Signature | Printed Name

Preliminary Examination, Part I

Thursday, April 28, 2022 | 9:30am-12:00pm

This examination is based on Penn's code of academic integrity

Instructions:

Sign and print your name above.

This part of the examination consists of six problems, each worth ten points. You should work all of the problems. Show all of your work. Try to keep computations well-organized and proofs clear and complete — and justify your assertions. Each problem should be given its own page (or more than one page, if necessary).

If a problem has multiple parts, earlier parts may be useful for later parts. Moreover, if you skip some part, you may still use the result in a later part.

Be sure to write your name both on the exam and on any extra sheets you may submit.

Score (for faculty use only)	
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Grader	

1. Compute

$$\oint_C \left(\cos(x) + \sin(y^2)\right)\,dx + \left(2xy\cos(y^2) + xy^3\right)\,dy,$$

where C is the triangle in the xy-plane with vertices $(0,1), (1,0), (1,2)$ oriented counterclockwise.

Solution: Let C_I denote the interior of the triangle. By Green's Theorem, the integral equals

$$\begin{aligned}\int\int_{C_I} \partial_x &\left(2xy\cos(y^2) + xy^3\right) - \partial_y\left(\cos(x) + \sin(y^2)\right)\,dA \\ &= \int_0^1 \int_{1-x}^{1+x} y^3\,dy\,dx = \frac{1}{4}\int_0^1 (1+x)^4 - (1-x)^4\,dx \\ &= \frac{1}{20}\left((1+x)^5 + (1-x)^5\right)\Big|_0^1 = \frac{1}{20}(32-2) = \frac{3}{2}.\end{aligned}$$

2. Suppose that

$$\lim_{x\to a^+} f'(x)$$

exists and is finite, for some $a \in \mathbb{R}$. Prove that $\lim_{x\to a^+} f(x)$ also exists and is finite.

Solution: Since $\lim_{x\to a^+} f'(x)$ is a finite number, there exists $\delta_0, M > 0$ such that $|f'(x)| \le M$ for all $x \in (a, a+\delta_0)$. For all $\epsilon > 0$, let $\delta = \min(\frac{\epsilon}{M}, \delta_0)$. Then according to the Mean value theorem, for all $x_1, x_2 \in (a, a+\delta)$, there exists some ξ between x_1, x_2 such that

$$|f(x_1) - f(x_2)| \le |f'(\xi)||x_1 - x_2| \le M\delta < \epsilon.$$

Hence, according to Cauchy convergence criterion, the right limit of $f(x)$ also exists at a and is finite.

3. Let $\mathcal{M}_{2\times 2}$ be the space of matrices of size 2×2. Let $\mathcal{P}_2$ be the space of polynomials of degree up to 2. Let L be a linear transformation from $\mathcal{M}_{2\times 2}$ to $\mathcal{P}_2$ such that:

$$L\left(\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}\right) = 1 + x, \quad L\left(\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}\right) = x^2, \quad L\left(\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}\right) = 1 + 2x + x^2.$$

   (a) Compute $L\left(\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\right)$ and $L\left(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)$.

   (b) Determine the rank and nullity of L.

(c) Take bases

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

and

$$\mathcal{C} = \{1, x, x^2\}$$

of $\mathcal{M}_{2\times 2}$ and $\mathcal{P}_2$, respectively. Recover as many matrix elements of $[L]_{\mathcal{C}\leftarrow\mathcal{B}}$ as you can. If you don't get all of them, explain why you cannot recover more.

(d) Can you describe $\operatorname{Ker} L$? If yes, do it; if not, explain.

Solution

(a)

$$L\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = L\left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\right) - L\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right) = 1 + 2x + x^2 - x^2 = 1 + 2x.$$

$$L\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = L\left(\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\right) - L\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = 1 + x - (1 + 2x) = -x.$$

(b) The image of L is spanned by $1+x, -x, x^2$, so L is surjective, hence $\operatorname{rk} L = 3$. By rank-nullity theorem, we have $\operatorname{nul} L = 4 - 3 = 1$.

(c)

$$[L]_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 0 & 0 & 1 & ? \\ 0 & -1 & 2 & ? \\ 1 & 0 & 0 & ? \end{bmatrix}$$

Cannot recover more because any choice of the fourth column provides a matrix that satisfies the provided conditions.

(d) No, because varying the fourth column changes $\operatorname{Ker} L$.

4. Compute

$$\int_0^\infty \frac{\cos x}{x^2 + a^2}\, \mathrm{d}x$$

for $a > 0$.

Solution: Consider the contour integral of $f(z) = \frac{e^{iz}}{z^2+a^2}$ along the contour $\{(x, y) : |x + iy| = R, y > 0\} \cup [-R, R]$. Sending $R \to \infty$, since $f(z)$ decays as $O(R^{-2})$ along the semi-circle, this contour integral converges to $\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+a^2}\, \mathrm{d}x$. By the Cauchy Residue Theorem,

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2 + a^2}\, \mathrm{d}x = 2\pi i \operatorname{Res}(f; ia) = 2\pi i \lim_{x \to ia} \frac{e^{ix}}{x + ia} = \frac{\pi e^{-a}}{a}.$$

So

$$\int_0^{\infty} \frac{\cos x}{x^2+a^2}\,\mathrm{d}x = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2}\,\mathrm{d}x = \frac{\pi e^{-a}}{2a}.$$

5. Let R be a commutative ring (recall that we assume that R has a multiplicative identity $1 \neq 0$). Prove that if the polynomial ring $R[X]$ is a PID, then R is a field.

   Solution: Suppose $0 \neq a \in R$ and let I be the ideal $R[X] \cdot a + R[X] \cdot X$. Since $R[X]$ is a PID, $I = R[X] \cdot f(X)$ for some $f(X) = c_n X^n + \cdots c_0 \in R[X]$ with $c_i \in R$ and $c_n \neq 0$. Then

   $$0 \neq a = (c_n X^n + \cdots c_0) \cdot (b_m X^m + \cdots b_0) = c_n b_m X^{n+m} + t(X)$$

   for some $b_j \in R$ with $b_m \neq 0$ and some $t(X) \in R[X]$, where $t(X)$ has degree less than $n+m$ if $n+m > 0$ and $t(X) = 0$ if $n+m = 0$. Since R is a PID it is an integral domain. Therefore $c_n b_m \neq 0$ because $c_n \neq 0 \neq b_m$, so this forces $n+m = 0$ and $n = m = 0$. Thus $f(X) = c_0 \in R$ and $c_0 b_0 = a$. But now $X \in I$ means $X = f(X) \cdot y(Y) = c_0 \cdot y(X)$ for some $y(X) \in R[X]$. Again using that R is an integral domain, we conclude that $y(X) = d_1 X$ for some $d_1 \in R$ with $c_0 d_1 = 1$. Then $I = R[X] \cdot f(x) = R[X] \cdot c_0$ contains 1, so $I = R[X]$. Then $1 \in I = R[X] \cdot a + R[X] \cdot X$ forces $1 = az(X) + d(X)X$ for some $z(X), d(X) \in R[X]$, and setting $X = 0$ gives $1 = aZ(0)$ with $Z(0) \in R$. Thus every non-zero element a of R is invertible, so R is a field.

6. Prove that for all positive integer n,

   $$\sum_{k=1}^{n} a_k^2 \leq \int_{-1}^{1} [f(x)]^2 dx, \quad a_k = \int_{-1}^{1} f(x)\sin(k\pi x)dx,$$

   where $f(x)$ is odd and piecewise continuous in $(-1,1)$. *Hint:* Start by rewriting the inequality

   $$\int_{-1}^{1} \left[f(x) - \sum_{k=1}^{n} a_k \sin(k\pi x) \right]^2 dx \geq 0$$

   as

   $$2\int_{-1}^{1} f(x) S_n(x) dx - \int_{-1}^{1} [S_n(x)]^2 dx \leq \int_{-1}^{1} [f(x)]^2 dx, \tag{1}$$

   where

   $$S_n(x) = \sum_{k=1}^{n} a_k \sin(k\pi x)$$

is the partial sum of the Fourier series corresponding to $f(x)$. Then evaluate each of the two integrals on the left side of (**??**).

Solution: Let

$$S_n(x) = \sum_{k=1}^{n} a_k \sin(k\pi x),$$

which is the partial sum of the Fourier series corresponding to $f(x)$. We have

$$\int_{-1}^{1} [f(x) - S_n(x)]^2 \, dx \geq 0,$$

since the integrand is non-negative. Expanding the integrand, we obtain

$$2\int_{-1}^{1} f(x)S_n(x)dx - \int_{-1}^{1} [S_n(x)]^2 dx \leq \int_{-1}^{1} [f(x)]^2 dx.$$

Multiplying $f(x)$ to the partial sum and integrating gives

$$\int_{-1}^{1} f(x)S_n(x)dx = \sum_{k=1}^{n} a_k^2.$$

Also, using the orthogonality of the sine functions yields

$$\int_{-1}^{1} [S_n(x)]^2 dx = \sum_{k=1}^{n} a_k^2.$$

Substituting this into the inequality gives the required result.

Signature	Printed Name

Preliminary Examination, Part II

Thursday, April 28, 2022 2:00pm-4:30pm

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7. (a) (4 points) Fill in the blanks in the following sentence: The Heine-Borel Theorem says that a set $S \subseteq \mathbb{R}^n$ is compact if and only if it is ____________ and ____________.

   (b) (6 points) Let S be a subset of $\mathbb{R}^n$. Let w be a limit point of S, and let C be a finite family of open sets with the property that each open set $U \in C$ is disjoint from some neighborhood V_U of w. Prove that C fails to be a cover of S.

   Solution: a) The Heine-Borel Theorem says that a set in $S \subseteq \mathbb{R}^n$ is compact iff it is closed and bounded.

   b) The intersection of the finite family of sets V_U is a neighborhood W of w in $\mathbb{R}^n$. Since w is a limit point of S, W must contain a point x in S. This $x \in S$ is not covered by the family C, because every U in C is disjoint from V_U and hence disjoint from W, which contains x.

8. Suppose a function f on $[a, b]$ is Riemann integrable. Let g be a function on $[a, b]$ that differs from f at only finitely many points. Prove that g is also Riemann integrable and $\int_a^b g(x)\,dx = \int_a^b f(x)\,dx$.

   Solution: Consider function $h = f - g$. By linearity of Riemann integral, it suffices to show that h is Riemann integrable on $[a, b]$ and has integral zero. This follows from the definition of Riemann integration. Indeed, suppose h is nonzero only at points $a_1, \cdots, a_k \in [a, b]$. Without loss of generality, assume $h(a_i) > 0, \forall i = 1, \cdots, k$. Let $\mathcal{P} = \{I_j\}$ be any partition of $[a, b]$ into subintervals of length less than δ, then the Riemann sum corresponding to this partition is bounded between zero and $\sum_{i=1}^k h(a_i)|I_i| \leq \delta \sum_{i=1}^k h(a_i)$. Here, I_i denotes the unique subinterval in the partition that contains a_i. As $\delta \to 0$, the limit of the Riemann sum is obviously zero.

9. Is this R-module M free?

   (a) $R = \mathbb{Z}$, $M = \mathbb{Q}$

   (b) $R = \mathbb{Q}$, $M = \mathbb{R}$

   Solution:

   (a) No, because $\mathbb{Q}$ is clearly not free of rank one, and any two nonzero rational numbers are linearly dependent over $\mathbb{Z}$.

   (b) Yes, because every vector space is a free module.

10. Construct the conformal map ψ from the unit disk $\{z \in \mathbb{C} : |z| < 1\}$ to the first quadrant $\{x + iy : x > 0, y > 0\}$, such that $\psi(0) = 1 + i$ and $\lim_{z\to 1} \psi(z) = 0$.

Solution: Let $\phi(z) = 2i\frac{1-z}{1+z}$. Then ϕ is the conformal map from the unit disk to the upper half plane with $\lim_{z\to 1} \phi(z) = 0$. Therefore, $\psi(z) = \sqrt{\phi(z)}$ is the desired map to the first quadrant.

11. Let V be a complex vector space of dimension 7 with basis $v_1, \ldots, v_7$. Let $H : V \to V$ be the linear map defined as $H(v_k) = v_{k+1}$ for $k = 1, \ldots, 6$ and $H(v_7) = 0$. Find the Jordan canonical form of the map $T = I + H^2 + H^4$, where $I : V \to V$ is the identity map.

Solution: A matrix for H relative to the basis $\{v_7, v_6, \ldots, v_1\}$ is a single Jordan block with eigenvalue 0. The matrix of T is then upper triangular with 1's down the diagonal, so all eigenvalues of T equal 1. Let $n_1, \cdots, n_m$ be the sizes of the Jordan blocks of T. Then these are also the sizes of the Jordan blocks of $T - I$, and $\sum_i n_i = 7$. We know m is dimension of the null space of $T - I$ since all blocks of $T - I$ have eigenvalue 0. Now

$$T - I = H^2 + H^4 = H^2(I + H^2)$$

has the property that $I + H^2$ is invertible, while H^2 is 0 on the two dimensional space $\mathbb{C}v_7 \oplus \mathbb{C}v_6$ and is injective on $\oplus_{i=1}^5 \mathbb{C}v_i$. So $T-I$ has null space of dimension 2 and there are $m = 2$ Jordan blocks of sizes $n_1 \le n_2$ with $n_1+n_2 = 7$. A Jordan block matrix M_i of size n_i with eigenvalue 0 has the property that the smallest power of M_i that is the 0-matrix is M^{n_i} Thus $n = \max\{n_1, n_2\}$ is the smallest integer such that

$$(T - I)^n = (H^2 + H^4)^n = H^{2n}(1 + H^2)^n$$

is zero. Here $1 + H^2$ is invertible and $H^{2n}(v_i) = v_{i+2n}$ for $1 \le i \le 7 - 2n$ and $H^{2n}(v_i) = 0$ if $i > 7 - 2n$. So $n = 4 = \max\{n_1, n_2\}$. The only positive integers $n_1 \le n_2$ with $1 \le n_1 \le n_2 \le 4$ and $n_1 + n_2 = 7$ are $n_1 = 3$ and $n_2 = 4$. So the Jordan form of T has two Jordan blocks of sizes 3 and 4 with eigenvalue 1.

12. Let M be an invertible $n \times n$ matrix of real numbers. Let $F(\mathbf{x}) = M\mathbf{x} + p(\mathbf{x})$ where $\mathbf{x} \in \mathbb{R}^n$, p is a continuously differentiable function from $\mathbb{R}^n \to \mathbb{R}^n$ and $\|p(\mathbf{x})\| \le C\|\mathbf{x}\|^m$ for some $m > 1$. Here $\|\mathbf{x}\| = \sqrt{\sum_{i=1}^n x_i^2}$ is the usual Euclidean norm.

   (a) Let $\mathbf{u} \in \mathbb{R}^n$, $\|\mathbf{u}\| = 1$. Recall that the definition of the directional derivative $D_{\mathbf{u}}(p(\mathbf{x}))$ at $\mathbf{0}$ is

$$\lim_{\epsilon\to 0} \frac{p(\epsilon\mathbf{u}) - p(\mathbf{0})}{\epsilon}.$$

Show that $D_{\mathbf{u}}(p(\mathbf{x}))$ at $\mathbf{0}$ equals 0, and explain why this implies the gradient of p at $\mathbf{0}$ equals 0, i.e. $\nabla p(\mathbf{0}) = 0$.

(b) Using part (a) above (even if you couldn't prove it), show the gradient (or Jacobian) of F at $\mathbf{x} = \mathbf{0}$ equals M, and use this to conclude that there is a neighborhood of $\mathbf{x} = \mathbf{0}$ such that F is invertible.

Solution:

(a) The directional derivative of p at zero satisfies

$$\nabla p(\mathbf{0}) \cdot \mathbf{u} = \lim_{\epsilon \to 0} \frac{p(\epsilon \mathbf{u})}{\epsilon}, \quad \|\mathbf{u}\| = 1,$$

since $p(\mathbf{0}) = 0$. This derivative is zero since

$$\|\nabla p(\mathbf{0}) \cdot \mathbf{u}\| = \lim_{\epsilon \to 0} \left\| \frac{p(\epsilon \mathbf{u})}{\epsilon \mathbf{u}} \right\| \leq C \lim_{\epsilon \to 0} \epsilon^{m-1} = 0,$$

and so $\nabla p(\mathbf{0}) = 0$ due to arbitrary $\mathbf{u}$.

(b) The gradient (or Jacobian) of F at $\mathbf{x} = \mathbf{0}$ satisfies

$$\nabla F(\mathbf{0}) = M + \nabla p(\mathbf{0}) = M.$$

Since $\nabla F(\mathbf{0}) = M$ is invertible and F is continuously differentiable, by the inverse function theorem, there is a neighborhood of $\mathbf{x} = \mathbf{0}$ such that F is invertible.