F)$ and $\mathrm{TF}(F^{PV}/F, M)$.

2. Let H be a differential Hopf algebra over R. Let S/R be a differential H-Hopf-Galois extension. Then S is a (S/R)-twisted form of H via the differential isomorphism can_S.

Differential torsors are another important example of twisted forms. Given a differential ring extension S/R, and an algebraic group G over C, we say that a differential G_R-torsor X is a (S/R)-*twisted form* of a differential G_R-torsor Y if X_S is differentially isomorphic to Y_S as differential G_S-torsors; equivalently, $R[X]$ is a (S/R)-twisted form of $R[Y]$ as differential $R[G]$-Hopf-Galois extensions. For a given differential G_R-torsor X, we let $\mathrm{TF}(S/R, X)$

denote the set of differential isomorphism classes of (S/R)-twisted forms of X.

Proposition 7. *Let G be a linear algebraic group over C. Then any differential G_F-torsor is a (F^{PV}/F)-twisted form of the trivial differential G_F-torsor G_F. In particular, we have a bijection $G\text{-}\mathbf{tors}^{\partial}(F) \cong \mathrm{TF}(F^{PV}/F, G_F)$.*

Proof. Let $H = F[G]$ and let X be a differential G_F-torsor over F. By Bachmayr et al. 2018, Proposition 1.15, there exists a closed subgroup G' of G and a simple differential G'_F-torsor Y with $C_{\mathrm{Frac}(F[Y])} = C$ such that $X \cong \mathrm{Ind}_{G'_F}^{G_F}(Y)$ as differential torsors. By Proposition 1.12(b), $F[Y]/F$ is a G'_F-Picard-Vessiot extension. Therefore $F[Y]/F$ is a (F^{PV}/F)-twisted form of H as a differential $F[G']$-Hopf-Galois extension. By Remark 1.11, $F[X] \cong (F[Y] \otimes_C C[G])^{G'}$ and so

$$\begin{aligned} F[X] \otimes_F F^{PV} &\cong (F[Y] \otimes_F F^{PV} \otimes_C C[G])^{G'} \\ &\cong (F[G'] \otimes_F F^{PV} \otimes_C C[G])^{G'} \\ &\cong F^{PV} \otimes_C C[G] \\ &\cong F[G] \otimes_F F^{PV}. \end{aligned}$$

Therefore $F[X]$ is a (F^{PV}/F)-twisted forms of H.

The final assertion of the proposition is now clear. □

4.3. Cohomology

We now define a cohomology set that we will use to classify the twisted forms of the previous section. Recall that to specify a morphism of varieties over an algebraically closed field C, it suffices to do so on the C-points of the varieties.

Definition 6. Let Γ and G be algebraic groups over C. Suppose that G is a group object in the category of affine varieties with Γ-action, i.e., there is a morphism of varieties $\Gamma \times G \to G$ defining a Γ-action on G that is compatible with the group structure on G. A *1-cocycle* is a morphism of varieties $a : \Gamma(C) \to G(C)$ such that the following condition holds: for any $\sigma \in \Gamma(C)$ we let $a_\sigma := a(\sigma)$ and require that $a_{\sigma\tau} = a_\sigma \cdot \sigma(a_\tau)$ holds for all $\sigma, \tau \in \Gamma(C)$. Two

1-cocycles a and b are *equivalent* if there exists $c \in G(C)$ such that $a_\sigma = c \cdot b_\sigma \cdot c^{-1}$. We define $H^1(\Gamma, G)$ to be the set of 1-cocycles $\Gamma \to G$ modulo equivalence.

Note that the cocycles we have just defined are morphisms of varieties and not merely maps of sets, unlike the case of (finite) Galois cohomology. Note also that the cohomology set is functorial in both Γ and G: homomorphisms of algebraic groups $\Gamma' \to \Gamma$ and $G \to G'$ over C induce maps $H^1(\Gamma', G) \to H^1(\Gamma, G)$ and $H^1(\Gamma, G) \to H^1(\Gamma, G')$.

4.4. Cohomology classifies twisted forms.

We now discuss how to classify the twisted forms by the cohomology set we introduced in the last section. Consider the following setup.

Let Γ be an algebraic group over C with Hopf algebra H_0' and let S/R be a differential $(H_0' \otimes_C R)$-Hopf-Galois extension. Let $(M, \{\Phi_i\}_{i \in I})$ be a $\mathbf{\Phi}$-structure over R and set $M_S := M \otimes_R S$. Furthermore assume that the automorphism group of M is represented by an algebraic group G over C, i.e., there exists an isomorphism

$$\mathrm{Aut}(M_S \otimes_C D) \cong G(D) \tag{4.5}$$

that holds for every C-algebra D and is functorial in D. We will always identify the two groups in (4.5).

Remark 5. Note that (4.5) gives a sequence of isomorphisms

$$\begin{aligned} \mathrm{Aut}(M_S \otimes_C H_0'^{\otimes n}) &\cong G(H_0'^{\otimes n}) \qquad (4.6) \\ &\cong \mathrm{Hom}_{C-\mathrm{Algs}}(C[G], H_0'^{\otimes n}) \cong \mathrm{Mor}_{C-\mathrm{Sch}}(\Gamma^n, G) \end{aligned}$$

which takes an element $f \in \mathrm{Aut}(M_S \otimes_C H_0'^{\otimes n})$ to the morphism $\Gamma^n \to G$ given on C-points by

$$\begin{aligned} \Gamma^n(C) &\to G(C) \\ (\sigma_1, ..., \sigma_n) &\mapsto (1_{M_S} \otimes \sigma_1 \otimes \cdots \sigma_n)_* f. \end{aligned} \tag{4.7}$$

Here $(1_{M_S} \otimes \sigma_1 \otimes \cdots \sigma_n)_* f$ is the morphism obtained by extension of scalars (see 2.1), such that the following diagram commutes:

$$\begin{tikzcd}
M_S \otimes_C {H'_0}^{\otimes n} \arrow[r, "f"] \arrow[d, "1_{M_S}\otimes\sigma_1\otimes\cdots\otimes\sigma_n"'] & M_S \otimes_C {H'_0}^{\otimes n} \arrow[d, "1_{M_S}\otimes\sigma_1\otimes\cdots\otimes\sigma_n"] \\
M_S \arrow[r, "(1_{M_S}\otimes\sigma_1\otimes\cdots\otimes\sigma_n)_* f"'] & M_S.
\end{tikzcd} \tag{4.8}$$

Remark 6. We can define an action of Γ on G in the following way. First we define an action of $\Gamma(C)$ on S by letting an element $\sigma \in \Gamma(C)$ act on S via the automorphism $\boldsymbol{\sigma} = (1_S \otimes \sigma) \circ \Delta_S$:

$$\boldsymbol{\sigma} : S \xrightarrow{\Delta_S} S \otimes_C H'_0 \xrightarrow{1_S\otimes\sigma} S. \tag{4.9}$$

This action extends to an action of $\Gamma(C)$ on $M_S = M \otimes_R S$. The action of $\Gamma(C)$ on M_S further gives an action of Γ on G on the C-points by conjugation: for any $\sigma \in \Gamma(C)$ and $\varphi \in G(C)$, we define the action to be $\sigma(\varphi) := \boldsymbol{\sigma} \circ \varphi \circ \boldsymbol{\sigma}^{-1}$. We will consider G with this Γ-action when discussing the cohomology set $H^1(\Gamma, G)$.

Remark 7. In the case $n = 1$ in Remark 5, an element $a \in \mathrm{Aut}(M_S \otimes_C H'_0)$ corresponds to an element of $\mathrm{Mor}_{C-\mathrm{Sch}}(\Gamma, G)$ which we again denote by a. If for each $\sigma \in \Gamma(C)$ we let $a_\sigma := a(\sigma)$, we have the equality

$$(1_{M_S} \otimes \sigma)_* a = a_\sigma.$$

The commutativity of the diagram

$$\begin{tikzcd}
M_S \arrow[r, "\Delta_{M_S}"] \arrow[dr, "\boldsymbol{\sigma}"'] & M_S \otimes_C H'_0 \arrow[r, "a"] \arrow[d, "1\otimes\sigma"] & M_S \otimes_C H'_0 \arrow[d, "1\otimes\sigma"] \\
& M_S \arrow[r, "a_\sigma"'] & M_S
\end{tikzcd}$$

further gives the equality

$$(1_{M_S} \otimes \sigma) \circ (a \circ \Delta_{M_S}) = a_\sigma \circ \boldsymbol{\sigma} \tag{4.10}$$

which we will later use in Lemma 9.

Construction 1. We define a map

$$\mathcal{F} : \mathrm{TF}(S/R, M) \to H^1(\Gamma, G)$$

as follows. Let (N, φ) be a twisted form of M. We define $\mathcal{F}(N, \varphi)$ to be the cocycle $a : \Gamma(C) \to G(C)$ which sends an element σ in $\Gamma(C)$ to the element

$$a_\sigma := \varphi \circ \boldsymbol{\sigma}(\varphi) = \varphi \circ \boldsymbol{\sigma} \circ \varphi^{-1} \circ \boldsymbol{\sigma}^{-1} \text{ in } G(C).$$

We must verify that $\mathcal{F}$ is well-defined. That a is a cocycle follows from the standard computation in $G(C)$:

$$\begin{aligned}
a_\sigma \cdot \sigma(a_\tau) &= a_\sigma \circ \boldsymbol{\sigma} \circ a_\tau \circ \boldsymbol{\sigma}^{-1} \\
&= (\varphi \circ \boldsymbol{\sigma} \circ \varphi^{-1} \circ \boldsymbol{\sigma}^{-1}) \circ \boldsymbol{\sigma} \circ (\varphi \circ \boldsymbol{\tau} \circ \varphi^{-1} \circ \boldsymbol{\tau}^{-1}) \circ \boldsymbol{\sigma}^{-1} \\
&= \varphi \circ (\boldsymbol{\sigma} \circ \boldsymbol{\tau}) \circ \varphi^{-1} \circ (\boldsymbol{\sigma} \circ \boldsymbol{\tau})^{-1} \\
&= a_{\sigma\tau}
\end{aligned}$$

which holds for all $\sigma, \tau \in \Gamma(C)$.

Next let (N', ψ) be differentially isomorphic to (N, φ) as twisted forms of M and let $b =$

$\mathcal{F}(N', \psi)$. Setting $c = \psi \circ \varphi^{-1}$, we have

$$\begin{aligned} c^{-1} \circ b_\sigma \circ \sigma(c) &= c^{-1} \circ b_\sigma \circ \boldsymbol{\sigma} \circ c \circ \boldsymbol{\sigma}^{-1} \\ &= (\varphi \circ \psi^{-1}) \circ (\psi \circ \boldsymbol{\sigma} \circ \psi^{-1} \circ \boldsymbol{\sigma}^{-1}) \circ \boldsymbol{\sigma} \circ (\psi \circ \varphi^{-1}) \circ \boldsymbol{\sigma}^{-1} \\ &= \varphi \circ \boldsymbol{\sigma} \circ \varphi^{-1} \circ \boldsymbol{\sigma}^{-1} \\ &= a_\sigma \end{aligned}$$

for all $\sigma \in \Gamma(C)$. Thus $\mathcal{F}$ takes equivalent twisted forms to equivalent cocycles. We conclude that $\mathcal{F}$ is well-defined.

Construction 2. We define a map

$$\mathcal{G} : H^1(\Gamma, G) \to \mathrm{TF}(S/R, M) \tag{4.11}$$

as follows. Given a cocycle a representing an element of $H^1(\Gamma, G)$, we define $\mathcal{G}(a)$ to be the differential R-module

$$N := \{ m \in M_S \mid (a_\sigma \circ \boldsymbol{\sigma})(m) = m \text{ for all } \sigma \in \Gamma(C) \}\,. \tag{4.12}$$

We will soon check that $\mathcal{G}$ is a well-defined map in Lemma 9. Our proofs of Lemma 9 and Theorem 10 below follow that of Nuss and Wambst 2007, Theorem 2.6 where a cohomology set was introduced to classify Hopf-Galois extensions for noncommutative rings. The following lemma allows us to convert from their "cochains" which are maps $M_S \to M_S \otimes_C H_0'$ to our cochains which are morphisms $\Gamma \to G$.

Lemma 8. *Let M be a C-vector space, X an algebraic variety over C, and $f, g \in M \otimes_C C[X]$. If $(1_M \otimes \sigma) f = (1_M \otimes \sigma) g$ for all $\sigma \in X(C)$ then $f = g$ in $M \otimes_C C[X]$.*

Proof. Since X is an algebraic variety over an algebraically closed field C, equality of functions on $X(C)$ implies equality of elements in $C[X]$. This gives the case $M = C$.

For a general M, let $\{m_i\}_{i\in I}$ be a basis of M over C and write $f = \sum m_i \otimes f_i$ and $g = \sum m_i \otimes g_i$ for some $f_i, g_i \in C[X]$. For all $\sigma \in X(C)$, we have $(1_M \otimes \sigma)(f) = (1_M \otimes \sigma)(g)$ hence $\sum m_i \otimes \sigma(f_i) = \sum m_i \otimes \sigma(g_i)$. The linear independence of $\{m_i\}_{i\in I}$ over C and the previous paragraph now give $f_i = g_i$ for all $i \in I$. $\square$

Lemma 9. *The map $\mathcal{G}$ in Construction 2 is well-defined.*

Proof. Step 1. We first check that given a cocycle a, $N = \mathcal{G}(a)$ is a twisted form of M. First note that a satisfies the following properties:

(a) $(a_\sigma \circ \boldsymbol{\sigma})(ms) = (a_\sigma \circ \boldsymbol{\sigma})(m)\boldsymbol{\sigma}(s)$ for all $\sigma \in \Gamma(C)$, $m \in M_S$, and $s \in S$;

(b) $a_1 = 1_{M_S}$;

(c) $a_{\sigma\tau} = a_\sigma \circ \boldsymbol{\sigma} \circ a_\tau \circ \boldsymbol{\sigma}^{-1}$ for all $\sigma, \tau \in \Gamma(C)$.

Here (c) is the cocycle condition for a, (b) follows from (c) by letting $\sigma = \tau = 1$ in $\Gamma(C)$, and (a) follows from the S-linearity of a_σ.

We claim that the composite map Δ' given by $a \circ \Delta_{M_S} : M_S \to M_S \otimes_C H'_0$ defines a coaction on M_S making M_S a differential (H'_0, S)-Hopf module. In other words, we must verify the following properties:

(A) $\Delta'(ms) = \Delta'(m)\Delta(s)$ for all $m \in M_S, s \in S$;

(B) $(1_{M_S} \otimes \epsilon_{H'_0}) \circ \Delta' = 1_{M_S}$;

(C) $(\Delta' \otimes 1_{H'_0}) \circ \Delta' = (1_{M_S} \otimes \Delta_{H'_0}) \circ \Delta'$.

Since $\epsilon_{H'_0} : H'_0 \to C$ corresponds to $1 \in \Gamma(C)$, (B) follows from (b) by (4.10).

To show (A) and (C), by Lemma 8, it suffices to show that the equalities obtained by applying $(1_{M_S} \otimes \sigma)$ to (A) and $(1_{M_S} \otimes \sigma \otimes \tau)$ to (C) hold for all $\sigma, \tau \in \Gamma(C)$. Applying

$(1_{M_S} \otimes \sigma)$ to (A) and simplifying by (4.10) gives (a). Thus (A) holds. Similarly, applying $1 \otimes (\sigma \circ \tau)$ to the right side of (C) gives

$$(1_{M_S} \otimes \sigma \otimes \tau) \circ (1_{M_S} \otimes \Delta_{H_0'}) \circ \Delta' = (1_{M_S} \otimes (\sigma \circ \tau)) \circ \Delta' = a_{\sigma\tau} \circ (\boldsymbol{\sigma} \circ \boldsymbol{\tau}). \tag{4.13}$$

Applying $1_{M_S} \otimes (\sigma \circ \tau)$ to the left side of (C) gives

$$a_\sigma \circ \boldsymbol{\sigma} \circ a_\tau \circ \boldsymbol{\sigma}^{-1} \tag{4.14}$$

since the following diagram commutes:

$$\begin{tikzcd}
M_S \arrow[d, "\Delta'"'] \arrow[drr, "a_\tau \circ \boldsymbol{\tau}"] & & \\
M_S \otimes_C H_0' \arrow[rr, "1 \otimes \tau"] \arrow[d, "\Delta' \otimes 1_{H_0'}"'] \arrow[dr, "(a_\sigma \circ \boldsymbol{\sigma}) \otimes 1_{H_0'}"] & & M_S \arrow[d, "a_\sigma \circ \boldsymbol{\sigma}"] \\
M_S \otimes_C H_0' \otimes_C H_0' \arrow[r, "1 \otimes \sigma \otimes 1"'] & M_S \otimes_C H_0' \arrow[r, "1 \otimes \tau"'] \arrow[ur, phantom, "(I)"] & M_S.
\end{tikzcd}$$

Here, the region (I) commutes by the computation

$$\begin{aligned}
((a_\sigma \circ \boldsymbol{\sigma}) \circ (1 \otimes \tau))(m \otimes h) &= ((a_\sigma \circ \boldsymbol{\sigma})(m \cdot \tau(h)) \\
&= ((a_\sigma \circ \boldsymbol{\sigma})(m)) \cdot \tau(h) \\
&= ((1 \otimes \tau)\sigma(a_\sigma \circ \boldsymbol{\sigma}))(m \otimes h).
\end{aligned}$$

Equating (4.13) with (4.14) gives (c). Thus (C) holds. This concludes checking that Δ' defines a coaction on M_S making M_S a differential (H_0', S)-Hopf module.

We are almost done with Step 1. Theorem 6 implies that the coinvariant module $(M_S)^{\mathrm{co}\,\Delta'}$ is a twisted form of M over R. Therefore it suffices to show that N equals $(M_S)^{\mathrm{co}\,\Delta'}$. Any $m \in (M_S)^{\mathrm{co}\,\Delta'}$ satisfies $\Delta'(m) = m \otimes 1$. For any $\sigma \in \Gamma(C)$, applying $(1 \otimes \sigma)$ to $\Delta'(m) = m \otimes 1$ and simplifying by (4.10) gives $(a_\sigma \circ \sigma)(m) = m$. Thus $(M_S)^{\mathrm{co}\,\Delta'} \subset N$. Invoking Lemma 8 gives the reverse inclusion and so $(M_S)^{\mathrm{co}\,\Delta'} = N$.

Step 2. The map $\mathcal{G}$ takes equivalent cocycles to isomorphic twisted forms. If b is a cocycle equivalent to a, there exists $c \in G(C) = \mathrm{Aut}(M_S)$ such that $b = c \circ a \circ c^{-1}$. Let $N_b = \mathcal{G}(b)$. The automorphism $c : M_S \to M_S$ restricts to an isomorphism $N_a \cong N_b$, as desired.

□

We now prove the main theorem of this section.

Theorem 10. *Consider the above setup. The maps $\mathcal{F}$ and $\mathcal{G}$ are inverses. Hence there is a bijection between the two sets* $\mathrm{TF}(S/R, M)$ *and* $H^1(\Gamma, G)$.

Proof. We first check $\mathcal{G} \circ \mathcal{F} = 1$. Let (N, φ) be a twisted form of M_S with associated cocycle $a = \mathcal{F}(N, \varphi)$. Set $P := \mathcal{G}(a)$. We want to show $P \cong N$. Clearly the isomorphism $\varphi : N \otimes S \to M_S$ has image in P, so φ restricts to $\varphi|_N : N \to P$. Consider the commutative diagram

$$\begin{tikzcd} N \otimes S \arrow[r, "\varphi|_N \otimes 1_S"] \arrow[dr, "\varphi"', "\cong"] & P \otimes S \arrow[d, "\text{mult.} \cong"] \\ & M_S. \end{tikzcd}$$

By definition of twisted form, the vertical map is an isomorphism. Thus $\varphi|_N \otimes 1_S$ is an isomorphism. By faithful flatness of S/R, $\varphi|_N$ is an isomorphism.

We next check $\mathcal{F} \circ \mathcal{G} = 1$. Let a be a cocycle, $(N, \varphi) := \mathcal{G}(a)$, and $b := \mathcal{F}(N, \varphi)$. Given $\sigma \in \Gamma(C)$, consider the diagram

$$\begin{tikzcd} M_S \arrow[rrr, "b_\sigma \circ \boldsymbol{\sigma}"] \arrow[ddd, equal] & & & M_S \arrow[ddd, equal] \\ & N \arrow[ul, "\varphi"'] \arrow[dl, "\varphi"] \arrow[r, "\boldsymbol{\sigma}"] & N \arrow[ur, "\varphi"] \arrow[dr, "\varphi"'] & \\ & & & \\ M_S \arrow[rrr, "a_\sigma \circ \boldsymbol{\sigma}"'] & & & M_S \end{tikzcd}$$

where the triangles trivially commute and the upper trapezoid commutes by definition of b_σ. The bottom trapezoid commutes by the following two computations. For all $n \in N$ and $s \in S$, we have

$$\begin{aligned}\varphi(\boldsymbol{\sigma}(n \otimes s)) &= \varphi((n \otimes 1)(1 \otimes \sigma(s))) \\ &= n\sigma(s)\end{aligned}$$

where the last equality uses the isomorphism $N \otimes S \cong M_S$ given by scalar multiplication. Also

$$\begin{aligned}(a_\sigma \circ \boldsymbol{\sigma} \circ \varphi)(n \otimes s) &= (a_\sigma \circ \boldsymbol{\sigma})(ns) \\ &= (a_\sigma \circ \boldsymbol{\sigma})(n)\sigma(s) \\ &= n\sigma(s)\end{aligned}$$

where the second equality uses property (a) in the proof of Lemma 9 and the third equality uses the definition of N. Therefore the diagram above is commutative so $b_\sigma \circ \boldsymbol{\sigma} = a_\sigma \circ \boldsymbol{\sigma}$ for all $\sigma \in \Gamma(C)$. By (4.10) and Lemma 8, we have $b = a$. □

4.5. Absolute cohomology

In this section we provide a variant to Theorem 10 which is more convenient to use in practice. While Theorem 10 is stated for twisted forms over differential Hopf-Galois extensions, which includes Picard-Vessiot ring extensions, sometimes it is easier to work with twisted forms over Picard-Vessiot field extensions.

Proposition 11. *Let R/F be a Picard-Vessiot ring extension with differential Galois group Γ, $K = \mathrm{Frac}(R)$, $\boldsymbol{\Phi}$ a tensor-type, and M a $\boldsymbol{\Phi}$-structure over F. Suppose that the automorphism group of $M \otimes_F R$ is representable by an algebraic group G over C. Then the inclusion map $\mathrm{TF}(R/F, M) \to TF(K/F, M)$ is a bijection. In particular, there is a bijection of the two sets $\mathrm{TF}(K/F, M)$ and $H^1(\Gamma, G)$.*

Proof. Given a (K/F)-twisted form of M, we obtain a cocycle $\mathrm{Gal}(K/F) \to G(C)$ as in Construction 1. This gives a map $\mathrm{TF}(K/F, M) \to H^1(\mathrm{Gal}(K/F), M)$ which is part of a commutative diagram

$$\begin{array}{ccc} \mathrm{TF}(R/F, M) & \longrightarrow & \mathrm{TF}(K/F, M) \\ {\scriptstyle\cong}\big\downarrow & & \big\downarrow \\ H^1(\mathrm{Gal}(R/F), M) & \xrightarrow[\cong]{} & H^1(\mathrm{Gal}(K/F), M). \end{array}$$

In this diagram, the top arrow is an injection, the left vertical arrow is a bijection by Theorem 10, and the bottom arrow is a bijection since $\mathrm{Gal}(K/F) = \mathrm{Gal}(R/F)$. Therefore all arrows in the diagram are bijections. $\square$

In Example 10 we considered (F^{PV}/F)-twisted forms. We can these with the following cohomology set. Any Picard-Vessiot field extension L/F with a Picard-Vessiot subextension K/F gives a map $\mathrm{Gal}(L/F) \to \mathrm{Gal}(K/F)$ which in turn induces a map

$$H^1(\mathrm{Gal}(K/F), G) \to H^1(\mathrm{Gal}(L/F), G). \tag{4.15}$$

The maps of the form (4.15) forms a direct system. We call its direct limit

$$H^1(F, G) = \varinjlim H^1(\mathrm{Gal}(L/F), G)$$

the *absolute cohomology set over F with values in G.*

Proposition 12. *Let $\mathbf{\Phi}$ be a tensor-type and M a $\mathbf{\Phi}$-structure over F. Suppose that the automorphism group of $M \otimes_F F^{PV}$ is representable by an algebraic group G over C. Suppose also that the map*

$$\varinjlim \mathrm{TF}(K/F, M) \to \mathrm{TF}(F^{PV}/F, M) \tag{4.16}$$

induced by inclusion maps $\mathrm{TF}(K/F, M) \to \mathrm{TF}(F^{PV}/F, M)$, where K/F are Picard-Vessiot extensions over F, is a bijection. Then there is a bijection of the two sets $\mathrm{TF}(F^{PV}/F, M)$

and $H^1(F, G)$.

Proof. For any Picard-Vessiot field extension K/F, Proposition 11 gives a bijection $\mathrm{TF}(K/F, M) \cong H^1(\mathrm{Gal}(K/F), G)$. Now take direct limits over the Picard-Vessiot field extensions K/F. $\square$

Proposition 12 lets us classify the classes of objects we encountered in Example 10.

Corollary (Differential modules)**.** *There is a bijection* $\mathbf{Diff}_n(F) \cong H^1(F, \mathrm{GL}_n)$.

Proof. By Example 10, it suffices to show $\mathrm{TF}(F^{PV}/F, M) \cong H^1(F, \mathrm{GL}_n)$ where M be the trivial differential module over F of rank n. We first verify the hypotheses of Proposition 12 hold. Let D be a C-algebra. The elements of $\mathrm{Aut}(M \otimes_C D)$ consist of differential $F \otimes_C D$-linear isomorphisms and so lie in $\mathrm{GL}(M \otimes_C D) \cong \mathrm{GL}_n(F \otimes_C D)$. Since these maps commute with the trivial derivation on $M \otimes_C D$, they lie in the subgroup $\mathrm{GL}_n(C \otimes_C D) = \mathrm{GL}_n(D)$. This proves $\mathrm{Aut}(M \otimes_C D) \cong \mathrm{GL}_n(D)$.

Next a (F^{PV}/F)-twisted form N of M gives a differential isomorphism $\varphi : N \otimes_F F^{PV} \to M \otimes_F F^{PV}$. Pick F-bases $\{n_i\}$ and $\{m_j\}$ for N and M. Then $\varphi(n_i) = \sum c_{ij} m_j$ for some $c_{ij} \in F^{PV}$. Therefore φ restricts to a differential isomorphism $N \otimes_F K \cong M \otimes_F K$, where K is the smallest Picard-Vessiot extension in F^{PV} generated by the n^2 coefficients c_{ij}. Thus (4.16) is surjective.

Since all hypotheses are verified, we may apply Proposition 12 to get the desired bijection. $\square$

Corollary (Differential torsors)**.** *Let G be a linear algebraic group over C and consider G_F as a trivial differential G_F-torsor. Then there is a bijection G-$\mathbf{tors}^{\partial}(F) \cong H^1(F, G)$.*

Proof. By Proposition 7, it suffices to show $\mathrm{TF}(F^{PV}/F, G_F) \cong H^1(F, G)$.

Let D be a C-algebra. The automorphisms of $G_{F \otimes_C D}$ as a $G_{F \otimes_C D}$-torsor are elements of $G(F \otimes_C D)$; the *differential* automorphism of the trivial differential $G_{F \otimes_C D}$-torsor $G_{F \otimes_C D}$

lie in the subgroup $G(C \otimes_C D) = G(D)$. Thus $\mathrm{Aut}(G(F \otimes_C D)) \cong G(D)$.

Next if X is a (F^{PV}/F)-twisted form of G_F, we get an isomorphism of differential Hopf-Galois extensions $F[X] \otimes_F F^{PV} \cong F[G] \otimes_F F^{PV}$. Since $F[X]$ and $F[G]$ are finitely generated as F-algebras, by a similar argument to the proof of Corollary 4.5, this isomorphism restricts to one over a Picard-Vessiot field extension K/F. Therefore X is a (K/F)-twisted form of G_F. We conclude that (4.16) is surjective.

Since all hypotheses are verified, we may apply Proposition 12 to get the desired bijection.

$\square$

Remark 8. The bijection in Corollary 4.5 can be made explicit by working with differential Hopf-Galois extensions. Here is a more straightforward description in terms of differential torsors. A differential G-torsor X in $\mathrm{TF}(F^{PV}/F, G_F)$ lies in $\mathrm{TF}(K/F, G_F)$ for some Picard-Vessiot field extension K/F. For any such K, $X_K \cong G_K$ and so has a differential K-rational point x. Any $\sigma \in \mathrm{Gal}(K/F)$ defines an automorphism on $X(K)$, and we set $a_\sigma \in G(K)$ be the unique element such that $\sigma(x) = x \cdot a_\sigma$ in $X(K)$. We can check that $\sigma \mapsto a_\sigma$ defines a cocycle $a \in H^1(\mathrm{Gal}(K/F), G)$, and so we have a map $\mathrm{TF}(F^{PV}/F, G_F) \to H^1(\mathrm{Gal}(K/F), G)$. The cocycle constructed is compatible with the maps $H^1(\mathrm{Gal}(K/F), G) \to H^1(\mathrm{Gal}(L/F), G)$ and so we get an induced map $\mathrm{TF}(F^{PV}/F, G_F) \to H^1(F, G)$. One checks that this map coincides with the other description.

CHAPTER 5

Versal Differential Torsors

In this chapter, we fix a base differential field F and assume that its constant field C is algebraically closed and of characteristic zero. Following Berhuy and Favi 2003, Sections 4 and 6, we define and prove properties about versal differential torsors and generic differential torsors.

Let G be an algebraic group over C. Roughly speaking, a "versal differential G-torsor" should be a differential G-torsor that can "specialize" to any other differential G-torsor defined over fields. Here "specializing" a differential torsor should mean pulling back the differential torsor to one defined over a point.

Formally, let $f : X \to Y$ be a differential G-torsor and let K be in $\text{Fields}^{\partial}_{F,C}$. We define a map

$$Y(K) \to G\textbf{-tors}^{\partial}(K) \tag{5.1}$$

which takes $y \in Y(K)$ to the fiber $X_y \to y$. By Corollary 4.5, this can be rewritten as

$$Y(K) \to H^1(K, G) \tag{5.2}$$

which takes y to a cocycle a that corresponds to X_y according to Remark 8. We will refer to both (5.1) and (5.2) as the *specialization map*.

Definition 7. Let G be an algebraic group over C. A differential G-torsor $f : X \to Y$ is *versal* for G if for any K in $\text{Fields}^{\partial}_{F,C}$ and any $Z \in G\textbf{-tors}^{\partial}(K)$, under the specialization map

$$s : Y(K) \to G\textbf{-tors}^{\partial}(K) \tag{5.3}$$

the fiber $s^{-1}(Z)$ is Kolchin dense in Y. The generic fiber of a versal differential G-torsor is called a *generic differential G-torsor.*

Equivalently, a differential G-torsor $X \to Y$ is versal if for any nonempty open differential subscheme W of Y and any K in $\mathrm{Fields}^{\partial}_{F,C}$, the specialization map

$$W(K) \to G\text{-}\mathbf{tors}^{\partial}(K) \tag{5.4}$$

is surjective.

Versal differential torsors arise in the following way.

Proposition 13. *Let G be an algebraic group over C. Let G_F act linearly on $\mathbb{A}^n_F$. Suppose that U is a G-invariant open differential subscheme of $\mathbb{A}^n$. Suppose also that $\pi : U \to Y$ is a differential G-torsor. Then $\pi : U \to Y$ is a versal differential G-torsor.*

Proof. Let K be in $\mathrm{Fields}^{\partial}_{F,C}$ and W an open differential subscheme of Y. We must show the map

$$W(K) \to H^1(K, G) \tag{5.5}$$

is surjective. Let $a \in H^1(K, G)$. We define the twisted action of Γ_K on $V(K^{PV})$ given by $\boldsymbol{\sigma}(v) := \sigma(v) \cdot a_\sigma^{-1}$ for all $v \in V(K^{PV})$ and $\sigma \in \Gamma_K$. Viewing $V(K)$ as the trivial differential K-module, $V(K^{PV})^{\boldsymbol{\Gamma_K}}$ is then a twisted form of $V(K)$ and so by Corollary 4.5, $V(K^{PV})^{\boldsymbol{\Gamma_K}} \otimes_K K^{PV} \cong V(K^{PV})$. Thus $V(K^{PV})^{\boldsymbol{\Gamma_K}}$ is a n-dimensional K-linear subspace of $V(K^{PV})$, hence Kolchin dense in $V(K^{PV})$.

Since $\pi^{-1}(W)$ is open in V, there exists a point $x \in V(K^{PV})^{\boldsymbol{\Gamma_K}} \cap \pi^{-1}(W)(K^{PV})$. For any $\sigma \in \Gamma_K$, we then have $x = \boldsymbol{\sigma}(x) = \sigma(x) \cdot a_\sigma^{-1}$, giving

$$\sigma(\pi(x)) = \pi(\sigma(x)) = \pi(x \cdot a_\sigma^{-1}) = \pi(x).$$

Therefore $\pi(x)$ lies in $W(K^{PV})^{\Gamma} = W(K)$ and $\pi(x)$ maps to a under (5.5). Since $a \in H^1(K, G)$ is arbitrary, the map (5.5) is surjective. □

To construct versal differential torsors explicitly we specialize our situation further:

Lemma 14. *Let $R = F\{x_1, ..., x_n\}$ and $w = \mathrm{wr}(x_1, ..., x_n)$. Let GL_n act by right matrix multiplication on $\mathbb{A}^n = \mathrm{Spec}_D R$ and let G be a closed subgroup of GL_n. If $R\{1/w\}^G$ is differentially finitely generated over F and has field of fractions $\mathrm{Frac}(R)^G$, then*

$$\mathrm{Spec}_D R\{1/w\} \to \mathrm{Spec}_D R\{1/w\}^G$$

is a versal differential G-torsor.

Proof. Let $B := R\{1/w\}$ and $A := R\{1/w\}^G$. Note that $\mathrm{Spec}_D B$ is a G-invariant open subset of $\mathbb{A}^n$. By Proposition 13, it suffices to show that $\mathrm{Spec}_D B \to \mathrm{Spec}_D A$ is a differential torsor.

Let $K = \mathrm{Frac}(R)$. By Van der Put and Singer 2012, Exercise 1.35(4), K/K^{GL_n} is a GL_n-Picard-Vessiot field extension for the differential equation

$$p(y) = \mathrm{wr}(y, x_1, ..., x_n)/w.$$

By the differential Galois correspondence, K/K^G is a G-Picard-Vessiot field extension for $p(y)$ over K^G. The Picard-Vessiot ring T for the extension K/K^G is then generated by the solutions $x_1, ..., x_n$ for $p(y)$ and the element w^{-1} over K^G. In particular, T contains R. The fact that $\mathrm{Spec}_D T \to \mathrm{Spec}_D K^G$ is a differential G-torsor gives a differential isomorphism

$$T \otimes_{K^G} T \cong T \otimes_F F[G] \tag{5.6}$$

given explicitly as follows. Let $F[\mathrm{GL}_n] = F[z_{ij}, 1/\det(Z)]$ where $Z = (z_{ij})$ is a $n \times n$ matrix of indeterminates. The closed embedding $G \to \mathrm{GL}_n$ induces a surjective F-algebra homomorphism $F[\mathrm{GL}_n] \to F[G] : z_{ij} \mapsto \overline{z_{ij}}$. Then (5.6) is determined by

$$\begin{array}{rcll} h \otimes 1 & \mapsto & h \otimes 1 & \text{for all } h \in T, \\ 1 \otimes x_j & \mapsto & \sum_{i=1}^n x_i \otimes \overline{z_{ij}} & \text{for } 1 \leq i \leq n, \\ 1 \otimes w^{-1} & \mapsto & w^{-1} \otimes \det(\overline{z_{ij}})^{-1}. & \end{array}$$

This restricts to a differential isomorphism

$$B \otimes_A B \cong B \otimes_F F[G] \tag{5.7}$$

which shows that $\mathrm{Spec}_D\, B \to \mathrm{Spec}_D\, A$ is a differential G-torsor. □

We now write down explicit examples of versal differential G-torsors.

Proposition 15. *Let $R = F\{x_1, ..., x_n\}$ and $w = \mathrm{wr}(x_1, ..., x_n)$. Let GL_n act by right matrix multiplication on $\mathbb{A}^n = \mathrm{Spec}_D\, R$. Then*

$$\mathrm{Spec}_D\, R\{1/w\} \to \mathrm{Spec}_D\, R\{1/w\}^G$$

is a versal differential G-torsor for $G = \mathrm{GL}_n$ and $\mathbb{G}_m^n$.

Proof. By Lemma 14, it suffices to show that $R\{1/w\}^G$ is differentially finitely generated over F. Let $K = \mathrm{Frac}(R)$.

Let $G = \mathrm{GL}_n$. The proof for this case is by Juan and A. R. Magid 2007 which we now reproduce. By Van der Put and Singer 2012, Exercise 1.35(4), K/K^G is a G-Picard-Vessiot field extension for the differential equation

$$p(y) = \mathrm{wr}(y, x_1, ..., x_n)/w = y^{(n)} + w_{n-1}y^{(n-1)} + \cdots + w_0 y.$$

Therefore the coefficients $w_0, ..., w_{n-1}$ are differentially algebraically independent over F.

Since $p(x_i) = 0$ for $i = 1, ..., n$,

$$\begin{aligned} R\{1/w\} &= F\{w_0, ..., w_{n-1}, 1/w\}[x_i^{(j-1)}]_{1 \le i,j \le n} \\ &\cong F\{w_0, ..., w_{n-1}\} \otimes_F F[x_i^{(j-1)}, 1/w]_{1 \le i,j \le n}. \end{aligned} \tag{5.8}$$

We identify $F[x_i^{(j-1)}, 1/w]_{1 \le i,j \le n}$ with $F[\mathrm{GL}_n]$ and $F\{w_0, ..., w_{n-1}\}$ with the coordinate ring of a scheme V over F defined as follows. For any algebra S over F, we set

$$V(S) = \oplus_{j=1}^{\infty} \oplus_{i=1}^{n} S \cdot x_i^{(j-1)}$$

to be the free S-module on basis elements $x_i^{(j-1)}$.

For each j, we define a (right) $G(S)$-action on the free S-module $\oplus_{i=1}^{n} S \cdot x_i^{(j-1)}$ of rank n by right matrix multiplication in the coordinates $x_1^{(j-1)}, ..., x_n^{(j-1)}$. We define V_{triv} to be V with trivial G-action.

Considering $V \times G$ and $V_{triv} \times G$ with the diagonal G-action, we have a G-equivariant isomorphism

$$V \times G \to V_{triv} \times G$$

given by $(v, g) \mapsto (v \cdot g^{-1}, g)$. This induces an isomorphism on coordinate rings

$$R\{1/w\} \cong F[V \times G] \xrightarrow{\sim} F[V_{triv} \times G] \cong F[V_{triv}] \otimes F[G] \tag{5.9}$$

which restricts to G-invariants

$$R\{1/w\}^G \xrightarrow{\sim} F[V_{triv} \times G]^G = F[V_{triv}]. \tag{5.10}$$

Under (5.9), $F\{w_0, ..., w_{n-1}\}$ maps to $F[V_{triv}]$ while $F[x_i^{(j-1)}]_{1 \le i,j \le n}$ maps to $F[G]$, so by (5.8), $F\{w_0, ..., w_{n-1}\}$ is precisely the preimage of $F[V_{triv}]$. Comparing with (5.10) we see that $R\{1/w\}^G = F\{w_0, ..., w_{n-1}\}$ is differentially finitely generated over F.

A similar proof works for $G = \mathbb{G}_m^n$. Let $G = \mathbb{G}_m^n$. Then K is a $\mathbb{G}_m^n$-Picard-Vessiot extension over $F\langle w_1, ..., w_n\rangle$ where $w_i = \partial x_i / x_i$. Since the x_i are solutions to the differential equation $\partial Y_i / Y_i = w_i$ for $i = 1, ..., n$, we have

$$\begin{aligned} R\{1/w\} &= F\{w_1, ..., w_n, 1/w\}[x_1, ..., x_n] \\ &\cong F\{w_1, ..., w_n\} \otimes_F F[x_1, ..., x_n, 1/w]. \end{aligned} \tag{5.11}$$

where again we identify $F[x_1, ..., x_n, 1/w]$ with $F[G]$ and $F\{w_1, ..., w_n\}$ with V as defined above. Arguing as before gives $R\{1/w\}^G = F\{w_1, ..., w_n\}$. $\square$

Now that we have exhibited versal differential torsors, our next goal is to understand the differential essential dimension of their generic fibers.

Definition 8. Given differential G-torsors $f : X \to Y$ and $f' : X' \to Y'$, we say that f' is a *compression* of f if there exists a commutative diagram

$$\begin{tikzcd} X \arrow[r, dashed, "g"] \arrow[d, "f"] & X' \arrow[d, "f'"] \\ Y \arrow[r, dashed, "h"] & Y' \end{tikzcd} \tag{5.12}$$

where g is a G-equivariant rational dominant morphism and h is a rational morphism.

Proposition 16. *The compression of a versal differential torsor is again versal.*

Proof. Let $f' : X' \to Y'$ be a compression of $f : X \to Y$ and let g and h be as in (5.12). Let K be in $\mathrm{Fields}^{\partial}_{F,C}$ and let $a \in H^1(K, G)$. Let $Z = \{y \in Y(K) \mid y \mapsto a\}$ and $Z' = \{y' \in Y'(K) \mid y' \mapsto a\}$. The domain of definition of h contains an affine open subset U of Y. For any $y \in U$, the fiber of f at y is isomorphic to the fiber of f' at $h(y)$, so $Z \cap U \subset h^{-1}(Z')$. To show that f' is versal differential torsor, we must show that Z is dense in $Y'(K)$. Let V' be an open subset of $Y'(K)$. We must check that $V' \cap Z'$ is nonempty. Well $h^{-1}(V' \cap Z') = h^{-1}(V') \cap h^{-1}(Z') \supset h^{-1}(V') \cap (Z \cap U)$. Since f is versal, Z is dense in $Y(K)$, and so the intersection of Z with the open set $h^{-1}(V') \cap U$ is nonempty. $\square$

Proposition 17. *Let $X \to Y$ be a versal differential G-torsor. Suppose its generic fiber $X_0 \to Y_0$, where $Y_0 = \mathrm{Spec}_D(F(Y))$, is defined over a subfield as $X_0' \to Y_0'$. Then there exists a differential G-torsor $X' \to Y'$ together with a compression*

$$\begin{tikzcd} X \arrow[r, dashed, "g"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ Y \arrow[r, dashed, "h"'] & Y' \end{tikzcd}$$

such that at the generic fibers we get

$$\begin{tikzcd} X_0 \arrow[r, "g"] \arrow[d, "f"'] & X_0' \arrow[d, "f'"] \\ Y_0 \arrow[r, "h"'] & Y_0'. \end{tikzcd}$$

Proof. Let $X, Y, X_0, Y_0, X_0', Y_0'$ be the differential spectra of B, A, P, K, P', K', respectively. Proving the proposition reduces to finding:

(a) a differential subring A' of K' such that $\mathrm{Frac}\, A' = K'$;

(b) a differential algebra B' over A' satisfying $P' \cong B' \otimes_{A'} K'$ and such that the map $\mathrm{Spec}_D\, B' \to \mathrm{Spec}_D\, A'$, induced from the structure map $A \to B$, is a differential G-torsor; and

(c) elements $u \in K$ and $p \in P$ and a commutative diagram

$$\begin{tikzcd} B' \arrow[r, "g^{\#}"] \arrow[d, "f"'] & B\{1/p\} \arrow[d, "f'"] \\ A' \arrow[r, "h^{\#}"'] & A\{1/u\}. \end{tikzcd} \tag{5.13}$$

Since $\mathrm{trdeg}_F^{\partial} K$ and $\mathrm{trdeg}_F^{\partial} K'$ are finite and since P/K and P'/K' are algebras of finite type, we have $K = F\langle \boldsymbol{x} \rangle$, $K' = F\langle \boldsymbol{x}' \rangle$, $P = K[\boldsymbol{y}]$, and $P' = K'[\boldsymbol{y}']$ for some finite subsets $\boldsymbol{x} \subset K$, $\boldsymbol{x}' \subset K'$, $\boldsymbol{y} \subset P$, and $\boldsymbol{y}' \subset P'$.

The isomorphism

$$P' \otimes_{K'} P' \cong P' \otimes_F F[G] \tag{5.14}$$

is determined by $f \otimes 1 \mapsto f \otimes 1$ for $f \in P'$ and $1 \otimes y_j \mapsto \sum f_{ij} y_i \otimes c_{ij}$ for some $c_{ij} \in F[G]$, $f_{ij} \in K'$. Since any element of K' can be written as a ratio of elements of $F\{\boldsymbol{x}'\}$, we let f be the product the denominators of the terms f_{ij} which are nonzero. Set $A' = F\{\boldsymbol{x}', 1/f\}$, and $B' = A'[\boldsymbol{y}']$. Then (5.14) restricts to an isomorphism

$$B' \otimes_{A'} B' \cong B' \otimes_F F[G].$$

This gives (a) and (b).

For (c), since A' is differentially of finite type over F, by the same reasoning as above, the composite map

$$A' \hookrightarrow K' \overset{h^\#}{\to} K$$

has image inside $F\{\boldsymbol{x}, 1/u\}$ for some $u \in K$. Likewise $A = F\{\boldsymbol{x}, 1/v\}$ for some $v \in K$. Thus we have a sequence of maps

$$A' \overset{h^\#}{\to} F\{\boldsymbol{x}, 1/u\} \hookrightarrow F\{\boldsymbol{x}, 1/u, 1/v\} = A\{1/u\}.$$

Similarly we get a map $B' \to B\{1/p\}$ compatible with this map, for some $p \in P$. This gives the commutative diagram (5.13). $\square$

Combining Propositions 16 and 17 we get the following:

Corollary. *If a generic differential G-torsor X descends to a differential G-torsor X', then X' is again a generic differential G-torsor.*

Corollary. *Let X be a generic differential G-torsor. Then* $\mathrm{ed}_F^\partial(X) = \mathrm{ed}_F^\partial(G\text{-}\mathbf{tors}^\partial)$.

Proof. Clearly $\mathrm{ed}_F^\partial(X) \le \mathrm{ed}_F^\partial(G\text{-}\mathbf{tors}^\partial)$. The differential torsor X descends to a differential torsor $X_0 \to \operatorname{Spec}_D K$ for which $\operatorname{trdeg}_F^\partial K = \operatorname{ed}_F^\partial(X)$. Propositions 16 and 17 guarantee that X_0 is the generic fiber of some versal differential torsor $X' \to Y'$. By the definition of versal differential torsor, the specialization map

$$Y' \to G\text{-}\mathbf{tors}^\partial$$

is a surjective natural transformation. Therefore by 4, we have

$$\operatorname{ed}_F^\partial(X) = \operatorname{trdeg}_F^\partial K = \dim Y' \geq \operatorname{ed}_F^\partial(G\text{-}\mathbf{tors}^\partial).$$

□

With this, we have finished the most technical part of this thesis. Our next task is to use this corollary to give bounds on the number $\operatorname{ed}_F^\partial(G\text{-}\mathbf{tors}^\partial)$.

CHAPTER 6

Applications

6.1. Bounds on differential essential dimension

Proposition 18. *We have* $\mathrm{ed}_F^{\partial}(\mathrm{GL}_n) = \mathrm{ed}_F^{\partial}(\mathbb{G}_m^n) = n$.

Proof. Let $G = \mathrm{GL}_n$, $H = \mathbb{G}_m^n$, $R = F\{x_1, ..., x_n\}$, $K = \operatorname{Frac} R$, and $w = \mathrm{wr}(x_1, ..., x_n)$. By Proposition 15, $\operatorname{Spec}_D R\{1/w\} \to \operatorname{Spec}_D R\{1/w\}^G$ is a versal differential G-torsor for $G = \mathrm{GL}_n$ and $\mathbb{G}_m^n$. Their generic fibers are the differential $F[G]$-Hopf Galois extension S/K^G and the differential $F[H]$-Hopf Galois extension T/K^H for some differential subrings S and T in R, respectively. By looking at our construction, we see that S/K^G is a G-Picard-Vessiot extension and T/K^H is a H-Picard-Vessiot extension. By Corollary 5, $\mathrm{ed}^{\partial}(S/K^G) = \mathrm{ed}^{\partial}(G\text{-}\mathbf{tors}^{\partial})$ so S/K^G descends to a differential $F[G]$-Hopf Galois extension S_0/K_0^G, $K_0 = \mathrm{Frac}(S_0)$, satisfying $\mathrm{trdeg}_F^{\partial} K_0 = \mathrm{trdeg}_F^{\partial} K_0^G = \mathrm{ed}^{\partial}(G\text{-}\mathbf{tors}^{\partial})$. Since S has no new constant, neither does S_0 and so S_0/K_0^G is a G-Picard-Vessiot extension. In turn, the G-Picard-Vessiot field extension K/K^G descends to K_0/K_0^G and so the H-Picard-Vessiot field extension K/K^H descends to K_0/K_0^H. Let T_0 be the Picard-Vessiot ring for the extension K_0/K_0^H. Then $T_0 \otimes_{K_0^H} K^H$ is the H-Picard-Vessiot ring of K/K^H and so must coincide with T. Therefore T/K^H descends to T_0/K_0^H and so $\mathrm{ed}^{\partial}(T/K^H) \leq \mathrm{trdeg}(K_0)$. By invoking Corollary 5 again, we have

$$\mathrm{ed}^{\partial}(H\text{-}\mathbf{tors}^{\partial}) = \mathrm{ed}^{\partial}(T/K^H) \leq \mathrm{trdeg}(K_0) = \mathrm{ed}^{\partial}(G\text{-}\mathbf{tors}^{\partial}). \tag{6.1}$$

By Proposition 5, $\mathrm{ed}^{\partial}(H\text{-}\mathbf{tors}^{\partial}) \geq n$. We also know that $\mathrm{trdeg}(K_0) \leq \mathrm{trdeg}^{\partial} K = n$ so all the terms in (6.1) equal n. □

6.2. General differential equations

Proposition 19. *Let $a_0, ..., a_{n-1}$ be differential indeterminates over F. Consider the general differential equation $p(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0$ over $L = F\langle x_0, ..., x_{n-1}\rangle$ as an element* *of* $\mathbf{DiffEq}_n(L)$*. Then* $\mathrm{ed}_F^\partial(p(y)) = n$*.*

Proof. Let $K = F\langle x_1, ..., x_n\rangle$ and $w = \mathrm{wr}(x_1, ..., x_n)$. By Van der Put and Singer 2012, Exercise 1.35(4), the coefficients of the differential equation

$$\mathrm{wr}(y, x_1, ..., x_n)/w = y^{(n)} + w_{n-1}y^{(n-1)} + \cdots + w_0$$

are differential algebraically independent over F and so we may regard $p(y)$ as this differential equation in the coefficients w_i. Let $K_0 = K^{\mathrm{GL}_n}$. By the short discussion after Example 6, $p(y)$ corresponds to the differential module $N = K_0[\partial]/K_0[\partial]p(y)$ under the isomorphism $\mathbf{DiffEq}_n \cong \mathbf{Diff}_n$. By 4.5, and 4.5, there is a sequence of bijections

$$\mathbf{Diff}_n(K) \cong H^1(K^{PV}/K, \mathrm{GL}_n) \cong \mathrm{GL}_n\text{-}\mathbf{tors}^\partial(K). \tag{6.2}$$

Since (6.2) defines an isomorphism of functors $\mathbf{Diff}_n \cong \mathrm{GL}_n\text{-}\mathbf{tors}^\partial$, it suffices to show that the differential module N corresponds to the generic differential GL_n-torsor S/K_0 (as constructed above; S is the Picard-Vessiot ring of the extension K/K_0) under this correspondence, for then

$$\mathrm{ed}_F^\partial(p(y)) = \mathrm{ed}_F^\partial(N) = \mathrm{ed}_F^\partial(S/K_0) = n.$$

First view N as the differential submodule $\bigoplus_{i=0}^{n-1} K_0 x_1^{(i)}$ of S. Consider any differential isomorphism $\varphi : S \otimes_{K_0} K \to K[\mathrm{GL}_n]$. Since $N \otimes_{K_0} K$ is a trivial differential module, its image is also a trivial differential module M_1 over K. For any $\sigma \in \mathrm{Gal}(K/K_0)$, consider the following diagram:

$$\begin{tikzcd}
 & & & S \arrow[r, "\varphi"] \arrow[d, dotted, "\sigma"] & K[\mathrm{GL}_n] \arrow[d, "\sigma"] \\
N_K \arrow[r, "\varphi"] \arrow[d, "\sigma"] \arrow[urrr, hook] & M_1 \arrow[d, "\sigma"] \arrow[urrr, hook] & & S \arrow[r, dotted, "\varphi"] & K[\mathrm{GL}_n] \\
N_K \arrow[r, "\varphi"] \arrow[urrr, hook, dotted] & M_1 \arrow[urrr, hook] & & &
\end{tikzcd}$$

All the faces of this rectangular prism are commutative except the front and the back. Therefore the cocycle corresponding to the back face restricts to the the cocycle which is front face. Therefore these define the equivalent cocycles in $H^1(K, \mathrm{GL}_n)$ once we fix identifications $\mathrm{Aut}(M_1) \cong \mathrm{GL}_n(C)$ and $\mathrm{Aut}(K[\mathrm{GL}_n]) \cong \mathrm{GL}_n(C)$.

$\square$

6.3. Generic Picard-Vessiot extensions

In this section, we define and prove lower bounds on generic Picard-Vessiot extensions. We follow the method of Buhler and Reichstein 1997, Section 7.

Definition 9. Let $K = F\langle y_1, ..., y_n\rangle$ with indeterminates $y_1, ..., y_n$ over F. Let $p_{y_1,...,y_n}(y)$ be a homogeneous linear differential equation over K. Suppose that $p_{y_1,...,y_n}(y)$ determines a Picard-Vessiot extension R/K with differential Galois group G. Then R/K is said to be a *generic G-Picard-Vessiot extension* if for every G-Picard-Vessiot extension R'/K', there exist elements $a_1, ..., a_n \in K'$ such that R'/K' is the Picard-Vessiot extension for the differential equation $p_{a_1,...,a_n}(y)$.

Proposition 20. *Let G be a linear algebraic group over C, and let R/K be a generic G-Picard-Vessiot extension. Suppose that at least one generic differential G-torsor exists and corresponds to a G-Picard-Vessiot extension. Then* $\mathrm{trdeg}^{\partial}_F K \geq \mathrm{ed}^{\partial}_F(G\text{-}\mathbf{tors}^{\partial})$.

Proof. Let R'/K' be a Picard-Vessiot extension corresponding to a generic differential G-torsor. By the definition of generic G-Picard-Vessiot extension, R/K is the Picard-Vessiot extension for some differential equation $p_{y_1,...,y_n}(y)$ over K, and there exist $a_1, ..., a_n \in K'$ such that R'/K' is the Picard-Vessiot extension for $p_{a_1,...,a_n}(y)$ over K'.

We claim that R'/K' is defined over the differential subfield $K'' := F\langle a_1, ..., a_n\rangle$. Indeed, $p_{a_1,...,a_n}(y)$ is defined over K'' and has a full set of solutions within R', thus determining a G-Picard-Vessiot ring R'' (over K'') inside R'. Since $R'' \otimes_{K''} K'$ determines a G-Picard-Vessiot subextension of R' over K', we must have $R'' \otimes_{K''} K' = R'$. Therefore R'/K' descends to

R''/K''.

Now $K'' = F\langle a_1, ..., a_n\rangle$ implies that $\operatorname{trdeg}^{\partial}_F K = n \geq \operatorname{trdeg}^{\partial}_F K''$. Moreover R'/K' corresponds to a generic differential G-torsor so by Corollary 5, R''/K'' also corresponds to a generic differential G-torsor. Thus $\operatorname{trdeg}^{\partial}_F K'' \geq \operatorname{ed}^{\partial}_F(R'') = \operatorname{ed}^{\partial}(G\textbf{-tors}^{\partial})$. Combining the inequalities gives $\operatorname{trdeg}^{\partial}_F K \geq \operatorname{ed}^{\partial}(G\textbf{-tors}^{\partial})$.

□

Corollary. *Let G be either GL_n or $\mathbb{G}_m^n$. Let R/K be a generic G-Picard Vessiot extension. Then $\operatorname{trdeg}^{\partial}_F K \geq n$.*

Proof. We have directly constructed versal differential torsors for these groups G, the proof of which already showed their generic fibers corresponded to G-Picard-Vessiot extensions. Thus Proposition 20 applies. □

In the case of $G = \mathbb{G}_m^n$, the extension $F\langle x_1, ..., x_n\rangle / F\langle x_1, ..., x_n\rangle^G$ is a generic Picard-Vessiot extension, and so the lower bound given in Proposition 20 is sharp. For an arbitrary group G, however, we do not know if there exists a generic G-Picard-Vessiot extension R/K for which equality holds in $\operatorname{trdeg}^{\partial}_F K \geq \operatorname{ed}^{\partial}_F(G\textbf{-tors}^{\partial})$.

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