DISTRIBUTION OF THE SUCCESSIVE MINIMA OF THE PETERSSON NORM ON CUSP FORMS

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To my parents, Sanjoy and Sikha Purohit.

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Figure 1: Infty

ABSTRACT

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Given an arithmetic variety $\mathscr{X}$ and a hermitian line bundle $\overline{\mathscr{L}}$, the arithmetic Hilbert-Samuel theorem describes the asymptotic behavior of the co-volumes of the lattices $H^0(\mathscr{X}, \mathscr{L}^{\otimes k})$ in the normed spaces $H^0(\mathscr{X}, \mathscr{L}^{\otimes k}) \otimes \mathbb{R}$ as $k \to \infty$. Using his work on quasi-filtered graded algebras, Chen proved a variant of the arithmetic Hilbert-Samuel theorem which studies the asymptotic behavior of the successive minima of the lattices above. Chen's theorem, however, requires that the metric on $\overline{\mathscr{L}}$ is continuous, and hence does not apply to automorphic vector bundles for which the natural metrics are often singular. In this thesis, we discuss a version of Chen's theorem for the line bundle of modular forms for a finite index subgroup $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{Z})$ endowed with the logarithmically singular Petersson metric. This generalizes work of Chinburg, Guignard, and Soulé addressing the case $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$.

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CHAPTER 1

Background

1.1. Arithmetic Hilbert-Samuel theorems

Let X be a projective variety of dimension d over a field K, and L an ample line bundle on X. Classical intersection theory over a projective variety can be used to study the dimension of the spaces of global sections $H^0(X, L^{\otimes k})$. The *Hirzebruch-Riemann-Roch* theorem expresses the Euler characteristic $\chi(L^{\otimes k})$ of $L^{\otimes k}$ in terms of intersection theory, which can then be used to deduce the classical *Hilbert-Samuel theorem*

$$\lim_{k\to\infty} \frac{\chi(L^{\otimes k})}{k^d/d!} = c_1(L)^d,$$

where $c_1(L)$ denotes the first Chern class. Since L is ample, for k large enough, note that $\chi(L^{\otimes k}) = \dim_K H^0(X, L^{\otimes k})$.

An *arithmetic variety* is an integral scheme $\mathscr{X}$ equipped with a flat, projective morphism $\pi : \mathscr{X} \to \operatorname{Spec}(\mathbb{Z})$ with a smooth generic fiber $\mathscr{X}_{\mathbb{Q}}$. Classical intersection theory turns out to be inadequate over such varieties, since unlike $\operatorname{Spec}(K)$, $\operatorname{Spec}(\mathbb{Z})$ is not proper. The analog of the fact that the sum of the orders of vanishing of a meromorphic function is zero on a projective curve over a field is the product formula for number fields. This suggests that the archimedean place of $\mathbb{Q}$, which is not reflected by the points of $\operatorname{Spec}(\mathbb{Z})$, must also be taken into account in order to have a meaningful intersection theory over $\operatorname{Spec}(\mathbb{Z})$.

In his seminal papers (Arakelov, 1974, 1975), Arakelov proposed such an intersection theory for arithmetic surfaces. His work was generalized by Faltings (Faltings, 1984), Deligne (Deligne, 1987), and eventually to arbitrary dimensions (i.e. arbitrary arithmetic varieties) by Gillet and Soulé (Gillet and Soulé, 1990; Gillet and Soulé, 1992). The resulting theory is often called *arithmetic intersection theory*. The key, as alluded to above, is to keep track of the contribution to intersection theory coming from the archimedean places, which involves some analytic computations on the complex manifold $\mathscr{X}(\mathbb{C})$, in addition to the contributions from the usual "algebraic" intersection theory.

For arithmetic intersection theory, we work with *hermitian line bundles.* These are given by pairs $(\mathscr{L}, h)$ where $\mathscr{L}$ is a line bundle - i.e., an invertible sheaf - on the arithmetic variety $\mathscr{X}$, along with a hermitian metric h on the complex line bundle $\mathscr{L}_{\mathbb{C}}$ over the complex analytic manifold $\mathscr{X}(\mathbb{C})$ (note that $\mathscr{X}(\mathbb{C})$ is a smooth complex analytic manifold by our assumption that the generic fiber of $\mathscr{X}$ is smooth). This means that for each point $x \in \mathscr{X}(\mathbb{C})$, the fiber $\mathscr{L}_{\mathbb{C}}(x)$ is equipped with a hermitian inner product h_x that varies smoothly with x (we further require that these inner products are invariant under the action of complex conjugation on $\mathscr{X}(\mathbb{C})$, but we will not belabor this point). The idea of the hermitian inner product is to give an analog of "integral models" and "integral sections" over archimedean places. For brevity, we will denote a hermitian line bundle $(\mathscr{L}, h)$ by $\overline{\mathscr{L}}$.

Given a hermitian line bundle $\overline{\mathscr{L}}$, the space of complex global sections $H^0(\mathscr{X}_{\mathbb{C}}, \mathscr{L}_{\mathbb{C}}^{\otimes k})$ can be equipped with two natural norms (we do not distinguish between the complex projective variety $\mathscr{X}_{\mathbb{C}}$ with the complex analytic manifold $\mathscr{X}(\mathbb{C})$). Given $s \in H^0(\mathscr{X}_{\mathbb{C}}, \mathscr{L}_{\mathbb{C}}^{\otimes k})$,

1. we have the L^∞ (or sup) norm:

$$\|s\|_k := \sup_{x \in \mathscr{X}(\mathbb{C})} |s(x)|_x,$$

   where $s(x)$ is the image of s in the fiber $\mathscr{L}_{\mathbb{C}}(x)$, and $|s(x)|_x := \sqrt{h_x(s(x), s(x))}$;

2. and the L^2 norm:

$$\|s\|_k^2 := \int_{\mathscr{X}(\mathbb{C})} |s(x)|_x^2 \, d\mu,$$

   where $d\mu$ is the volume form associated to a Kähler metric on $\mathscr{X}(\mathbb{C})$ (invariant under complex conjugation).

The space of integral sections $H^0(\mathscr{X}, \mathscr{L}^{\otimes k})$ is a free $\mathbb{Z}$-module, and equipped with either of the above norms $\|\cdot\|_k$ on $H^0(\mathscr{X}, \mathscr{L}^{\otimes k}) \otimes_{\mathbb{Z}} \mathbb{C} = H^0(\mathscr{X}_{\mathbb{C}}, \mathscr{L}_{\mathbb{C}}^{\otimes k})$ becomes a *metrized vector* *bundle over* $\operatorname{Spec}(\mathbb{Z})$. If $\|\cdot\|_k$ is the L^2 norm, then $H^0(\mathscr{X}, \mathscr{L}^{\otimes k})$ is a *hermitian* vector bundle over $\operatorname{Spec}(\mathbb{Z})$. We then define the *arithmetic Euler characteristic*

$$\widehat{\chi}\left(H^0(\mathscr{X}, \mathscr{L}^{\otimes k}), \|\cdot\|_k\right) := \log \frac{\operatorname{vol}\left(\{v \in H^0(\mathscr{X}_{\mathbb{R}}, \mathscr{L}_{\mathbb{R}}^{\otimes k}) : \|v\|_k \leq 1\}\right)}{\operatorname{vol}\left(H^0(\mathscr{X}_{\mathbb{R}}, \mathscr{L}_{\mathbb{R}}^{\otimes k})/H^0(\mathscr{X}, \mathscr{L}^{\otimes k})\right)},$$

where $H^0(\mathscr{X}_{\mathbb{R}}, \mathscr{L}_{\mathbb{R}}^{\otimes k}) := H^0(\mathscr{X}, \mathscr{L}^{\otimes k}) \otimes \mathbb{R}$ is equipped with the restriction of the norm $\|\cdot\|_k$. Here, $\operatorname{vol}(\cdot)$ denotes any Haar measure on $H^0(\mathscr{X}_{\mathbb{R}}, \mathscr{L}_{\mathbb{R}}^{\otimes k})$.

If the arithmetic variety is of relative dimension d (i.e. if $d = \dim \mathscr{X}(\mathbb{C})$) then the *arithmetic Hilbert-Samuel theorem* says that

$$\lim_{k \to \infty} \frac{\widehat{\chi}\left(H^0(\mathscr{X}, \mathscr{L}^{\otimes k}), \|\cdot\|_k\right)}{k^{d+1}/(d+1)!} = \widehat{c}_1(\overline{\mathscr{L}})^{d+1},$$

under suitable (arithmetic) ampleness assumptions on $\overline{\mathscr{L}}$. Here $\widehat{c}_1(\overline{\mathscr{L}})$ denotes the *arithmetic first Chern class* associated to $\overline{\mathscr{L}}$, and $\widehat{c}_1(\overline{\mathscr{L}})^{d+1}$ denotes the intersection product in the *arithmetic Chow ring* as defined by Gillet and Soulé (followed by a pushforward to $\widehat{\mathrm{CH}}^1(\operatorname{Spec}(\mathbb{Z})) \cong \mathbb{R}$, the first arithmetic Chow group of $\operatorname{Spec}(\mathbb{Z})$).

The arithmetic Hilbert-Samuel theorem was first proved by Gillet and Soulé in (Gillet and Soulé, 1992) as a consequence of their Riemann-Roch theorem. Abbes and Bouche (Abbes and Bouche, 1995), gave a simpler proof without using intersection theory. Then various generalizations were proved by many people, some of which are (Zhang, 1995), (Randriambololona, 2006), (Yuan, 2008, 2009), (Chen, 2010), and (Chen and Moriwaki, 2022). There are also adelic generalizations in the works of Lau-Rumely-Varley (Rumely et al., 2000) and Chinburg-Lau-Rumely (Chinburg et al., 2003). The interest in this theorem stems from its various applications to arithmetic geometry: in particular, Vojta's proof of Mordell's conjecture, and Zhang's proof of the generalized Bogomolov conjecture.

1.2. Successive minima and Chen's version of arithmetic Hilbert-Samuel

In this thesis, we will focus on one of Chen's versions of arithmetic Hilbert-Samuel theorem. Before describing Chen's work, we recall one of Minkowski's theorems about successive minima. Suppose $(V, \|\cdot\|)$ is a normed $\mathbb{R}$-vector space of dimension n, and $\Gamma \subseteq V$ is a lattice (free $\mathbb{Z}$-module of rank n). Let $B_n := \{v : \|v\| \leq 1\}$ denote the closed unit ball. For $i = 1, \ldots, n$, then *ith successive minima*, denoted μ_i, is defined to be the infimum over all real numbers μ with the property that $\Gamma \cap \mu B_n$ contains at least i linearly independent elements. So, for instance, μ_1 is the length of the shortest non-zero lattice point. We remark that these successive minima are in general very difficult to compute. In fact, there are proposed post-quantum cryptography algorithms based on the difficulty of finding the shortest non-zero lattice vectors.

Minkowski's theorem on successive minima states that for any Haar measure $\mathrm{vol}(\cdot)$ on V,

$$\frac{2^n}{n!} \leq \mu_1\mu_2 \ldots \mu_n \, \frac{\mathrm{vol}(B_n)}{\mathrm{vol}(V/\Gamma)} \leq 2^n.$$

Applying $-\log(\cdot)$ to the above inequality, we get

$$\sum_{i=1}^{n} -\log(\mu_i) = \widehat{\chi}\,(\Gamma, \|\cdot\|) + O(n\log(n)).$$

This suggests that from the point of view of Arakelov theory, the quantities $-\log(\mu_i)$ are more natural. These are called the *successive maxima* of the (normed) lattice, which we will denote by $\lambda_i := -\log(\mu_i)$.

While the previous versions of the arithmetic Hilbert-Samuel theorems describe the asymptotic behavior of the $\widehat{\chi}\,(\Gamma, \|\cdot\|)$ over a family of lattices, Chen's version, roughly speaking, can be thought of as describing the asymptotic behavior of the summands λ_i of the Euler-characteristic (normalized appropriately).

More precisely, Chen's theorem states the following. Let $\mathscr{X}$ be an arithmetic variety, and let

$\overline{\mathscr{L}}$ be a smooth hermitian line bundle on $\mathscr{X}$ such that $\mathscr{L}_{\mathbb{Q}}$ is ample (in fact, it suffices for the metric on $\overline{\mathscr{L}}$ to be continuous - meaning that the hermitian norms vary *continuously* over the fibers). Equip $H^0(\mathscr{X}_{\mathbb{C}}, \mathscr{L}_{\mathbb{C}}^{\otimes k})$ with either the L^∞ or the L^2 norm, $\|\cdot\|_k$. Suppose $d_k := \mathrm{rank}_{\mathbb{Z}}\, H^0(\mathscr{X}, \mathscr{L}^{\otimes k})$, and suppose $\lambda_{k,i}$ are the successive maxima of the lattice $H^0(\mathscr{X}, \mathscr{L}^{\otimes k})$ in the normed space $H^0(\mathscr{X}_{\mathbb{R}}, \mathscr{L}_{\mathbb{R}}^{\otimes k})$. Let δ_x denote the Dirac-delta function supported on the point $x \in \mathbb{R}$, and let

$$\nu_k := \frac{1}{d_k} \sum_{i=1}^{d_k} \delta_{\frac{1}{k}\lambda_{k,i}}$$

denote the discrete probability measure on $\mathbb{R}$ supported on the normalized successive maxima $\frac{1}{k}\lambda_{k,i}$. Then Chen's theorem 4.1.8 in Chen (2010) implies

Theorem 1.2.1 (Chen)**.** *The sequence of discrete probability measures ν_k converges weakly to a probability measure ν on $\mathbb{R}$ with compact support.*

1.3. Singular metrics

The metrics we have considered thus far are smooth (or continuous). It turns out, however, that many natural metrics of arithmetic interest are not smooth or continuous - in fact, they are not even defined at every point $x \in \mathscr{X}(\mathbb{C})$. Automorphic vector bundles on Shimura varieties provide a rich source of examples of such metrics. Mumford studied one such class of such metrics in (Mumford, 1977), which he called *good metrics*. These metrics are examples of metrics with *logarithmic singularities*. Mumford showed that the smooth metrics on a certain class of automorphic vector bundles can always be extended to a toroidal compactification of the Shimura variety in question, and that the extended metric is good.

Good metrics turn out to be inadequate for developing an analog of arithmetic intersection theory for singular line bundles. Kühn (Kühn, 2001), and Burgos Gil-Kramer-Kühn (Burgos Gil et al., 2007, 2005), define a more general notion of singular metrics that they use to develop a generalized arithmetic intersection theory suitable for working with hermitian line bundles with singular metrics, generalizing many results of Gillet and Soulé to the singular setting.

Working with (a generalization of) this singular setting, Berman and Montplet (Berman and Montplet, 2012) proved an arithmetic Hilbert-Samuel theorem for singular hermitian line bundles in adjoint form. Let $\mathscr{L}$ be a suitably singular hermitian line bundle (for example, with logarithmic singularities, but can be more general) on an arithmetic variety $\mathscr{X}$ of relative dimension d. The line bundles $\mathscr{L}^{\otimes k} \otimes \mathscr{K}$ are said to be in adjoint form, where $\mathscr{K}$ is an integral model of the canonical bundle of $\mathscr{X}_{\mathbb{Q}}$. The space of global sections $H^0(\mathscr{X}_{\mathbb{C}}, \mathscr{L}_{\mathbb{C}}^{\otimes k} \otimes \mathscr{K}_{\mathbb{C}})$ comes equipped with a natural L^2 hermitian inner product, call it $\|\cdot\|_k$. Then theorem 1.1 (Berman and Montplet, 2012) shows that the limit

$$\lim_{k\to\infty} \frac{(d+1)!}{k^{d+1}} \widehat{\chi}\left(H^0(\mathscr{X}, \mathscr{L}^{\otimes k} \otimes \mathscr{K}), \|\cdot\|_k\right)$$

exists, and is equal to a (generalized) arithmetic intersection number (under suitable ampleness hypotheses on $\mathscr{L}$). The same paper also shows a similar, more general, result if we restrict to line bundles with logarithmic singularities along a normal crossings divisor $D \subseteq \mathscr{X}_{\mathbb{Q}}$ (theorem 1.2).

1.4. Analog of Chen's theorem for singular metrics?

Motivated by all of this, we ask the following general question. Let $\mathscr{X}$ be an arithmetic variety, $\overline{\mathscr{L}}$ a hermitian line bundle with logarithmic singularities along a normal crossings divisor $D \subseteq \mathscr{X}_{\mathbb{Q}}$. The log-singularity assumption makes it so that the space of global sections $H^0(\mathscr{X}_{\mathbb{C}}, \mathscr{L}_{\mathbb{C}}^{\otimes k}(-D_{\mathbb{C}}))$ can be endowed with either the L^∞ or L^2 norm, $\|\cdot\|_k$ (for the L^2 norm, we must choose a Kähler metric on $\mathscr{X}(\mathbb{C})$ invariant under complex conjugation, as before). We have the lattice

$$H^0(\mathscr{X}, \mathscr{L}^{\otimes k}) \cap H^0(\mathscr{X}_{\mathbb{Q}}, \mathscr{L}_{\mathbb{Q}}^{\otimes k}(-D))$$

in the normed $\mathbb{R}$-vector space $H^0(\mathscr{X}_{\mathbb{R}}, \mathscr{L}_{\mathbb{R}}^{\otimes k}(-D_{\mathbb{R}}))$. Let $\lambda_{k,i}$ denote the successive maxima, and let

$$\nu_k := \frac{1}{d_k} \sum_{i=1}^{d_k} \delta_{\frac{1}{k}\lambda_{k,i}}$$

denote the discrete probability measure on $\mathbb{R}$ associated to the normalized successive maxima.

Problem 1.4.1. Do the ν_k converge weakly to a probability measure? If so, what can we say about the support of the limit measure?

In this thesis, we address this problem in the case $\mathscr{X}$ is an *arithmetic surface associated to (a finite index subgroup)* $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{Z})$, $D \subseteq \mathscr{X}_{\mathbb{Q}}$ is the divisor of cusps, and $\overline{\mathscr{L}}$ is the hermitian line bundle associated to the modular forms of weight 12, equipped with the Petersson metric, which turns out to have logarithmic singularities along D (as well as at the elliptic points). See chapter 3 for descriptions of these objects, and theorems 2.0.1 and 2.0.2 for the precise statements.

This thesis is motivated by work of Chinburg-Guignard-Soulé (Chinburg et al., 2018), in which they addressed the case $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$.

1.5. Outline of this thesis

In chapter 2, we give precise statements of our main results (theorems 2.0.1 and 2.0.2), and discuss some key differences between our work and those of (Chinburg et al., 2018) and (Chen, 2010).

In chapter 3, we start with some background on complex modular forms and modular curves. Then, following (Kühn, 2001), we define integral models for modular curves and the line bundle of modular forms in §3.3, and describe the Petersson metric on this bundle in §3.4. We then introduce the notion of adelic vector bundles, as in definition 3.1 of (Gaudron, 2008), their successive maxima, and apply these concepts to the space of rational cusp forms of weight $12k$ for finite index subgroups of $\mathrm{PSL}_2(\mathbb{Z})$. This lets us define a decreasing $\mathbb{R}$-filtration on each S_k, which enables us to use results of Chen on quasi-filtered algebras in order to prove our results.

In chapter 4, we prove theorem 2.0.1. In §4.1, we prove a uniform upper bound on normalized successive maxima using (generalized) intersection theory of Kühn and Bost-Gillet-Soulé, which is then used in §4.2 along with Chen's theorem on quasi-filtered algebras to deduce *vague* convergence of measures (for the sequence ν_k as in problem 1.4.1). Then in §4.3 we show that the vague convergence is *weak* by doing explicit calculations with bases of cusp forms of large weights.

Finally, in chapter 5, we prove theorem 2.0.2. After setting up some notation in §5.1, we prove in §5.2 some results comparing the $\mathbb{R}$-filtrations on various spaces of cusp forms. These comparison results are used in §5.3 to prove theorem 2.0.2. Finally, in corollary 5.3.1, we prove that the support of the limit measure from theorem 2.0.1 is unbounded below.

CHAPTER 2

Statements of main results

We follow the setup in §4.11 of (Kühn, 2001). Let $\Gamma(1) := \mathrm{PSL}_2(\mathbb{Z})$, and identify the complex modular curve $X(1)_{\mathbb{C}}$ associated to $\Gamma(1)$ with $\mathbb{P}^1_{\mathbb{C}}$ via the modular j-function (see example 3.1.1). Let $\Gamma \subseteq \Gamma(1)$ be a finite index subgroup. The modular curve $X(\Gamma)_{\mathbb{C}}$ and the natural map $\pi_{\Gamma,\mathbb{C}} : X(\Gamma)_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ are defined over a number field E (see §3.3). Let $X(\Gamma)$ be a (smooth projective geometrically connected) model of $X(\Gamma)_{\mathbb{C}}$ over E, let $\pi_{\Gamma,E} : X(\Gamma) \to \mathbb{P}^1_E$ be the model of $\pi_{\Gamma,\mathbb{C}}$ over E, and let $\pi_\Gamma : X(\Gamma) \to \mathbb{P}^1_{\mathbb{Q}}$ denote the composition of $\pi_{\Gamma,E}$ with the natural map from $\mathbb{P}^1_E$ to $\mathbb{P}^1_{\mathbb{Q}}$. Let $\infty \in \mathbb{P}^1_{\mathbb{Q}}$ correspond to the unique pole of the j-function, and let $D \in \mathrm{Div}(X(\Gamma))$ denote the sum of the points in $\pi_\Gamma^{-1}(\infty)$. We call D the *divisor of cusps of* $X(\Gamma)$.

Let $\mathscr{X}(\Gamma)$ be *an arithmetic surface associated to* Γ (see §3.3 and §4.11 of (Kühn, 2001)). Then $\mathscr{X}(\Gamma)$ is a regular projective arithmetic surface with generic fiber $X(\Gamma)$, and comes with a morphism $\pi_{\Gamma,\mathbb{Z}} : \mathscr{X}(\Gamma) \to \mathbb{P}^1_{\mathbb{Z}}$ extending π_Γ. Let $\overline{\mathscr{L}}$ denote the metrized line bundle on $\mathscr{X}(\Gamma)$ associated to modular forms of level Γ and weight 12 endowed with the Petersson metric (see §3.3 and §3.4, and §4.12 of (Kühn, 2001)). As mentioned before, this metric has logarithmic singularities at elliptic points and cusps (proposition 4.9 in (Kühn, 2001)).

Theorem 2.0.1. *Let $\Gamma \subseteq \Gamma(1)$ be a finite index subgroup. Let $\mathscr{X}(\Gamma)$ be an arithmetic surface associated to Γ, $X(\Gamma)$ its generic fiber, and let D denote the divisor of cusps of $X(\Gamma)$. Let $\overline{\mathscr{L}}$ be the line bundle on $\mathscr{X}(\Gamma)$ associated to modular forms of level Γ and weight 12 endowed with the Petersson metric. For every $k \geq 1$, let*

$$\mathscr{S}_k := H^0(\mathscr{X}(\Gamma), \mathscr{L}^{\otimes k}) \cap H^0(X(\Gamma), \mathscr{L}_{\mathbb{Q}}^{\otimes k}(-D))$$

denote the euclidean lattice of integral cusp forms of level Γ and weight $12k$ with respect to the Petersson inner product. Let $\mu_{k,i}$ denote the successive minima of $\mathscr{S}_k$, and let $\lambda_{k,i} :=$

$-\log(\mu_{k,i})$ *denote the successive maxima. Let* $d_k := \mathrm{rk}_{\mathbb{Z}}\mathscr{S}_k$*, and let*

$$\nu_k := \frac{1}{d_k}\sum_{i=1}^{d_k}\delta_{\frac{1}{k}\lambda_{k,i}}$$

denote the probability measure on $\mathbb{R}$ *associated to the normalized successive maxima of* $\mathscr{S}_k$*. Then the* ν_k *converge weakly to a Borel probability measure* ν *on* $\mathbb{R}$*.* ν *has support bounded above and unbounded below.*

A natural question is: given finite index subgroups $\Gamma' \subseteq \Gamma$ of $\Gamma(1)$, how do the successive maxima of their integral cusp forms, and the associated limit measures compare? This is addressed by the following theorem.

Theorem 2.0.2. *Let* $\Gamma' \subseteq \Gamma$ *be finite index subgroups of* $\Gamma(1)$*. Let* $\mathscr{X}(\Gamma')$ *(resp.* $\mathscr{X}(\Gamma)$*) be an arithmetic surface associated to* Γ' *(resp.* Γ*) with generic fiber* $X(\Gamma')$ *(resp.* $X(\Gamma)$*). Suppose there is a* $\mathbb{Q}$*-morphism* $\pi_{\Gamma',\Gamma} : X(\Gamma') \to X(\Gamma)$ *that is a model over* $\mathbb{Q}$ *of the natural map* $X(\Gamma')_{\mathbb{C}} \to X(\Gamma)_{\mathbb{C}}$ *of complex modular curves.*

Let ν'_k *(resp.* ν_k*) denote the probability measure on* $\mathbb{R}$ *associated to the normalized successive maxima of the euclidean lattice of integral cusp forms of level* Γ' *(resp.* Γ*) and weight* $12k$ *with respect to* $\mathscr{X}(\Gamma')$ *(resp.* $\mathscr{X}(\Gamma)$*) as in theorem 2.0.1. Suppose that* $\nu'_k \to \nu'$ *(resp.* $\nu_k \to \nu$*) weakly for a Borel probability measure* ν' *(resp.* ν*) on* $\mathbb{R}$ *(by theorem 2.0.1). Then there is a Borel probability measure* ω *on* $\mathbb{R}$ *such that*

$$\nu' = \frac{1}{\deg(\pi_{\Gamma',\Gamma})}\cdot\nu + \left(1 - \frac{1}{\deg(\pi_{\Gamma',\Gamma})}\right)\cdot\omega.$$

2.1. Remarks

Remark 2.1.1. Theorem 2.0.1 generalizes theorem 3.2.2 (i), (ii) of (Chinburg et al., 2018), which addresses the case $\Gamma = \Gamma(1)$ for the associated arithmetic surface $\mathscr{X}(1) = \mathbb{P}^1_{\mathbb{Z}}$ (identification coming from the j-function). We remark that our overall approach is similar to that of (Chinburg et al., 2018), with a couple of key differences. First, the approach in

(Chinburg et al., 2018) uses various properties of q-expansions of modular forms for $\Gamma(1)$, including integrality of the coefficients of q-expansions of modular forms over the integral modular curve $\mathscr{X}(1)$ to obtain (lower) bounds on the Petersson norms of integral cusp forms (see for instance, proposition 3.3.1, lemma 3.3.1, theorem 3.4.1, and lemma 3.4.2 in (Chinburg et al., 2018)). For a general (in particular, non-congruence) finite index subgroup $\Gamma \subseteq \Gamma(1)$, we do not have access to such integrality properties for the q-expansion coefficients of integral modular forms.

Instead, following (Chen, 2010), we use results from the intersection theory of logarithmically singular line bundles on arithmetic surfaces as developed by Kühn in (Kühn, 2001), along with height formulas developed by Bost, Gillet, and Soulé in (Bost et al., 1994) to obtain corresponding (lower) bounds for the sup norms of integral cusp forms (see proposition 4.1.3, which is the counterpart to lemma 3.4.2 in (Chinburg et al., 2018)). We then appeal to a "Gromov's lemma" type result in our log-singular setting (see proposition 4.1.2, page 1 of (Friedman et al., 2013), and theorem 1.7 in (Auvray et al., 2016)), comparing sup norms to L^2 norms for sections in increasing powers of the line bundle of modular forms, to get analogous (lower) bounds on the L^2 (i.e. Petersson) norms of integral cusp forms for Γ. The estimates obtained from the intersection theory approach, however, are not as sharp as those obtained from the q-expansions, and this ultimately prevents us from showing directly that the support of the limit measure ν in theorem 2.0.1 is unbounded below. Instead, we deduce unboundedness (in corollary 5.3.1) from theorem 2.0.2 along with the unboundedness in the case $\Gamma = \Gamma(1)$ proved in part (ii) of theorem 3.2.2 of (Chinburg et al., 2018).

Another key difference with (Chinburg et al., 2018) is that, unlike the $\Gamma = \Gamma(1)$ case considered there, it is difficult to write down explicit bases for the spaces of cusp forms for general Γ, which are then used for getting bounds on the successive maxima. In proposition 4.3.1, we construct less explicit bases that nevertheless have the same general shape as those in (Chinburg et al., 2018) theorem 3.4.1 and lemma 3.4.4, and that turns out to be enough to yield the desired bounds on the successive maxima in our case.

Remark 2.1.2. An important feature of theorem 2.0.1 and theorem 3.2.2 of (Chinburg et al., 2018) is the fact that the limit measures have support unbounded below, in stark contrast to Chen's theorem for smooth or continuous metrics mentioned before, where the limit measure has compact support. This is explained, very roughly, as follows. When $\mathscr{L}$ has continuous metric, then at least if $\mathscr{L}_{\mathbb{Q}}$ is ample,

$$\bigoplus_{k\geq 0} H^0(\mathscr{X}_{\mathbb{Q}}, \mathscr{L}_{\mathbb{Q}}^{\otimes k})$$

is a finitely generated quasi-filtered algebra over $\mathbb{Q}$ (in the sense of definition 3.2.1 of (Chen, 2010)). The convergence result for the measures then follows from a more general result of Chen (theorem 3.4.3 (Chen, 2010)) concerning the distribution of successive maxima for such algebras, which shows, in particular, that the limiting distribution has compact support.

In our setting, and that of (Chinburg et al., 2018), the Petersson metric on the line bundle $\mathscr{L}$ of modular forms is logarithmically singular, and hence to make sense of the norms at the archimedean places, we restricted to the subspace $H^0(\mathscr{X}_{\mathbb{Q}}, \mathscr{L}_{\mathbb{Q}}^{\otimes k}(-D))$ of sections vanishing along the cusps. The graded $\mathbb{Q}$-algebra

$$B := \bigoplus_{k\geq 0} H^0(\mathscr{X}_{\mathbb{Q}}, \mathscr{L}_{\mathbb{Q}}^{\otimes k}(-D))$$

is not finitely generated. However, we can "approximate" B by a sequence of finite-type quasi-filtered $\mathbb{Q}$ algebras B_L indexed by integers L (as in §4.2, and §3.6 of (Chinburg et al., 2018)). Then applying theorem 3.4.3 of (Chen, 2010) to each B_L, we get a sequence of compactly supported measures $\nu_{L,\infty}$, where the support of each $\nu_{L,\infty}$ is bounded above by a constant independent of L, but the lower bound for the support goes to $-\infty$ as $L \to \infty$. We then show that the $\nu_{L,\infty}$ converge to ν as $L \to \infty$, which accounts for the support of ν being unbounded below.

CHAPTER 3

Background on integral modular forms

In this section, we start with some background on complex modular forms and modular curves. Then, following (Kühn, 2001), we define integral models for modular curves and the line bundle of modular forms in §3.3, and describe the Petersson metric on this bundle in §3.4. We then introduce the notion of adelic vector bundles, as in definition 3.1 of (Gaudron, 2008), their successive maxima, and apply these concepts to the space of rational cusp forms of weight $12k$ for finite index subgroups of $\mathrm{PSL}_2(\mathbb{Z})$.

3.1. Modular curves and modular forms over $\mathbb{C}$

Let Γ be a finite index subgroup of $\Gamma(1) := \mathrm{PSL}_2(\mathbb{Z})$. Then Γ acts on the (complex) extended upper half plane $\mathfrak{h}^* := \mathfrak{h} \cup \mathbb{P}^1(\mathbb{Q})$ by linear fractional transformations

$$\Gamma \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}.$$

The quotient

$$X(\Gamma)_{\mathbb{C}} := \Gamma \backslash (\mathfrak{h} \cup \mathbb{P}^1(\mathbb{Q})).$$

is a compact Riemann surface called the *modular curve associated to* Γ. The inclusion $\Gamma \subseteq \Gamma(1)$ induces a holomorphic map $\pi_{\Gamma,\mathbb{C}} : X(\Gamma)_{\mathbb{C}} \to X(\Gamma(1))_{\mathbb{C}}$ of Riemann surfaces of degree $[\Gamma(1) : \Gamma]$.

For a point $z \in \mathfrak{h}^*$, let

$$\Gamma_z := \{\tau \in \Gamma : \tau \cdot z = z\}$$

denote the stabilizer of z in Γ. This group is either trivial, finite, or infinite cyclic, and we call the image of z in $X(\Gamma)_{\mathbb{C}}$ an ordinary point, elliptic point, or cuspidal point (or just a cusp) in these cases, respectively. For $\Gamma(1)$, the elliptic points are the images of i and $e^{2\pi i/3}$ and there is a unique cusp corresponding to the image of (any) point of $\mathbb{P}^1(\mathbb{Q})$.

For a general Γ, the elliptic points and cusps of $X(\Gamma)_{\mathbb{C}}$ map to elliptic points and cusp of $X(\Gamma(1))_{\mathbb{C}}$, respectively.

Given $\gamma \in \mathrm{PSL}_2(\mathbb{R})$, the *weight k slash operator* $(\cdot|_k\gamma)$ on holomorphic functions on $\mathfrak{h}$ is given by

$$f(z) \mapsto (f|_k\gamma)(z) := (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right)$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. For $\Gamma \subseteq \Gamma(1)$, a *meromorphic modular form of level* Γ *and weight* k is a meromorphic function $f : \mathfrak{h} \to \mathbb{C}$ such that

(a) $(f|_k\gamma)(z) = f(z)$ for all $\gamma \in \Gamma$, and

(b) f is meromorphic at the cusps of Γ. This means that at each cusp, if t denotes a local parameter, then f admits a Fourier expansion of the form $f(t) = \sum_n a_n t^n$, with $a_n = 0$ for sufficiently small n.

Meromorphic modular forms of weight 0 are called *modular functions* - these are simply the rational functions of $X(\Gamma)_{\mathbb{C}}$. A *modular form of weight k for* Γ is a meromorphic modular form that is holomorphic everywhere, including at the cusps. A *cusp form of weight k for* Γ is a modular form that is zero at every cusp (i.e., the a_0 coefficient in the local Fourier expansion at every cusp is zero). The space of modular (resp. cusp) forms of level Γ and weight k is denoted by $M_k(\Gamma)_{\mathbb{C}}$ (resp. $S_k(\Gamma)_{\mathbb{C}}$) - these are finite dimensional vector spaces over $\mathbb{C}$.

Example 3.1.1. (i) The *classical j-invariant function* is a modular function for $\Gamma(1)$ given by

$$j(z) = \frac{1}{q} + 744 + 196884q + \ldots,$$

where $q = e^{2\pi i z}$ is the local parameter at the unique cusp S_∞ of $X(1)_{\mathbb{C}} := X(\Gamma(1))_{\mathbb{C}}$. Note that j has a unique pole of order 1 at S_∞, so that the induced morphism $j : X(1)_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ is an isomorphism. Henceforth, we identify $X(1)_{\mathbb{C}}$ with $\mathbb{P}^1_{\mathbb{C}}$ via j.

(ii) The *classical discriminant function* Δ is a cusp form of weight 12 for $\Gamma(1)$ given by

$$\Delta(z) = q\prod_{n=1}^{\infty}(1-q^n)^{24},$$

where as before, $q = e^{2\pi i z}$.

The functional equations satisfied by meromorphic modular forms show that they are sections of a line bundle $\mathfrak{M}_k(\Gamma)_{\mathbb{C}}$ on $X(\Gamma)_{\mathbb{C}}$, and the meromorphic modular forms that are zero at every cusp form a sub-bundle $\mathfrak{S}_k(\Gamma)_{\mathbb{C}}$. The following is proposition 4.7 in (Kühn, 2001):

Proposition 3.1.2. *Let $\Gamma \subseteq \Gamma(1)$ be a finite index subgroup, and let $k \geq 1$ be a positive integer. Let $S_1, \ldots, S_t \in X(\Gamma)_{\mathbb{C}}$ denote the cusps, and let $\pi_{\Gamma,\mathbb{C}} : X(\Gamma)_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ denote the canonical map. Let $D_{\mathbb{C}} := S_1 + \cdots + S_t \in \mathrm{Div}(X(\Gamma)_{\mathbb{C}})$. Then we have an isomorphism of line bundles:*

$$\mathfrak{M}_{12k}(\Gamma)_{\mathbb{C}} \to \pi^*_{\Gamma,\mathbb{C}}\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(\infty)^{\otimes k} : f \mapsto f/\Delta^k,$$

which restricts to an isomorphism

$$\mathfrak{S}_{12k}(\Gamma)_{\mathbb{C}} \to \left(\pi^*_{\Gamma,\mathbb{C}}\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(\infty)^{\otimes k}\right)(-D_{\mathbb{C}}).$$

3.2. Petersson metric on complex modular forms

The *Petersson metric* on $\mathfrak{M}_{12k}(\Gamma)$ is defined as follows. Given a section f of $\mathfrak{M}_{12k}(\Gamma)$ over the open subset $U \subseteq X(\Gamma)_{\mathbb{C}}$ and a point $z \in \mathfrak{h}$ corresponding to a point of U, we set

$$|f|_{\mathrm{Pet}}(z)^2 := |f(z)|^2(4\pi\,\mathrm{im}(z))^{12k}. \tag{3.2.1}$$

This metric has *logarithmic singularities along the set of elliptic points and cusps of* Γ in the sense of definition 3.1 of (Kühn, 2001), which we now recall. Let X be a compact Riemann surface, $S = \{S_1, \ldots, S_r\}$ a finite set of points of X, and $Y := X \setminus S$. A hermitian line bundle $\overline{L} = (L, \|\cdot\|)$ on X is said to be *logarithmically singular along* S if h is smooth on

Y, and for every i, in a local coordinate z centered at S_i, and for any meromorphic section ℓ of L, we have

$$\|\ell(z)\| = -\log(|z|^2)^\alpha \cdot |z|^{\mathrm{ord}_{S_j}(\ell)} \cdot \varphi(z), \tag{3.2.2}$$

for some real number α, and a positive continuous function $\varphi(z)$, smooth away from S_i, such that the first and second (mixed) partials of φ are bounded by certain powers of $1/|z|$ away from S_i. α, φ, the powers of $1/|z|$, etc. all depend on $\overline{L}, S_i$, and ℓ. The bounds on the second partials ensure local integrability of the first Chern form associated to the metric. We remark that in Mumford's definition of a good metric (Mumford, 1977), φ is smooth on the whole coordinate neighborhood around S_i.

Proposition 4.9 of (Kühn, 2001) shows that the Petersson metric is logarithmically singular along the elliptic points and cusps. Briefly, a local chart around the cusp ∞ is given by $z \mapsto e^{2\pi i z/w} =: q$, where $z \in \mathfrak{h}$, and w is the width of ∞ in Γ. In the coordinate q, the (square of the) Petersson metric is given by

$$|f|_{\mathrm{Pet}}(q)^2 = |f(q)|^2 \cdot (-w \log|q|^2)^{12k}.$$

Any other cusp S_i can be brought to ∞ by an element of $\Gamma(1)$, so the local computation at S_i reduces to that around ∞.

Meanwhile, a local chart around an elliptic point that is the image of $z_0 \in \mathfrak{h}$ is given by

$$z \mapsto \left(\frac{z - z_0}{z - \overline{z_0}}\right)^n =: t,$$

where $n := |\Gamma_{\tau_0}|$. The (square of the) Petersson metric in the local coordinate t is given by

$$|f|_{\mathrm{Pet}}(t)^2 = |f(t^{1/n})|^2 \cdot \left(\frac{1 - |t^{1/n}|^2}{|1 - t^{1/n}|^2} \cdot 4\pi \mathrm{im}(\tau_0)\right)^{12k}.$$

(The f on the right side above denotes the function on the unit disk induced by the modular form f under the isomorphism $z \mapsto (z - z_0)/(z - \overline{z_0})$.)

By comparing both of these local expressions with (3.2.2), we see that the Petersson metric is, indeed, logarithmically singular at the cusps and elliptic points.

Remark 3.2.1. The Petersson metric defined above in equation (3.2.1) differs from the classical Petersson metric by a factor of $(4\pi)^{12k}$. The reason for this factor is roughly as follows - see (Kühn, 2001) p. 227-228 for more information. Suppose Γ is a congruence group such that the moduli functor representing "elliptic curves with Γ-level structure" is representable, and let $\mathscr{X}$ be the compactified moduli space as in (Deligne and Rapoport, 1973). Let $v : \mathscr{E} \to \mathscr{X}$ denote the universal elliptic curve, and let $e : \mathscr{X} \to \mathscr{E}$ denote the zero section. The sheaf

$$\omega_{\mathscr{E}/\mathscr{X}} := e^*\Omega^1_{\mathscr{E}/\mathscr{X}}$$

has a natural hermitian metric given as follows. For a complex point $x \in \mathscr{U}(\mathbb{C}) \subseteq \mathscr{X}(\mathbb{C})$, where $\mathscr{U} \subseteq \mathscr{X}$ is the dense open subscheme representing elliptic curves (rather than generalized elliptic curves), we have a canonical isomorphism

$$\omega_{\mathscr{E}/\mathscr{X}}(x) \cong H^0(\mathscr{E}_x, \Omega^1_{\mathscr{E}_x}),$$

where $\omega_{\mathscr{E}/\mathscr{X}}(x)$ (resp. $\mathscr{E}_x$, and $\Omega^1_{\mathscr{E}_x}$) denotes the pullback of $\omega_{\mathscr{E}/\mathscr{X}}$ (resp. $\mathscr{E}$, and $\Omega^1_{\mathscr{E}/\mathscr{X}}$) by $x : \mathrm{Spec}(\mathbb{C}) \to \mathscr{X}$. Given $\alpha \in H^0(\mathscr{E}_x, \Omega^1_{\mathscr{E}_x})$, define

$$|\alpha|_{L^2}(x)^2 := \frac{i}{2\pi}\int_{\mathscr{E}_x} \alpha \wedge \overline{\alpha}.$$

This metric, defined above over $\mathscr{U}$, extends to a logarithmically singular metric on $\omega_{\mathscr{E}/\mathscr{X}}$ over $\mathscr{X}$. It turns out that with the added $(4\pi)^{12k}$ factor in the Petersson metric, there is an isometry of line bundles

$$\left(\omega^{\otimes 12k}_{\mathscr{E}/\mathscr{X}}, |\cdot|_{L^2}\right) \cong \left(\mathfrak{M}_{12k}(\Gamma), |\cdot|_{\mathrm{Pet}}\right),$$

where $\omega^{\otimes 12k}_{\mathscr{E}/\mathscr{X}}$ is given the product metric.

Petersson inner product of cusp forms

Let $d_\Gamma := [\Gamma(1) : \Gamma]$, and let $\mathcal{F}_\Gamma$ denote a fundamental domain for the action of Γ on $\mathfrak{h}$. Define the *Petersson inner product* on $S_{12k}(\Gamma)_\mathbb{C}$ by

$$\langle f, g\rangle_{\mathrm{Pet}} := \frac{1}{d_\Gamma}\int_{\mathcal{F}_\Gamma} f(z)\overline{g(z)}(4\pi\,\mathrm{im}(z))^{12k}\,\frac{dx\,dy}{y^2} \qquad (f, g \in S_{12k}(\Gamma)_\mathbb{C}),$$

where $z = x + iy \in \mathfrak{h}$. This is a Hermitian inner product on $S_{12k}(\Gamma)_\mathbb{C}$, and the norm of $f \in S_{12k}(\Gamma)_\mathbb{C}$ is given by $\|f\|_{\mathrm{Pet}} := \langle f, f\rangle_{\mathrm{Pet}}^{1/2}$, which we note is simply the L^2 norm of the Petersson metric (equation (3.2.1)) with respect to the hyperbolic volume form $d\mu_\Gamma$ on $X(\Gamma)_\mathbb{C}$:

$$\|f\|^2_{\mathrm{Pet}} = \frac{1}{d_\Gamma}\int_{X(\Gamma)} |f|_{\mathrm{Pet}}(z)^2\,d\mu_\Gamma.$$

3.3. Integral models of modular curves and modular forms

We follow the setup in §4.11 of (Kühn, 2001) (with slightly modified notation). Let $\Gamma \subseteq \Gamma(1)$ be a finite index subgroup, and let $\pi_{\Gamma,\mathbb{C}} : X(\Gamma)_\mathbb{C} \to \mathbb{P}^1_\mathbb{C}$ denote the natural map given by the j-function. The branch points of $\pi_{\Gamma,\mathbb{C}}$ are contained in $\{0, 1728, \infty\} \subseteq \mathbb{P}^1_\mathbb{C}$, and hence $X(\Gamma)_\mathbb{C}$ and $\pi_{\Gamma,\mathbb{C}}$ are defined over a number field E. For such an E, let $X(\Gamma)_E$ be a smooth projective geometrically connected curve over E with base change to $\mathbb{C}$ (via the implicit embedding of E into $\mathbb{C}$) isomorphic to $X(\Gamma)_\mathbb{C}$, and let $\pi_{\Gamma,E} : X(\Gamma)_E \to \mathbb{P}^1_E$ be an E-morphism such that its base change to $\mathbb{C}$ (again, via the implicit embedding) coincides with $\pi_{\Gamma,\mathbb{C}}$. To simplify notation, we will refer to $X(\Gamma)_E$ by $X(\Gamma)$. Let $\pi_\Gamma : X(\Gamma) \to \mathbb{P}^1_\mathbb{Q}$ denote the composition of $\pi_{\Gamma,E}$ with the natural map $\mathbb{P}^1_E \to \mathbb{P}^1_\mathbb{Q}$.

Let $X(\Gamma)_\mathbb{Z}$ denote the normalization of $\mathbb{P}^1_\mathbb{Z}$ in $X(\Gamma)$ under the natural map $X(\Gamma) \to \mathbb{P}^1_\mathbb{Q} \to \mathbb{P}^1_\mathbb{Z}$. Then $X(\Gamma)_\mathbb{Z}$ is a normal arithmetic surface with $X(\Gamma)$ as generic fiber. By results of Lipman (see (Lipman, 1978)), there exists a *desingularization* of $X(\Gamma)_\mathbb{Z}$. Namely, there exists a regular projective arithmetic surface with a proper birational morphism to $X(\Gamma)_\mathbb{Z}$. Let $\mathscr{X}(\Gamma) \to X(\Gamma)_\mathbb{Z}$ denote such a desingularization, and let the composite map $\mathscr{X}(\Gamma) \to X(\Gamma)_\mathbb{Z} \to \mathbb{P}^1_\mathbb{Z}$ be denoted by $\pi_{\Gamma,\mathbb{Z}} : \mathscr{X}(\Gamma) \to \mathbb{P}^1_\mathbb{Z}$. As in §4.11 in (Kühn, 2001), we call any such $\mathscr{X}(\Gamma)$ an *arithmetic surface associated to* Γ. We remark that this definition differs slightly from that in §4.11 of (Kühn, 2001) in that we do not require the field of constants E of the generic fiber $X(\Gamma)$ of $\mathscr{X}(\Gamma)$ to be of minimal degree here.

Let $\infty \in \mathbb{P}^1_{\mathbb{Q}}$ correspond to the unique pole of the j-function, and let $\overline{\infty} \subseteq \mathbb{P}^1_{\mathbb{Z}}$ be its Zariski closure. Let $\mathscr{L} := \pi^*_{\Gamma,\mathbb{Z}}\mathcal{O}_{\mathbb{P}^1_{\mathbb{Z}}}(\overline{\infty})$.

Now,

$$\mathscr{X}(\Gamma)_{\mathbb{C}} := \mathscr{X}(\Gamma) \otimes_{\mathbb{Z}} \mathbb{C} = \bigsqcup_{\sigma : E \to \mathbb{C}} X(\Gamma) \otimes_{E,\sigma} \mathbb{C},$$

where the disjoint union is over all embeddings σ of E into $\mathbb{C}$. For each σ, the base change $X(\Gamma) \otimes_{E,\sigma} \mathbb{C}$ is also a modular curve, say associated to the group $\Gamma_\sigma \subseteq \Gamma(1)$. Then

$$\mathscr{X}(\Gamma)_{\mathbb{C}} \cong \bigsqcup_{\sigma : E \to \mathbb{C}} X(\Gamma_\sigma)_{\mathbb{C}}.$$

For each $k \geq 1$,

$$\mathscr{L}_{\mathbb{C}}^{\otimes k} = \bigoplus_{\sigma : E \to \mathbb{C}} \pi^*_{\Gamma_\sigma,\mathbb{C}}\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(\infty)^{\otimes k}$$

is identified with the sum of the line bundles of modular forms of level Γ_σ and weight $12k$ by proposition 3.1.2. Hence, we call $\mathscr{L}$ the *line bundle on* $\mathscr{X}(\Gamma)$ *associated to modular forms of level* Γ *and weight* 12.

Now let $\{T_1, \ldots, T_r\} \subseteq X(\Gamma)$ denote the preimage of $\infty \in \mathbb{P}^1_{\mathbb{Q}}$ under π_Γ, and let $D := T_1 + \cdots + T_r \in \mathrm{Div}(X(\Gamma))$ be the divisor of cusps of $X(\Gamma)$. Then, again by proposition 3.1.2,

$$\mathscr{L}_{\mathbb{C}}^{\otimes k}(-D_{\mathbb{C}}) := \mathscr{L}_{\mathbb{Q}}^{\otimes k}(-D) \otimes_{\mathbb{Q}} \mathbb{C}$$

is the sum of the bundles of cusp forms for the Γ_σ.

Remark 3.3.1. We will be interested in studying the submodule of $H^0(\mathscr{X}(\Gamma), \mathscr{L}^{\otimes k})$ associated to cusp forms (see §3.6). As such, it does not matter which desingularization $\mathscr{X}(\Gamma) \to X(\Gamma)_{\mathbb{Z}}$ we choose, since for any such desingularization, and for any vector bundle

E on $X(\Gamma)_{\mathbb{Z}}$, if $\mathscr{E}$ denotes its pullback to $\mathscr{X}(\Gamma)$, then the natural map $H^0(X(\Gamma)_{\mathbb{Z}}, E) \to H^0(\mathscr{X}(\Gamma), \mathscr{E})$ is an isomorphism.

3.4. Petersson metric on integral modular forms

For $k \geq 1$, we endow $\mathscr{L}^{\otimes k}$ with the *Petersson metric* as described in §3.2. Explicitly: for a section $f = (f_\sigma)_\sigma$ of $\mathscr{L}_{\mathbb{C}}^{\otimes k}$ over the open $\bigsqcup_\sigma U_\sigma \subseteq \bigsqcup_\sigma X(\Gamma_\sigma)_{\mathbb{C}}$,

$$|f|^2_{\mathrm{Pet}}(z) := |(f_\sigma \Delta^k)(z)|^2 (4\pi \operatorname{im}(z))^{12k},$$

where $z \in \mathfrak{h}^*$ corresponds to a point of $U_\sigma \subseteq X(\Gamma_\sigma)_{\mathbb{C}}$. We remark that even though there are choices involved in picking the groups Γ_σ such that $X(\Gamma) \otimes_{E,\sigma} \mathbb{C} \cong X(\Gamma_\sigma)_{\mathbb{C}}$, the resulting metric on $\mathscr{L}$ is independent of these choices. As already mentioned in §3.2, this metric is logarithmically singular at the elliptic points and cusps of the Γ_σ. We refer to $\mathscr{L}^{\otimes k}$ endowed with the Petersson metric by $\overline{\mathscr{L}}^{\otimes k}$.

For each σ, denote by $d\mu_\sigma$ the invariant measure on $X(\Gamma_\sigma)_{\mathbb{C}}$ induced by $\frac{dxdy}{y^2}$ on the upper half plane, where we use the coordinate $z = x + iy$, and let $d\mu$ denote the measure $\bigsqcup_\sigma d\mu_\sigma$ on $\bigsqcup_\sigma X(\Gamma_\sigma)_{\mathbb{C}}$.

For $f \in H^0(\mathscr{X}(\Gamma)_{\mathbb{C}}, \mathscr{L}_{\mathbb{C}}^{\otimes k}(-D_{\mathbb{C}}))$ a global section, define

$$\|f\|^2_{k,\infty} := \frac{1}{[E:\mathbb{Q}]d_\Gamma} \int_{\mathscr{X}(\Gamma)(\mathbb{C})} |f|^2_{\mathrm{Pet}} \, d\mu$$

to be the normalized L^2-norm of the Petersson metric. Note that for $f = (f_\sigma)$,

$$\|f\|^2_{k,\infty} = \frac{1}{[E:\mathbb{Q}]} \sum_\sigma \|f_\sigma \Delta^k\|^2_{\mathrm{Pet}},$$

where $\|\cdot\|_{\mathrm{Pet}}$ refers to the Petersson norm from §3.2.

3.5. Adelic vector bundles, heights, and successive maxima

Let K be a number field, and let Σ_K denote the set of all places of K. For $v \in \Sigma_K$, let $\mathbb{C}_v$ denote the completion of an algebraic closure of K_v. For v finite, let $|\cdot|_v$ denote the norm on $\mathbb{C}_v$ that extends the p-adic norm on $\mathbb{Q}_p$, where p is the rational prime lying under v. For v an archimedean place, we let $|\cdot|_v$ denote the usual norm on $\mathbb{C}_v = \mathbb{C}$.

A finite dimensional K-vector space V is called an *adelic vector bundle over* K if for each $v \in \Sigma_K$, $V \otimes_K \mathbb{C}_v$ is equipped with a norm $\|\cdot\|_v$ subject to the following conditions:

(i) There exists a K-basis $(s_1, \ldots, s_r)$ of V such that for all but finitely many finite places $v \in \Sigma_K$,

$$\|\alpha_1 s_1 + \cdots + \alpha_r s_r\|_v = \max(|\alpha_1|_v, \ldots, |\alpha_r|_v).$$

(ii) For every $v \in \Sigma_K$, $\|\cdot\|_v$ is invariant under the action of the group $\mathrm{Gal}(\mathbb{C}_v/K_v)$. Namely, if $(s_1, \ldots, s_r)$ is a K_v-basis of $E \otimes_K K_v$, and if $\alpha_1, \ldots, \alpha_r \in \mathbb{C}_v$, then

$$\|\tau(\alpha_1)s_1 + \cdots + \tau(\alpha_r)s_r\|_v = \|\alpha_1 s_1 + \cdots + \alpha_r s_r\|_v$$

for all $\tau \in \mathrm{Gal}(\mathbb{C}_v/K_v)$.

(iii) For $v \in \Sigma_K$ finite, $\|\cdot\|_v$ satisfies $\|s + s'\|_v \leq \max(\|s\|_v, \|s'\|_v)$.

Let V be an adelic vector bundle over K. The *naive adelic height* function on V is given by

$$\begin{aligned} \lambda : V &\to \mathbb{R} \\ s &\mapsto -\sum_{v \in \Sigma_K} k_v \log \|s\|_v, \end{aligned}$$

where $k_v := [K_v : \mathbb{Q}_p]$ for p the rational place lying under v. Using λ, we equip V with the following filtration: given $a \in \mathbb{R}$, set

$$V^a := \mathrm{span}_K\{s \in V : \lambda(s) \geq a\}.$$

This is a decreasing filtration on V indexed by the real numbers with the property that $V^a = V$ for $a \ll 0$, and $V^a = 0$ for $a \gg 0$. If $\dim_K V \geq 1$, the *naive adelic successive* *maxima* of V are the real numbers $\lambda_1, \ldots, \lambda_{\dim_K V}$, where

$$\lambda_i := \sup\{a \in \mathbb{R} : \dim_K V^a \geq i\}.$$

Example 3.5.1. Suppose $\mathscr{V}$ is a finite, free $\mathbb{Z}$-module of rank $d \geq 1$, equipped with a norm $\|\cdot\|_\infty$ on the $\mathbb{C}$-vector space $\mathscr{V} \otimes_{\mathbb{Z}} \mathbb{C}$. Then $V := \mathscr{V} \otimes_{\mathbb{Z}} \mathbb{Q}$ has a natural structure of an adelic vector bundle over $\mathbb{Q}$ as follows. For $v = \infty$, we use the given norm $\|\cdot\|_\infty$ on $V \otimes_{\mathbb{Q}} \mathbb{C}$, and for $v = p$ finite, define $\|\cdot\|_p : V \otimes_{\mathbb{Q}} \mathbb{C}_p \to \mathbb{R}$ by

$$\|s\|_p := \inf_{\alpha \in \mathbb{C}_p} \{|\alpha|_p : s \in \alpha\,(\mathscr{V} \otimes_{\mathbb{Z}} R_p)\}, \tag{3.5.1}$$

where R_p is the closed unit ball of $\mathbb{C}_p$. If $s_1, \ldots, s_d$ denotes any $\mathbb{Z}$-basis of $\mathscr{V}$, then if we express $s \in V \otimes_{\mathbb{Q}} \mathbb{C}_p$ as $s = \sum_i \alpha_i s_i$ with $\alpha_i \in \mathbb{C}_p$, we have $\|s\|_p = \max_i(|\alpha_i|_p)$.

If $\lambda_1, \ldots, \lambda_d$ denote the successive maxima for V with respect to the filtration induced by the naive adelic height as above, then

$$\lambda_i = -\log(\mu_i),$$

where $\mu_1, \ldots, \mu_d$ are the successive minima for the lattice $\mathscr{V} \subseteq \mathscr{V} \otimes_{\mathbb{Z}} \mathbb{R}$, where $\mathscr{V} \otimes_{\mathbb{Z}} \mathbb{R}$ is equipped with the norm induced from that on $\mathscr{V} \otimes_{\mathbb{Z}} \mathbb{C}$.

3.6. Adelic $\mathbb{Q}$-vector bundle structure on rational cusp forms

Let $\Gamma \subseteq \Gamma(1)$ be a finite index subgroup and let $X(\Gamma), \mathscr{X}(\Gamma), \mathscr{L}$, and D be as in §3.3. Let

$$\begin{aligned}
\mathscr{M}_k &:= H^0(\mathscr{X}(\Gamma), \mathscr{L}^{\otimes k}),\\
M_k &:= \mathscr{M}_k \otimes_{\mathbb{Z}} \mathbb{Q},\\
S_k &:= H^0(X(\Gamma), \mathscr{L}_{\mathbb{Q}}^{\otimes k}(-D)),\\
\mathscr{S}_k &:= \mathscr{M}_k \cap S_k,
\end{aligned}$$

where the last intersection takes place in M_k. These correspond to the spaces of integral modular forms, rational modular forms, rational cusp forms, and integral cusp forms of level Γ and weight $12k$, respectively.

Note that $\mathscr{S}_k$ is a finite free $\mathbb{Z}$-module with $S_k = \mathscr{S}_k \otimes_{\mathbb{Z}} \mathbb{Q}$. By example 3.5.1, S_k has a natural structure of an adelic vector bundle over $\mathbb{Q}$, with respect to the norms $\|\cdot\|_{k,p}$, as in equation 3.5.1 for a finite place p, and $\|\cdot\|_{k,\infty}$, the normalized L^2 norm from §3.4. We denote the naive adelic height function on S_k by λ_k, the induced filtration by $(S_k^a)_{a\in\mathbb{R}}$, and the associated successive maxima by $\lambda_{k,i}$ (for $i = 1, \ldots, \dim_{\mathbb{Q}} S_k$) (as in §3.5). These maxima are equal to the ones in theorem 2.0.1 by example 3.5.1.

CHAPTER 4

Proof of theorem 2.0.1

A key ingredient in the proof of theorem 2.0.1 is theorem 3.4.3 in (Chen, 2010) concerning quasi-filtered graded algebras. We apply this result to the graded algebras B_L in proposition 4.2.3. To do so, we need to get a uniform upper bound on the normalized height $\lambda_k(f)/k$ of any non-zero $f \in S_k$, and show that the algebra B_L is *quasi-filtered* with respect to an appropriate function (see definition 3.2.1 of (Chen, 2010)).

The uniform upper bound is proved in §4.1. The key ingredients for this are formulas for the intersection numbers of logarithmically singular line bundles as in (Kühn, 2001) and heights of cycles with respect to a smoothly metrized line bundle as in (Bost et al., 1994) (see proposition 4.1.3), along with a version of "Gromov's lemma" proved in (Friedman et al., 2013) for cusp forms with respect to the Petersson norm and later improved (and generalized) in (Auvray et al., 2016) (see proposition 4.1.2). Proposition 4.1.2 is also key to lemma 4.2.2, which in turn is used to show that the B_L are quasi-filtered. Using all of this, we show that the ν_k converge vaguely to a sub-probability Borel measure ν on $\mathbb{R}$.

Finally, in §4.3, we derive explicit lower bounds on the successive maxima $\lambda_{k,i}$ for k large by constructing a basis for S_k of a particular shape, and doing explicit calculations with it. These lower bounds then imply that ν is a probability measure (and hence the vague convergence is also weak convergence).

We follow the general structure of §3.6 and §3.7 of (Chinburg et al., 2018) in §4.2 and §4.3. Throughout this section, we keep the notation from §3.6.

4.1. Upper bound on heights

Lemma 4.1.1. *Let $f \in S_k$ be a non-zero element. There exists a rational number β such that $\beta f \in \mathscr{S}_k$ and $\beta f \notin B \cdot \mathscr{S}_k$ for all positive integers $B \geq 2$. Furthermore, for any $f \in \mathscr{S}_k$ such that $f \notin B \cdot \mathscr{S}_k$ for all integers $B > 1$, we have $\|f\|_{k,p} = 1$ for all finite places p, and* *hence,* $\lambda_k(f) = -\log \|f\|_{k,\infty}$.

Proof. Take any $\mathbb{Z}$-basis $f_1, \ldots, f_d$ of $\mathscr{S}_k$, and let $f = \sum_i \alpha_i f_i$ with $\alpha_i \in \mathbb{Q}$. Then it is clear that there is a rational number β such that $\beta\alpha_i \in \mathbb{Z}$ for all i and $\gcd(\beta\alpha_1, \ldots, \beta\alpha_d) = 1$. It is also clear that $\beta f \in \mathscr{S}_k$, and $\beta f \notin B \cdot \mathscr{S}_k$ for all integers $B > 1$.

Finally given any $f \in \mathscr{S}_k$ such that $f \notin B \cdot \mathscr{S}_k$ for all integers $B > 1$, writing $f = \sum_i \alpha_i f_i$, we have $\alpha_i \in \mathbb{Z}$ and $\gcd(\alpha_1, \ldots, \alpha_d) = 1$. By example 3.5.1, we conclude that $\|f\|_{k,p} = \max_i(|\alpha_i|_p) = 1$ for all finite places p. $\square$

Let $\Gamma \subseteq \Gamma(1)$ be a finite index subgroup, and let $S_{2k}(\Gamma)_{\mathbb{C}}$ denote the space of complex cusp forms for Γ of weight $2k$. Let $\langle \cdot, \cdot \rangle_{\text{Pet}}$ denote the Petersson inner product on $S_{2k}(\Gamma)_{\mathbb{C}}$ as in §3.2. For $f \in S_{2k}(\Gamma)$, denote by

$$\|f\|_{\sup} := \sup_{z \in \Gamma \backslash \mathfrak{h}} |f(z)|(4\pi \operatorname{im}(z))^k$$

the sup-norm of f with respect to the Petersson metric.

Proposition 4.1.2. *Let $\Gamma \subseteq \Gamma(1)$ be a finite index subgroup, and let $S_{2k}(\Gamma)_{\mathbb{C}}$ denote the space of complex cusp forms for Γ of weight $2k$. There exist positive constants c_1 and c_2, with c_2 independent of Γ, such that for any $0 \neq f \in S_{2k}(\Gamma)_{\mathbb{C}}$, we have*

$$c_1 \|f\|_{\text{Pet}} \leq \|f\|_{\sup} \leq c_2 k^{3/4} \|f\|_{\text{Pet}}.$$

Proof. Let μ_Γ be the measure on $X(\Gamma)_{\mathbb{C}}$ induced from the hyperbolic volume form $\frac{dxdy}{y^2}$ on $\mathfrak{h}$ with coordinate $z = x + iy$. For the first inequality, note that

$$\|f\|_{\text{Pet}}^2 = \frac{1}{d_\Gamma} \int_{X(\Gamma)_{\mathbb{C}}} |f(z)|^2 (4\pi \operatorname{im}(z))^{2k} \, d\mu_\Gamma(z) \leq \|f\|_{\sup}^2 \left(\frac{1}{d_\Gamma} \int_{X(\Gamma)_{\mathbb{C}}} d\mu_\Gamma \right),$$

so we may let $c_1 = (\frac{1}{d_\Gamma} \int_{X(\Gamma)_{\mathbb{C}}} d\mu_\Gamma)^{-1/2}$.

For the other direction, let $\{f_1, \ldots, f_d\}$ be an orthonormal basis for $S_{2k}(\Gamma)_{\mathbb{C}}$ for the Petersson inner product. Note that by our normalization of the Petersson inner product (i.e. the inclusion of the $(4\pi)^{2k}$ factor), $\{(4\pi)^k f_1, \ldots, (4\pi)^k f_d\}$ is an orthonormal basis for the classical Petersson inner product. Then, for $z \in \mathfrak{h}$, define

$$B_k^\Gamma(z) := \sum_{j=1}^d |(4\pi)^k f_j(z)|^2 \operatorname{im}(z)^{2k} = \sum_{j=1}^d |f_j(z)|^2 (4\pi \operatorname{im}(z))^{2k}$$

as in (Auvray et al., 2016). Note that for any $0 \neq f \in S_{2k}(\Gamma)_{\mathbb{C}}$, if $f = \sum_{j=1}^d \alpha_j f_j$, then $\|f\|_{\mathrm{Pet}}^2 = \sum_{j=1}^d |\alpha_j|^2$, and for any $z \in \mathfrak{h}$,

$$\begin{aligned} |f(z)|^2 (4\pi \operatorname{im}(z))^{2k} &= \left| \sum_{j=1}^d \alpha_j f_j(z) \right|^2 (4\pi \operatorname{im}(z))^{2k} \\ &\leq \left(\sum_{j=1}^d |\alpha_j|^2 \right) \left(\sum_{j=1}^d |f_j(z)|^2 \right) (4\pi \operatorname{im}(z))^{2k}. \end{aligned}$$

Hence,

$$\frac{|f(z)|^2 (4\pi \operatorname{im}(z))^{2k}}{\|f\|_{\mathrm{Pet}}^2} \leq B_k^\Gamma(z),$$

and

$$\frac{\|f\|_{\sup}^2}{\|f\|_{\mathrm{Pet}}^2} = \sup_{z \in \Gamma \backslash \mathfrak{h}} \frac{|f(z)|^2 (4\pi \operatorname{im}(z))^{2k}}{\|f\|_{\mathrm{Pet}}^2} \leq \sup_{z \in \Gamma \backslash \mathfrak{h}} B_k^\Gamma(z).$$

By theorem 1.7 in (Auvray et al., 2016),

$$\sup_{z \in \Gamma \backslash \mathfrak{h}} B_k^\Gamma(z) = \left(\frac{k}{\pi} \right)^{3/2} + O(k).$$

In particular, there exists a constant c_2 (independent of k and Γ) which gives us the desired bound. $\square$

We follow very closely the proof of lemma 4.1.7 in (Chen, 2010) for the following result.

Proposition 4.1.3. *There is a constant C such that for any $0 \neq f \in S_k$, $\lambda_k(f) \leq Ck$.*

Proof. Since $\lambda_k(f) = \lambda_k(\alpha f)$ for any non-zero $\alpha \in \mathbb{Q}^\times$, we may suppose, after appropriately scaling f, that $f \in \mathscr{S}_k$ and $f \notin B \cdot \mathscr{S}_k$ for any positive integer $B > 1$. Then by lemma 4.1.1, $\lambda_k(f) = -\log \|f\|_{k,\infty}$.

Taking any projective (closed) embedding $\Phi_\mathbb{Z} : \mathscr{X}(\Gamma) \hookrightarrow \mathbb{P}^N_\mathbb{Z}$, we let $\overline{L} := \Phi^*_\mathbb{Z}\overline{\mathcal{O}_{\mathbb{P}^N_\mathbb{Z}}(1)}$, where $\overline{\mathcal{O}_{\mathbb{P}^N_\mathbb{Z}}(1)}$ refers to $\mathcal{O}_{\mathbb{P}^N_\mathbb{Z}}(1)$ endowed with the Fubini-Study metric. Then $\overline{L}$ is arithmetically ample with $c_1(\overline{L}) > 0$. Now take any global section $\ell \in H^0(\mathscr{X}(\Gamma), L)$ such that $\mathrm{div}_L(\ell)$ and $\mathrm{div}_{\mathscr{L}^{\otimes k}}(f)$ don't share any common horizontal divisors (i.e. their divisors on the generic fiber $X(\Gamma)$ have disjoint support). Then the generalized intersection number $\overline{L} \cdot \overline{\mathscr{L}}^{\otimes k}$ in the sense of equation (3.10) in (Kühn, 2001) is given by

$$\overline{L} \cdot \overline{\mathscr{L}}^{\otimes k} = (\ell \cdot f)_{\mathrm{fin}} + \langle \ell \cdot f \rangle_\infty,$$

where $(\ell \cdot f)_{\mathrm{fin}} = (\mathrm{div}_L(\ell) \cdot \mathrm{div}_{\mathscr{L}^{\otimes k}}(f))_{\mathrm{fin}}$ is equal to

$$(\mathrm{div}_L(\ell) \cdot \mathrm{div}_{\mathscr{L}^{\otimes k}}(f))_{\mathrm{fin}} := \sum_{i,j=0}^{1} (-1)^{i+j} \log \# H^i \left(\mathscr{X}(\Gamma), \mathrm{Tor}_j^{\mathcal{O}_{\mathscr{X}(\Gamma)}} (\mathcal{O}_{\mathrm{div}_L(\ell)}, \mathcal{O}_{\mathrm{div}_{\mathscr{L}^{\otimes k}}(f)}) \right)$$

($\mathcal{O}_D$ denotes the structure sheaf for an effective divisor $D \subseteq \mathscr{X}(\Gamma)$), and $\langle \ell \cdot f \rangle_\infty$ is given in our case by

$$\langle \ell \cdot f \rangle_\infty = -\sum_{P \in \mathscr{X}(\Gamma)(\mathbb{C})} n_P \log |\ell(P)| - \int_{\mathscr{X}(\Gamma)(\mathbb{C})} \log |f|_{\mathrm{Pet}} \cdot c_1(\overline{L}),$$

if $\mathrm{div}_{\mathscr{L}_\mathbb{C}^{\otimes k}}(f) = \sum_{P \in \mathscr{X}(\Gamma)(\mathbb{C})} n_P P$ (lemma 3.9 in (Kühn, 2001)). Note that this expression makes sense since $\mathrm{div}_{\mathscr{L}_\mathbb{C}^{\otimes k}}(f)$ and $\mathrm{div}_{L_\mathbb{C}}(\ell)$ have disjoint support. Then, using bilinearity of the intersection pairing, we have

$$(\ell \cdot f)_{\mathrm{fin}} - \sum_{P \in \mathscr{X}(\Gamma)(\mathbb{C})} n_P \log |\ell(P)| = k(\overline{L} \cdot \overline{\mathscr{L}}) + \int_{\mathscr{X}(\Gamma)(\mathbb{C})} \log |f|_{\mathrm{Pet}} \cdot c_1(\overline{L}),$$

where the expression on the left is the height $h_{\overline{L}}(\mathrm{div}_{\mathscr{L}^{\otimes k}}(f))$ of the cycle $\mathrm{div}_{\mathscr{L}^{\otimes k}}(f)$ with respect to $\overline{L}$, as defined in 3.1.1 in (Bost et al., 1994) (see also §2.3.4 of (Bost et al., 1994), in particular, equation 2.3.17). Since $\overline{L}$ is arithmetically ample and $\mathrm{div}_{\mathscr{L}^{\otimes k}}(f)$ is effective, $h_{\overline{L}}(\mathrm{div}_{\mathscr{L}^{\otimes k}}(f)) \geq 0$ (proposition 3.2.4 in (Bost et al., 1994)). Hence

$$0 \leq k(\overline{L} \cdot \overline{\mathscr{L}}) + \int_{\mathscr{X}(\Gamma)(\mathbb{C})} \log |f|_{\mathrm{Pet}} \cdot c_1(\overline{L}),$$

and since $c_1(\overline{L}) > 0$, we get

$$\log \sup_{z \in \mathscr{X}(\Gamma)(\mathbb{C})} |f|_{\mathrm{Pet}}(z) \geq C' k$$

for $C' := -(\overline{L} \cdot \overline{\mathscr{L}}) / \int_{\mathscr{X}(\Gamma)(\mathbb{C})} c_1(\overline{L})$.

Denote the image of f under the map $H^0(\mathscr{X}(\Gamma), \mathscr{L}^{\otimes k}(-D)) \to H^0(\mathscr{X}(\Gamma)_{\mathbb{C}}, \mathscr{L}_{\mathbb{C}}^{\otimes k}(-D_{\mathbb{C}}))$ by $(f^\sigma)_\sigma$. Suppose $|f|_{\mathrm{Pet}}(z)$ achieves its supremum on the component $X(\Gamma_{\sigma'})_{\mathbb{C}}$, so that

$$\sup_{z \in \mathscr{X}(\Gamma)(\mathbb{C})} |f|_{\mathrm{Pet}}(z) = \sup_{z \in X(\Gamma_{\sigma'})_{\mathbb{C}}} |(f^{\sigma'} \Delta^k)(z)| (4\pi \operatorname{im}(z))^{6k} = \|f^{\sigma'} \Delta^k\|_{\mathrm{sup}}.$$

Then

$$\|f\|_{k,\infty}^2 = \frac{1}{[E:\mathbb{Q}]} \sum_\sigma \|f^\sigma \Delta^k\|_{\mathrm{Pet}}^2 \geq \frac{1}{[E:\mathbb{Q}]} \|f^{\sigma'} \Delta^k\|_{\mathrm{Pet}}^2 \geq \frac{1}{[E:\mathbb{Q}]} c_2^{-2} k^{-3/2} \|f^{\sigma'} \Delta^k\|_{\mathrm{sup}}^2,$$

where the last inequality uses proposition 4.1.2. We conclude that

$$\lambda_k(f) = -\log \|f\|_{k,\infty} \leq Ck$$

for a constant C independent of k. □

4.2. Cusp forms vanishing to increasing orders at the cusps

For an integer $L \geq 1$, define

$$\mathcal{B}_{L,k} := \mathscr{L}_{\mathbb{Q}}^{\otimes k}(-\lceil k/L \rceil D) \subseteq \mathscr{L}_{\mathbb{Q}}^{\otimes k}(-D).$$

Then for each $\sigma : E \to \mathbb{C}$,

$$(\mathcal{B}_{L,k})_{\mathbb{C}}|_{X(\Gamma_\sigma)_{\mathbb{C}}} = \mathscr{L}_{\mathbb{C}}^{\otimes k}(-\lceil k/L \rceil) D_{\mathbb{C}})\,|_{X(\Gamma_\sigma)_{\mathbb{C}}}$$

is identified with the line bundle of weight $12k$ cusp forms for $X(\Gamma_\sigma)_{\mathbb{C}}$ that vanish to order at least k/L at every cusp.

Let $B_{L,k} := H^0(X(\Gamma), \mathcal{B}_{L,k})$. We use the inclusion $B_{L,k} \subseteq S_k$ to define a filtration on $B_{L,k}$. Namely, for $a \in \mathbb{R}$, we set the ath filtered piece of $B_{L,k}$ to be

$$B_{L,k}^a := S_k^a \cap B_{L,k}.$$

This is a decreasing $\mathbb{R}$-filtration on $B_{L,k}$. For $i = 1, \ldots, \dim_{\mathbb{Q}} B_{L,k}$, let

$$\lambda_{L,k,i} = \sup\{a \in \mathbb{R} : \dim_{\mathbb{Q}} B_{L,k}^a \geq i\}$$

denote the ith successive maxima of $B_{L,k}$. Since these are the successive maxima associated to the subspace filtration, the multi-set $\{\lambda_{L,k,i}\}_{i=1}^{\dim_{\mathbb{Q}} B_{L,k}}$ is a sub multi-set of $\{\lambda_{k,i}\}_{i=1}^{\dim_{\mathbb{Q}} S_k}$.

Furthermore, let

$$\widetilde{\lambda}_k : S_k \to \mathbb{R} \cup \{\infty\}, \quad \widetilde{\lambda}_{L,k} : B_{L,k} \to \mathbb{R} \cup \{\infty\}$$

be defined by $\widetilde{\lambda}_k(f) := \sup\{a \in \mathbb{R} : f \in S_k^a\}$, and $\widetilde{\lambda}_{L,k}(f) := \sup\{a \in \mathbb{R} : f \in B_{L,k}^a\}$. These are called the *index functions* of S_k and $B_{L,k}$ for their respective filtrations (as in §2 of (Chen, 2007)). For $f \in B_{L,k}$, we have $\widetilde{\lambda}_{L,k}(f) = \widetilde{\lambda}_k(f)$.

Remark 4.2.1. We could have opted to use the filtration on $B_{L,k}$ obtained from the restriction of the height function λ_k on S_k, so that for $a \in \mathbb{R}$ the ath filtered piece of $B_{L,k}$ would be

$$B_{L,k,a} := \mathrm{span}_{\mathbb{Q}}\{s \in B_{L,k} : \lambda_k(s) \geq a\}.$$

This approach is taken in (Chinburg et al., 2018), §3.6 (see, in particular, the proof of lemma

3.6.2) and makes proving various estimates (see (Chinburg et al., 2018), lemma 3.7.1) rather tricky. By contrast, working with the subspace filtration simplifies these matters significantly - compare lemma 4.2.4 below to its counterpart, (Chinburg et al., 2018), lemma 3.7.1.

As mentioned at the start of this section, the following lemma is used to show the algebras B_L in proposition 4.2.3 are quasi-filtered.

Lemma 4.2.2. *Let* $\psi(k) := \frac{3}{4}\log(k) + \log(c_2) - \log(c_1)/2$ *where* c_1 *and* c_2 *are the constants from proposition 4.1.2 for* $\Gamma \subseteq \Gamma(1)$*. For any collection of elements* $f_i \in S_{k_i}$ *(*$i = 1, \ldots, n$*, with* $n \geq 2$*), we have*

$$\widetilde{\lambda}_{k_1+\cdots+k_n}(f_1 \cdots f_n) \geq \sum_{i=1}^{n} \left(\widetilde{\lambda}_{k_i}(f_i) - \psi(k_i)\right).$$

Proof. Note that c_1 was set to be $(\int_{X(\Gamma)_\mathbb{C}} d\mu_\Gamma)^{-1}$, which only depends on the index of Γ in $\Gamma(1)$. Since $[\Gamma(1) : \Gamma] = [\Gamma(1) : \Gamma_\sigma]$ for all σ, we may use the same c_1 for all the Γ_σ. Hence for any $f \in S_k$, proposition 4.1.2 gives

$$c_1^2 \|f\|_{k,\infty}^2 \leq \sum_\sigma \|f^\sigma \Delta^k\|_{\text{sup}}^2 \leq c_2^2 k^{3/2} \|f\|_{k,\infty}^2.$$

Then for $f_i \in S_{k_i}$ $(i = 1, \ldots, n)$, letting $K := k_1 + \ldots, k_n$, we get

$$\begin{aligned} c_1^2 \|f_1 \ldots, f_n\|_{K,\infty}^2 &\leq \sum_\sigma \|(f_1 \ldots f_n)^\sigma \Delta^K\|_{\text{sup}}^2 \\ &\leq \sum_\sigma \|f_1^\sigma \Delta^{k_1}\|_{\text{sup}}^2 \cdots \|f_n^\sigma \Delta^{k_n}\|_{\text{sup}}^2 \\ &\leq \left(\sum_\sigma \|f_1^\sigma \Delta^{k_1}\|_{\text{sup}}^2\right) \cdots \left(\sum_\sigma \|f_n^\sigma \Delta^{k_n}\|_{\text{sup}}^2\right) \\ &\leq c_2^{2n} k_1^{3/2} \cdots k_n^{3/2} \|f_1\|_{k_1,\infty}^2 \cdots \|f_n\|_{k_n,\infty}^2, \end{aligned}$$

and consequently,

$$\|f_1 \dots f_n\|_{K,\infty} \le e^{\psi(k_1)+\cdots+\psi(k_n)} \|f_1\|_{k_1,\infty} \cdots \|f_n\|_{k_n,\infty}.$$

(We use the fact that $c_1 < 1$ here.)

Pick any $\varepsilon > 0$. By definition, $f_i \in S_{k_i}^{\widetilde{\lambda}_{k_i}(f_i)-\varepsilon/n}$, and hence $f_i = \sum g_{i,j}$ for $g_{i,j} \in S_{k_i}$ with $\lambda_{k_i}(g_{i,j}) \ge \widetilde{\lambda}_{k_i}(f_i) - \varepsilon/n$. It is easy to see that for any finite place p,

$$\|g_{1,j_1} \cdots g_{n,j_n}\|_{K,p} \le \|g_{1,j_1}\|_{k_1,p} \cdots \|g_{n,j_n}\|_{k_n,p}.$$

This combined with the previous paragraph yields

$$\begin{aligned} \lambda_K(g_{1,j_1} \cdots g_{n,j_n}) &\ge \sum_{i=1}^{n} (\lambda_{k_i}(g_{i,j_i}) - \psi(k_i)) \\ &\ge \sum_{i=1}^{n} \left(\widetilde{\lambda}_{k_i}(f_i) - \psi(k_i)\right) - \varepsilon. \end{aligned}$$

Since ε is arbitrary, we conclude that

$$\widetilde{\lambda}_K(f_1 \cdots f_n) \ge \sum_{i=1}^{n} \left(\widetilde{\lambda}_{k_i}(f_i) - \psi(k_i)\right),$$

as required. □

Given a Borel measure ν, and an integrable function f on $\mathbb{R}$, let

$$\nu(f) := \int_{\mathbb{R}} f \; d\mu.$$

Proposition 4.2.3. *With the notation above, the sequence of probability measures*

$$\nu_{L,k} := \frac{1}{\dim_{\mathbb{Q}} B_{L,k}} \sum_{i=1}^{\dim_{\mathbb{Q}} B_{L,k}} \delta_{\frac{1}{k}\lambda_{L,k,i}}$$

converges weakly as $k \to \infty$ to a probability measure $\nu_{L,\infty}$ with compact support.

Proof. First, we consider the case $k = qL$ for integers $q \geq 1$. Let

$$B_L := \bigoplus_{q \geq 0} B_{L,qL}.$$

Since $\deg(\mathcal{B}_{L,L}) > 0$, $\mathcal{B}_{L,L}$ is ample, and hence B_L is a finitely generated $\mathbb{Q}$-algebra with $B_{L,qL} \neq 0$ for q large. We endow each $B_{L,qL}$ with the filtration $(B^a_{L,qL})_{a \in \mathbb{R}}$ discussed above. Then by lemma 4.2.2, B_L is ψ_L-quasi-filtered with $\psi_L(q) := \psi(qL)$, where ψ is as in lemma 4.2.2. By proposition 4.1.3, $\widetilde{\lambda}_{L,qL}(f) = \widetilde{\lambda}_{qL}(f) \leq CqL$ for all non-zero $f \in B_{L,qL}$. Hence by theorem 3.4.3 of (Chen, 2010) (and by rescaling the measures by $1/L$), the collection of measures

$$\nu_{L,qL} := \frac{1}{\dim_{\mathbb{Q}} B_{L,qL}} \sum_{i=1}^{\dim_{\mathbb{Q}} B_{L,qL}} \delta_{\frac{1}{qL}\lambda_{L,qL,i}}$$

converges weakly to a compactly supported probability measure $\nu_{L,\infty}$ on $\mathbb{R}$.

For general k, suppose that $k = qL + r$ with $0 \leq r < L$. For $f \in B_{L,k} \subseteq B_{L,(q+1)L}$, lemma 4.2.2 gives

$$\begin{aligned}\widetilde{\lambda}_{L,(q+1)L}(f) &\geq \widetilde{\lambda}_{L,k}(f) + \widetilde{\lambda}_{L,L-r}(1) - \psi(k) - \psi(L-r) \\ &\geq \widetilde{\lambda}_{L,k}(f) + c_3 \log(k),\end{aligned}$$

for some constant c_3 independent of k. This implies that for all $i = 1, \ldots, \dim_{\mathbb{Q}} B_{L,k}$,

$$\lambda_{L,(q+1)L,i} \geq \lambda_{L,k,i} + c_3 \log(k).$$

Then

$$\dim_{\mathbb{Q}} B_{L,(q+1)L} - \dim_{\mathbb{Q}} B_{L,k} = [E : \mathbb{Q}] d_{\Gamma}(L - r),$$

is independent of q. Hence, for any bounded increasing continuous function $f : \mathbb{R} \to \mathbb{R}$, we have

$$\nu_{L,k}(f) \leq \nu_{L,(q+1)L}(f) + o(1),$$

and hence,

$$\limsup_{q\to\infty} \nu_{L,qL+r}(f) \leq \nu_{L,\infty}(f).$$

Similarly, using the inclusion $B_{L,qL} \subseteq B_{L,k}$, we deduce that

$$\liminf_{q\to\infty} \nu_{L,qL+r}(f) \geq \nu_{L,\infty}(f),$$

and hence $\nu_{L,k}(f) \to \nu_{L,\infty}(f)$ for all bounded increasing continuous functions f. Since all measures involved are probability measures, this shows that $\nu_{L,k} \to \nu_{L,\infty}$ weakly. □

Lemma 4.2.4. *For every positive real number ε and bounded Lipschitz function $h : \mathbb{R} \to \mathbb{R}$, there is a constant $L_0 = L_0(\varepsilon, h)$ such that for all $L \geq L_0$, and for all $k \geq 1$, we have*

$$|\nu_k(h) - \nu_{L,k}(h)| \leq \epsilon.$$

Proof. Let $|h|_{\mathrm{Lip}} := \sup_{x\in\mathbb{R}} |h(x)| + \sup_{x,y\in\mathbb{R},x\neq y} \frac{|h(x)-h(y)|}{|x-y|} = M$. Recall that since $B_{L,k}$ is equipped with the subspace filtration coming from S_k, the multi-set $\{\lambda_{L,k,i}\}_{i=1}^{\dim_{\mathbb{Q}} B_{L,k}}$ is a sub multi-set of $\{\lambda_{k,i}\}_{i=1}^{\dim_{\mathbb{Q}} S_k}$. Let $d_{L,k} := \dim_{\mathbb{Q}} B_{L,k}$ and $d_k := \dim_{\mathbb{Q}} S_k$. Then

$$\begin{aligned}
|\nu_k(h) - \nu_{L,k}(h)| &= \left| \frac{1}{d_k} \sum_{i=1}^{d_k} h\left(\frac{1}{k}\lambda_{k,i}\right) - \frac{1}{d_{L,k}} \sum_{i=1}^{d_{L,k}} h\left(\frac{1}{k}\lambda_{L,k,i}\right) \right| \\
&\leq \frac{1}{d_k} \left| \sum_{i=1}^{d_k} h\left(\frac{1}{k}\lambda_{i,k}\right) - \sum_{i=1}^{d_{L,k}} h\left(\frac{1}{k}\lambda_{i,L,k}\right) \right| + \left(\frac{1}{d_{L,k}} - \frac{1}{d_k}\right) \left| \sum_{i=1}^{d_{L,k}} h\left(\frac{1}{k}\lambda_{i,L,k}\right) \right| \\
&\leq \frac{1}{d_k}(d_k - d_{L,k})M + \frac{d_k - d_{L,k}}{d_k d_{L,k}} d_{L,k} M \\
&\leq \frac{c_4\left(\lceil k/L \rceil - 1\right)}{d_k} \\
&\leq \frac{c_4}{L},
\end{aligned}$$

for some constant c_4. The conclusion follows at once. $\square$

Corollary 4.2.5. *Suppose $N \geq 2$. Let $h : \mathbb{R} \to \mathbb{R}$ be a bounded Lipschitz function. Then the sequences $(\nu_k(h))_k$ and $(\nu_{L,\infty}(h))_L$ are convergent and have the same limit.*

Proof. Let $\varepsilon > 0$ be any positive real number, and let $L_0 = L_0(\varepsilon, h)$ be as in the previous lemma. For any $L \geq L_0$, and any k, we have $|\nu_k(h) - \nu_{L,k}(h)| \leq \varepsilon$ from the lemma above, and hence

$$\nu_{L,k}(h) - \varepsilon \leq \nu_k(h) \leq \nu_{L,k}(h) + \varepsilon.$$

From this we get

$$\nu_{L,\infty}(h) - \varepsilon \leq \liminf_k \nu_k(h) \leq \limsup_k \nu_k(h) \leq \nu_{L,\infty}(h) + \varepsilon,$$

and hence

$$0 \leq \limsup_k \nu_k(h) - \liminf_k \nu_k(h) \leq 2\varepsilon.$$

Since ε is arbitrary, we conclude that $\lim_k \nu_k(h)$ exists. Moreover, since for all $L \geq L_0$, we have

$$|\nu_{L,\infty}(h) - \lim_k \nu_k(h)| \leq \varepsilon,$$

we conclude that $\lim_L \nu_{L,\infty}(h) = \lim_k \nu_k(h)$, as required. $\square$

Now by the Riesz representation theorem for measures, there exists a sub-probability Borel measure ν on $\mathbb{R}$ representing the positive linear functional $\lim_k \nu_k = \lim_L \nu_{L,\infty}$ on $C_c(\mathbb{R})$. Namely, for all $h \in C_c(\mathbb{R})$, we have

$$\nu(h) = \lim_k \nu_k(h) = \lim_L \nu_{L,\infty}(h). \tag{4.2.1}$$

Hence, ν_k converges *vaguely* to ν.

4.3. ν is a probability measure

We obtain lower bound estimates on the successive maxima $\lambda_{k,i}$ of S_k in proposition 4.3.1, which are then used in proposition 4.3.2 to show that ν is a probability measure.

Recall that $T_1, \ldots, T_r$ are the pre-images of $\infty \in \mathbb{P}^1_{\mathbb{Q}}$ under the map $\pi_\Gamma : X(\Gamma) \to \mathbb{P}^1_{\mathbb{Q}}$, and $D = T_1 + \cdots + T_r$. Let e_i denote the ramification index of T_i over ∞. Then

$$\mathscr{L}_{\mathbb{Q}}^{\otimes k}(-D) = \mathcal{O}_{X(\Gamma)}\left((ke_1 - 1)T_1 + \cdots + (ke_r - 1)T_r\right).$$

Let g denote the genus of $X(\Gamma)_{\mathbb{C}}$. Then the genus of $X(\Gamma)$ as a curve over E, and that of $X(\Gamma_\sigma)_{\mathbb{C}}$, for each σ, is also equal to g.

Proposition 4.3.1. *Fix an integer k_0 such that $k_0 e_p - 1 > 2g - 2$ for all $p = 1, \ldots, r$, and let $d_0 := \dim_{\mathbb{Q}} S_{k_0}$. For all $k > k_0$, if $\{\lambda_{k,i}\}_{i=1}^{\dim_{\mathbb{Q}} S_k}$ denote the successive maxima of S_k, then:*

- *For $1 \leq i \leq d_0$,*
$$\lambda_{k,i} \geq 6k \log\left(\frac{k - k_0}{k}\right) - C_4 k.$$

- *For $d_0 + 1 \leq i$,*
$$\lambda_{i,k} \geq 6k \log\left(\frac{k - k_0 - \left\lceil \frac{i - d_0}{d_\Gamma[E:\mathbb{Q}]} \right\rceil}{k}\right) - C_4 k.$$

For $i > d_0 + (k - k_0 - 1)d_\Gamma[E : \mathbb{Q}]$, we interpret the right hand side above as $-\infty$.

Proof. We construct an E-basis for S_k when $k \geq k_0$. We start by constructing a basis for S_{k_0+1}/S_{k_0}. Using proposition 1.40 in (Shimura, 1971), one easily checks that for $k \geq 1$, $\deg(\mathscr{L}_{\mathbb{Q}}^{\otimes k}(-D)) > 2g - 2$, and hence by Riemann-Roch,

$$\dim_E S_k = \dim_E H^0(X(\Gamma), \mathscr{L}_{\mathbb{Q}}^{\otimes k}(-D)) = \deg(\mathscr{L}_{\mathbb{Q}}^{\otimes k}(-D)) - g + 1.$$

for $k \geq 1$. Hence, for any $m \geq 0$,

$$\dim_E S_{k_0+m+1} - \dim_E S_{k_0+m} = \deg \mathscr{L}_{\mathbb{Q}} = d_\Gamma. \tag{4.3.1}$$

For $p = 1, \ldots, r$, and $q = 1, \ldots, e_p$, since

$$\dim_E H^0(X(\Gamma), \mathcal{O}((k_0e_p - 1 + q)T_p)) - \dim_E H^0(X(\Gamma), \mathcal{O}((k_0e_p - 1 + (q-1))T_p)) = \deg T_p,$$

there exist rational functions $b_{q,p,i}$ for $i = 1, \ldots, \deg T_p$ in $H^0(X(\Gamma), \mathcal{O}((k_0e_p - 1 + q)T_p))$ that are E-linearly independent, and that are regular everywhere on $X(\Gamma)$ except at T_p, where they have a pole of order $k_0e_p - 1 + q$.

Varying p, q, and i, there are d_Γ such functions $\{b_{q,p,i}\}_{q,p,i}$, and $b_{q,p,i} \in S_{k_0+1} \setminus S_{k_0}$. It is clear from construction that $\{b_{q,p,i}\}_{q,p,i} \subseteq S_{k_0+1}$ are E-linearly independent, which in light of equation 4.3.1 implies that their images constitute an E-basis of S_{k_0+1}/S_{k_0}.

For any $m \geq 0$, note that

$$\mathrm{ord}_{T_p}(j^m b_{q,p,i}) = -me_p - (k_0e_p - 1 + q) = -[(k_0 + m)e_p - 1 + q].$$

Since $\{j^m b_{q,p,i}\}_{q,p,i} \subseteq S_{k_0+m+1} \setminus S_{k_0+m}$ are E-linearly independent, equation 4.3.1 again implies that their images make up an E-basis of S_{k_0+m+1}/S_{k_0+m}. Now let $d_0' = \dim_E S_{k_0}$ and fix an E-basis $\{c_1, \ldots, c_{d_0'}\}$ of S_{k_0}. For $k > k_0$, the above discussion yields the E-basis

$$\{c_1, \ldots, c_{d_0'}\} \cup \bigcup_{t=0}^{k-k_0-1} \{j^t b_{q,p,i}\}_{q,p,i}$$

of S_k. By scaling the c_v's and the $b_{q,p,i}$'s by elements of $\mathbb{Q}^\times$, we may assume that they are integral (i.e., they lie in $\mathscr{S}_{k_0}$ and $\mathscr{S}_{k_0+1}$, respectively). Then since $j \in \mathscr{M}_1$ is integral, $j^t b_{q,p,i} \in \mathscr{S}_k$ are integral as well. Fixing a $\mathbb{Z}$-basis $\{\alpha_1, \ldots, \alpha_{[E:\mathbb{Q}]}\}$ of the ring of integers

$\mathcal{O}_E$, we get the $\mathbb{Q}$-basis of S_k:

$$\bigcup_{u=1}^{[E:\mathbb{Q}]} \left(\{\alpha_u c_1, \ldots, \alpha_u c_{d_0'}\} \cup \bigcup_{t=0}^{k-k_0-1} \{\alpha_u j^t b_{q,p,i}\}_{q,p,i} \right).$$

Since each $\mathscr{S}_k$ is an $\mathcal{O}_E$-module, the $\alpha_u c_v$ and $\alpha_u j^t b_{q,p,i}$ are integral as well.

Let $\alpha_u^\sigma c_v^\sigma$ and $\alpha_u^\sigma j^t b_{q,p,i}^\sigma$ denote the images of $\alpha_u c_v$ and $\alpha_v j^t b_{q,p,i}$, respectively, in $S_k \otimes_{E,\sigma} \mathbb{C}$. Each $\alpha_u^\sigma c_v^\sigma \Delta^{k_0} \in S_{12k_0}(\Gamma_\sigma)_\mathbb{C}$, and $\alpha_u^\sigma b_{q,p,i}^\sigma \Delta^{k_0+1} \in S_{12(k_0+1)}(\Gamma_\sigma)_\mathbb{C}$. There is a positive constant C_1 such that the estimates

$$\begin{aligned}
\sup_{z_\sigma \in \Gamma_\sigma \backslash \mathfrak{h}} |(\alpha_u^\sigma c_v^\sigma \Delta^{k_0})(z)| &\leq C_1 \\
\sup_{z_\sigma \in \Gamma_\sigma \backslash \mathfrak{h}} |(\alpha_u^\sigma b_{q,p,i}^\sigma \Delta^{k_0+1})(z)| &\leq C_1 \\
|j(z)| &\leq C_1 e^{2\pi y} \\
|\Delta(z)| &\leq C_1 e^{-2\pi y}
\end{aligned}$$

hold for all σ, u, v, and triples (q,p,i). Then

$$\begin{aligned}
\|\alpha_u^\sigma c_v^\sigma \Delta^k\|_{\mathrm{Pet}}^2 &= \frac{1}{d_{\Gamma_\sigma}} \int_{\mathcal{F}_{\Gamma_\sigma}} |(\alpha_u^\sigma c_v^\sigma \Delta^k)(z)|^2 (4\pi y)^{12k} \, \frac{dxdy}{y^2} \\
&\leq \frac{C_1^2}{d_{\Gamma_\sigma}} \int_{\mathcal{F}_{\Gamma_\sigma}} |\Delta^{k-k_0}(z)|^2 (4\pi y)^{12k} \, \frac{dxdy}{y^2} \\
&\leq \frac{C_1^{2(k-k_0)+2}}{d_{\Gamma_\sigma}} \int_{\mathcal{F}_{\Gamma_\sigma}} e^{-4\pi y(k-k_0)} (4\pi y)^{12k} \, \frac{dxdy}{y^2} \\
&\leq \frac{C_1^{2(k-k_0)+2} w}{d_{\Gamma_\sigma}} \int_0^\infty e^{-4\pi y(k-k_0)} (4\pi y)^{12k} \, \frac{dxdy}{y^2} \\
&\leq \left(\frac{k}{k-k_0} \right)^{12k} e^{C_2 k},
\end{aligned}$$

for some constant C_2, and where we pick a connected fundamental domain $\mathcal{F}_{\Gamma_\sigma}$ for Γ_σ contained in a vertical strip of the form $\{(x,y) : \beta \leq x \leq \beta + w, 0 < y\}$ for some $\beta \in \mathbb{R}$, where w is the width of the cusp ∞ for Γ_σ. Similarly, for $t = 0, \ldots, k - k_0 - 2$,

$$
\begin{aligned}
\|\alpha_u^\sigma b_{q,p,i}^\sigma j^t \Delta^k\|_{\mathrm{Pet}}^2 &= \frac{1}{d_{\Gamma_\sigma}} \int_{\mathcal{F}_{\Gamma_\sigma}} |(\alpha_u^\sigma b_{q,p,i}^\sigma j^t \Delta^k)(z)|^2 (4\pi y)^{12k} \ \frac{dxdy}{y^2} \\
&= \frac{1}{d_{\Gamma_\sigma}} \int_{\mathcal{F}_{\Gamma_\sigma}} |(\alpha_u^\sigma b_{q,p,i}^\sigma \Delta^{k_0+1})(z)|^2 |(j^t \Delta^{k-k_0-1})(z)|^2 (4\pi y)^{12k} \ \frac{dxdy}{y^2} \\
&\leq \frac{C_1^2}{d_{\Gamma_\sigma}} \int_{\mathcal{F}_{\Gamma_\sigma}} |(j^t \Delta^{k-k_0-1})(z)|^2 (4\pi y)^{12k} \ \frac{dxdy}{y^2} \\
&\leq \frac{C_1^{2t+2k-2k_0}}{d_{\Gamma_\sigma}} \int_{\mathcal{F}_{\Gamma_\sigma}} e^{-4\pi y(k-k_0-t-1)} (4\pi y)^{12k} \ \frac{dxdy}{y^2} \\
&\leq \frac{C_1^{2t+2k-2k_0} w}{d_{\Gamma_\sigma}} \int_0^\infty e^{-4\pi y(k-k_0-t-1)} (4\pi y)^{12k} \ \frac{dxdy}{y^2} \\
&\leq \left(\frac{k}{k-k_0-t-1} \right)^{12k} e^{C_3 k}
\end{aligned}
$$

for some constant C_3. Then

$$
\begin{aligned}
\|\alpha_u c_v\|_{k,\infty}^2 &= \frac{1}{[E:\mathbb{Q}]} \sum_\sigma \|\alpha_u^\sigma c_v^\sigma \Delta^k\|_{\mathrm{Pet}}^2 \leq \left(\frac{k}{k-k_0} \right)^{12k} e^{C_2 k}, \\
\|\alpha_u b_{q,p,i} j^t\|_{k,\infty}^2 &= \frac{1}{[E:\mathbb{Q}]} \sum_\sigma \|\alpha_u^\sigma b_{q,p,i}^\sigma j^t \Delta^k\|_{\mathrm{Pet}}^2 \leq \left(\frac{k}{k-k_0-t-1} \right)^{12k} e^{C_3 k}.
\end{aligned}
$$

Since $\alpha_u c_v$ and $\alpha_u b_{q,p,i} j^t$ are all integral and hence their p-norms are all at most 1 for every finite place p, we have

$$
\begin{aligned}
\lambda_k(\alpha_u c_v) &\geq -\log \|\alpha_u c_v\|_{k,\infty} \geq 6k \log \left(\frac{k-k_0}{k} \right) - C_4 k, \\
\lambda_k(\alpha_u b_{q,p,i} j^t) &\geq -\log \|\alpha_u b_{q,p,i} j^t\|_{k,\infty} \geq 6k \log \left(\frac{k-k_0-t-1}{k} \right) - C_4 k,
\end{aligned}
$$

for a sufficiently large constant C_4.

Now, let $d_0 = \dim_{\mathbb{Q}} S_{k_0} = [E:\mathbb{Q}] d_0'$. For $1 \leq i \leq d_0$, any i elements of the set

$$\bigcup_{u=1}^{[E:\mathbb{Q}]} \{\alpha_u c_1, \ldots, \alpha_u c_{d_0'}\}$$

are $\mathbb{Q}$-linearly independent. Hence, for $1 \leq i \leq d_0$,

$$\lambda_{k,i} \geq 6k \log\left(\frac{k-k_0}{k}\right) - C_4 k.$$

For $d_0 + t_0 d_\Gamma[E:\mathbb{Q}] + 1 \leq i \leq d_0 + (t_0+1)d_\Gamma[E:\mathbb{Q}]$, where $t_0 = 0, \ldots, k-k_0-2$, we can take the subset

$$\bigcup_{u=1}^{[E:\mathbb{Q}]} \left(\{\alpha_u c_1, \ldots, \alpha_u c_{d_0'}\} \cup \bigcup_{t=0}^{t_0-1} \{\alpha_u j^t b_{q,p,i}\}_{q,p,i} \right) \cup S,$$

where $S \subseteq \bigcup_{u=1}^{[E:\mathbb{Q}]} \{\alpha_u b_{q,p,i} j^{t_0}\}$ is any subset of cardinality $i - d_0 - t_0 d_\Gamma[E:\mathbb{Q}]$. (If $t_0 = 0$, then the set $\bigcup_{t=0}^{t_0-1} \{\alpha_u j^t b_{q,p,i}\}_{q,p,i}$ in the above union should be interpreted as the empty set.) This set of i $\mathbb{Q}$-linearly independent elements shows that for i in the above range,

$$\lambda_{k,i} \geq 6k \log\left(\frac{k-k_0-t_0-1}{k}\right) - C_4 k.$$

Noting that $t_0 + 1 = \left\lceil \frac{i-d_0}{d_\Gamma[E:\mathbb{Q}]} \right\rceil$, we conclude that:

- If $1 \leq i \leq d_0$, then
$$\lambda_{k,i} \geq 6k \log\left(\frac{k-k_0}{k}\right) - C_4 k.$$
- If $d_0 + 1 \leq i \leq d_0 + (k-k_0-1)d_\Gamma[E:\mathbb{Q}]$, then
$$\lambda_{k,i} \geq 6k \log\left(\frac{k-k_0-\left\lceil \frac{i-d_0}{d_\Gamma[E:\mathbb{Q}]} \right\rceil}{k}\right) - C_4 k.$$
- If $i > d_0 + (k-k_0-1)d_\Gamma[E:\mathbb{Q}]$, the above expression still holds if we interpret the right hand side to be $-\infty$.

This gives us the lower bounds in the proposition. $\square$

We now show that the sequence of measures $(\nu_k)_k$ is *uniformly tight*: namely, given any $\varepsilon > 0$, there exists a compact set $K \subseteq \mathbb{R}$ for which $\nu_k(\mathbb{R} \setminus K) \leq \varepsilon$ for all k.

Proposition 4.3.2. *The sequence of measures* $(\nu_k)_k$ *is uniformly tight, and hence* ν *is a probability measure and the vague convergence* $\nu_k \to \nu$ *is in fact weak convergence. Furthermore,* ν *has support bounded above.*

Proof. By proposition 4.1.3, there is a constant C (which we may take to be positive) such that $\lambda_{k,i}/k \leq C$ for all k and all i, meaning that the supports of the measures ν_k are all contained in $(-\infty, C]$. Hence for any positive real number a, $\nu_k(\mathbb{R} \setminus [-a, C]) = \nu_k((-\infty, -a)) + \nu_k((C, \infty)) = \nu_k((-\infty, -a))$. Thus, to show uniform tightness, it suffices to show that for any $\varepsilon > 0$, there is a positive real number a_1 and a positive integer k_1 such that for all reals $a \geq a_1$ and all integers $k \geq k_1$, $\nu_k((-\infty, -a)) < \varepsilon$. (For each $i < k_1$, there is some compact set K_i such that $\nu_i(\mathbb{R} \setminus K_i) < \varepsilon$. Letting $K' = K_1 \cup \cdots \cup K_{k_1 - 1} \cup [-a, C]$, we note that $\nu_k(\mathbb{R} \setminus K') \leq \varepsilon$ for all k.)

We keep the conventions from proposition 4.3.1. Take any $k > k_0$. Given a positive real number a,

$$\nu_k((-\infty, -a)) = \frac{\#\{i : \lambda_{k,i}/k < -a\}}{\dim_{\mathbb{Q}} S_k}.$$

We first restrict to counting only those i with $d_0 + 1 \leq i \leq d_0 + (k - k_0 - 1)d_\Gamma[E : \mathbb{Q}]$, noting that the remaining i's contribute at most $d_0 + d_\Gamma[E : \mathbb{Q}]$ to the count (which is a constant independent of k). Then for i in the above range, proposition 4.3.1 gives

$$6 \log\left(\frac{k - k_0 - \left\lceil \frac{i - d_0}{d_\Gamma[E:\mathbb{Q}]} \right\rceil}{k}\right) - C_4 \leq \lambda_{k,i}/k < -a,$$

which implies that

$$i > d_0 + (k - k_0 - 1 - C_5 k e^{-a/6}) d_\Gamma[E : \mathbb{Q}]$$

for $C_5 = e^{C_4/6}$. Hence,

$$\nu_k((-\infty, -a)) = \frac{\#\{i : \lambda_{k,i}/k < -a\}}{\dim_{\mathbb{Q}} S_k} \leq \frac{(C_5 k e^{-a/6} + 2) d_\Gamma[E : \mathbb{Q}] + d_0 + d_\Gamma[E : \mathbb{Q}]}{\dim_{\mathbb{Q}} S_k}.$$

Noting that $\dim_{\mathbb{Q}} S_k$ grows linearly with k, we may pick k_1 and a_1 large enough so that for any $k \geq k_1$ and $a \geq a_1$, $\nu_k((-\infty, -a)) < \varepsilon$. This concludes the proof of uniform tightness.

To see that ν is a probability measure, we note that by Prohorov's theorem, uniform tightness implies that $(\nu_k)_k$ admits a weakly convergent subsequence $(\nu_{k_m})_m$. If $\nu_{k_m} \to \omega$ weakly, then ω is also the vague limit of the ν_{k_m}. Then by uniqueness of vague limits, we conclude that $\omega = \nu$. Since ω is a probability measure, we conclude that ν is one as well. It is a standard result that if the limit measure is also a probability measure, then vague convergence is equivalent to weak convergence, and hence ν_k converges weakly to ν. Finally, it is clear that the support of ν is contained in $(-\infty, C]$. $\square$

CHAPTER 5

Comparison of measures - Proof of theorem 2.0.2

5.1. Notation

Given finite index subgroups $\Gamma' \subseteq \Gamma$ of $\Gamma(1)$, we say that $\Gamma' \subseteq \Gamma$ *is defined over* E, where E is a number field, if there exist smooth projective geometrically connected E-curves $X(\Gamma')$ and $X(\Gamma)$ such that:

- the base change of $X(\Gamma')$ and $X(\Gamma)$ to $\mathbb{C}$ give the modular curves $X(\Gamma')_{\mathbb{C}}$ and $X(\Gamma)_{\mathbb{C}}$, respectively, and
- there exist E-morphisms $\pi : X(\Gamma') \to X(\Gamma)$ and $\pi_\Gamma : X(\Gamma) \to \mathbb{P}^1_E$ with base change to $\mathbb{C}$ equal to the natural maps $X(\Gamma')_{\mathbb{C}} \to X(\Gamma)_{\mathbb{C}}$ and $X(\Gamma)_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ induced by the inclusion $\Gamma' \subseteq \Gamma$, and by the j-function, respectively.

In particular, there is also an E-morphism $\pi_\Gamma \circ \pi : X(\Gamma') \to \mathbb{P}^1_E$ with base change to $\mathbb{C}$ equal to the natural map $X(\Gamma')_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ given by the j-function.

Given a finite index subgroup $\Gamma \subseteq \Gamma(1)$, we also say that Γ *is defined over* E to mean that $\Gamma \subseteq \Gamma(1)$ is defined over E. In this case, the above conditions are equivalent to the existence of models $X(\Gamma)$ and $X(\Gamma) \to \mathbb{P}^1_E$ over E of $X(\Gamma)_{\mathbb{C}}$ and the j-function map $X(\Gamma)_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$, respectively. Note that if $\Gamma' \subseteq \Gamma$ is defined over E, then so are Γ' and Γ.

For the rest of this section, suppose that $\Gamma' \subseteq \Gamma$ is defined over a number field E, and let $X(\Gamma'), X(\Gamma)$, and $\pi : X(\Gamma') \to X(\Gamma)$ be as above. As in §3.3, let $X(\Gamma)_{\mathbb{Z}}$ denote the normalization of $\mathbb{P}^1_{\mathbb{Z}}$ in $X(\Gamma)$ under the natural map $X(\Gamma) \to \mathbb{P}^1_E \to \mathbb{P}^1_{\mathbb{Z}}$, and define $X(\Gamma')_{\mathbb{Z}}$ analogously. The natural map $\pi : X(\Gamma') \to X(\Gamma)$ induces a morphism $X(\Gamma')_{\mathbb{Z}} \to X(\Gamma)_{\mathbb{Z}}$ (over $\mathbb{P}^1_{\mathbb{Z}}$). For any choice of a desingularization $\mathscr{X}(\Gamma) \to X(\Gamma)_{\mathbb{Z}}$, there exists a desingularization $\mathscr{X}(\Gamma') \to X(\Gamma')_{\mathbb{Z}}$ along with a morphism $\pi_{\mathbb{Z}} : \mathscr{X}(\Gamma') \to \mathscr{X}(\Gamma)$ that extends $X(\Gamma')_{\mathbb{Z}} \to X(\Gamma)_{\mathbb{Z}}$.

Let $\mathscr{L}' := \pi_{\mathbb{Z}}^* \mathscr{L}$, where $\mathscr{L} = \pi_{\Gamma,\mathbb{Z}}^* \mathcal{O}_{\mathbb{P}^1_{\mathbb{Z}}}(\overline{\infty})$ (see §3.3). Keeping the same conventions as in §3.6, we let

$$\begin{aligned}
\mathscr{M}_k &:= H^0(\mathscr{X}(\Gamma), \mathscr{L}^{\otimes k}), & \mathscr{M}'_k &:= H^0(\mathscr{X}(\Gamma'), \mathscr{L}'^{\otimes k}) \\
M_k &:= \mathscr{M}_k \otimes_{\mathbb{Z}} \mathbb{Q}, & M'_k &:= \mathscr{M}'_k \otimes_{\mathbb{Z}} \mathbb{Q} \\
S_k &:= H^0(X(\Gamma), \mathscr{L}_{\mathbb{Q}}^{\otimes k}(-D)), & S'_k &:= H^0(X(\Gamma'), \mathscr{L}'^{\otimes k}_{\mathbb{Q}}(-D')) \\
\mathscr{S}_k &:= \mathscr{M}_k \cap S_k, & \mathscr{S}'_k &:= \mathscr{M}'_k \cap S'_k,
\end{aligned}$$

where D and D' are the formal sums of the pre-images of $\infty \in \mathbb{P}^1_E$ in $X(\Gamma)$ and $X(\Gamma')$ under the natural maps $X(\Gamma) \to \mathbb{P}^1_E$ and $X(\Gamma') \to \mathbb{P}^1_E$, respectively. We remark that these definitions are independent of the choices of the regular models $\mathscr{X}(\Gamma)$ and $\mathscr{X}(\Gamma')$.

Given an extension F/E of number fields, let $X(\Gamma)_F := X(\Gamma) \otimes_E F$ denote the base change, and let $X(\Gamma)_{F,\mathbb{Z}}$ denote the normalization of $\mathbb{P}^1_{\mathbb{Z}}$ under the natural map $X(\Gamma)_F \to \mathbb{P}^1_{\mathbb{Z}}$. The map $X(\Gamma)_F \to X(\Gamma)$ extends to a map between the normalizations $X(\Gamma)_{F,\mathbb{Z}} \to X(\Gamma)_{\mathbb{Z}}$. Next, let $\mathscr{X}(\Gamma)_F$ be a desingularization of $X(\Gamma)_{F,\mathbb{Z}}$ over F that admits a morphism to $\mathscr{X}(\Gamma)$ extending the natural map $X(\Gamma)_{\mathbb{Z},F} \to X(\Gamma)_{\mathbb{Z}}$. Let $\mathscr{M}_{F,k}$ denote the global sections of the pullback of $\mathscr{L}^{\otimes k}$ to $\mathscr{X}(\Gamma)_F$, and define $M_{F,k}, S_{F,k}$, and $\mathscr{S}_{F,k}$ in the obvious manner.

For any place v of $\mathbb{Q}$, we let $\|\cdot\|_{k,v}, \|\cdot\|'_{k,v}$, and $\|\cdot\|_{F,k,v}$ denote the local norms on $S_k \otimes_{\mathbb{Q}} \mathbb{C}_v, S'_k \otimes_{\mathbb{Q}} \mathbb{C}_v$, and $S_{F,k} \otimes_{\mathbb{Q}} \mathbb{C}_v$, respectively. Finally, let λ_k, λ'_k, and $\lambda_{F,k}$ be the naive adelic height functions on S_k, S'_k, and $S_{F,k}$, respectively.

We adopt the convention that we drop the subscript E for objects defined over E (as in §3.3), but include the subscript for objects over subfields or field extensions of E.

5.2. Main results

Lemma 5.2.1. *Let E be a number field, X and Y smooth projective integral curves over E, and $\pi : Y \to X$ a non-constant map of curves over E. Suppose $\mathscr{X}$ and $\mathscr{Y}$ are regular projective models of X and Y, respectively, over Spec$(\mathcal{O}_E)$, and let $\pi_{\mathbb{Z}} : \mathscr{Y} \to \mathscr{X}$ be a*

Spec$(\mathcal{O}_E)$*-morphism extending* π*. For any line bundle* $\mathscr{L}$ *on* $\mathscr{X}$ *with* $L := \mathscr{L}|_X$*, we have the equality*

$$H^0(\mathscr{X}, \mathscr{L}) = H^0(X, L) \cap H^0(\mathscr{Y}, \pi_{\mathbb{Z}}^* \mathscr{L}),$$

where the intersection takes place in $H^0(Y, \pi^* L)$*.*

Proof. First let $\mathscr{X}' \xrightarrow{\eta} \mathscr{X}$ denote the normalization of $\mathscr{X}$ in Y with respect to the map $Y \to X \to \mathscr{X}$. Then $\mathscr{X}'$ is a normal arithmetic surface, which admits a birational map $\mathscr{Y} \dashrightarrow \mathscr{X}'$ (inducing the identity on the generic fiber Y). By (Liu, 2002) chapter 9.2, theorem 2.7, there is a projective birational morphism $\mathscr{Y}' \xrightarrow{\alpha} \mathscr{Y}$, with $\mathscr{Y}'$ regular, and a birational morphism $\mathscr{Y}' \xrightarrow{\beta} \mathscr{X}'$ lifting $\mathscr{Y} \dashrightarrow \mathscr{X}'$. The natural pullback morphisms induce the equality $H^0(\mathscr{X}', \eta^* \mathscr{L}) = H^0(\mathscr{Y}', \beta^* \eta^* \mathscr{L}) = H^0(\mathscr{Y}, \pi_{\mathbb{Z}}^* \mathscr{L})$ (since $\eta \circ \beta = \pi_{\mathbb{Z}} \circ \alpha$) in $H^0(Y, \pi^* L)$. Hence, it suffices to show that $H^0(\mathscr{X}, \mathscr{L}) = H^0(X, L) \cap H^0(\mathscr{X}', \eta^* \mathscr{L})$.

We only need to show $\supseteq$. Take $s \in H^0(X, L) \cap H^0(\mathscr{X}', \eta^* \mathscr{L})$. Let $\mathrm{Spec}(A) \subseteq \mathscr{X}$ be an affine open subset where $\mathscr{L}$ is trivial. Then L is trivial on $A \otimes_{\mathbb{Z}} \mathbb{Q}$, and $\eta^* \mathscr{L}$ is trivial on $\eta^{-1}(\mathrm{Spec}(A)) = \mathrm{Spec}(A')$, where A' is the integral closure of A in the rational function field $\kappa(Y)$. The section s then corresponds to an element in $A' \cap (A \otimes_{\mathbb{Z}} \mathbb{Q}) = A \subseteq \kappa(X)$, since A is integrally closed in its fraction field $\kappa(X)$. □

Lemma 5.2.2. *Let* $\Gamma' \subseteq \Gamma$ *be finite index subgroups of* $\Gamma(1)$ *defined over a number field* E*, and let* F/E *be a finite extension. Then with the notation in §5.1, the local norms respect the inclusions*

(a) $S_k \otimes_{\mathbb{Q}} \mathbb{C}_v \to S'_k \otimes_{\mathbb{Q}} \mathbb{C}_v$*, and*

(b) $S_k \otimes_{\mathbb{Q}} \mathbb{C}_v \to S_{F,k} \otimes_{\mathbb{Q}} \mathbb{C}_v$

for all places v *of* $\mathbb{Q}$*.*

Proof. First let $v = p$ be a finite place. The inclusion of finite free $\mathbb{Z}$-modules $\mathscr{M}_k \subseteq \mathscr{M}'_k$ implies that there exists a $\mathbb{Z}$-basis $\{b_1, \ldots, b_{d'}\}$ of $\mathscr{M}'_k$ for which there exist integers $n_1, \ldots, n_d$

such that $n_1b_1, \dots, n_db_d$ is a $\mathbb{Z}$-basis for $\mathscr{M}_k$. Since $\mathscr{M}_k = M_k \cap \mathscr{M}'_k$ by lemma 5.2.1, we must have $b_i \in \mathscr{M}_k$, and hence $n_i = 1$ for all $i = 1, \dots, d$. Given $s \in M_k \otimes_{\mathbb{Q}} \mathbb{C}_p$, if $s = \sum_{i=1}^d \alpha_i b_i$ with $\alpha_i \in \mathbb{C}_p$, then $\|s\|_{k,p} = \|s\|'_{k,p} = \max\{|\alpha_i|\}_{i=1}^d$, by the definition of the local norms at p. A similar argument also shows that $\|s\|_{k,p} = \|s\|_{F,k,p}$.

Now suppose $v = \infty$. For any $\sigma : E \to \mathbb{C}$, the base change of $\pi : X(\Gamma') \to X(\Gamma)$ by σ gives $\pi_\sigma : X(\Gamma'_\sigma)_{\mathbb{C}} \to X(\Gamma_\sigma)_{\mathbb{C}}$ for finite index subgroups $\Gamma'_\sigma \subseteq \Gamma_\sigma \subseteq \Gamma(1)$. Now given any $f = (f_\sigma)_\sigma \in S_k \otimes_{\mathbb{Q}} \mathbb{C} \subseteq S'_k \otimes_{\mathbb{Q}} \mathbb{C}$, the Petersson norm of the cusp form $f_\sigma \Delta^k \in S_{12k}(\Gamma_\sigma)_{\mathbb{C}} \subseteq S_{12k}(\Gamma'_\sigma)_{\mathbb{C}}$ is independent of the groups Γ_σ and Γ'_σ, we get $\|f\|_{k,\infty} = \|f\|'_{k,\infty}$.

Finally, we note that for f as above,

$$\begin{aligned}
\|f\|^2_{F,k,\infty} &= \frac{1}{[F:\mathbb{Q}]} \sum_{\tau : F \to \mathbb{C}} \|f^\tau \Delta^k\|^2_{\mathrm{Pet}} \\
&= \frac{1}{[F:\mathbb{Q}]} \sum_{\sigma : E \to \mathbb{C}} \sum_{\substack{\tau \\ \tau|_E = \sigma}} \|f^\tau \Delta^k\|^2_{\mathrm{Pet}} \\
&= \frac{1}{[E:\mathbb{Q}]} \sum_{\sigma : E \to \mathbb{C}} \|f^\sigma \Delta^k\|^2_{\mathrm{Pet}} = \|f\|^2_{k,\infty}.
\end{aligned}$$

$\square$

Now suppose $\Gamma' \trianglelefteq \Gamma$ is normal, and that $\pi : X(\Gamma') \to X(\Gamma)$ is Galois over E. Since $X(\Gamma')_{\mathbb{Z}}$ is also the normalization of $X(\Gamma)_{\mathbb{Z}}$ in $X(\Gamma')$, the universal property of normalization implies that every element $\tau \in \mathrm{Aut}(X(\Gamma')/X(\Gamma))$ lifts uniquely to an element $\widetilde{\tau} \in \mathrm{Aut}(X(\Gamma')_{\mathbb{Z}}/X(\Gamma)_{\mathbb{Z}})$. Let the elements of $\mathrm{Aut}(X(\Gamma')/X(\Gamma))$ be denoted τ_j, and let $(\cdot)|_{\tau_j}$ denote the pullback map on sections induced by τ_j.

Suppose also that F/E is a Galois extension of number fields. Since $X(\Gamma)$ is geometrically integral over E, $\mathrm{Gal}(X(\Gamma)_F/X(\Gamma)) = \mathrm{Gal}(F/E)^{\mathrm{op}}$ (where the "op" means that we take the opposite group). For $\alpha_j \in \mathrm{Gal}(F/E)$, let α_j^* denote the corresponding element of $\mathrm{Gal}(X(\Gamma)_F/X(\Gamma))$. Again by the universal property of normalization, α_j^* extends uniquely to an automorphism $\mathrm{Aut}(X(\Gamma)_{\mathbb{Z},F}/X(\Gamma)_{\mathbb{Z}})$. We denote by $(\cdot)^{\alpha_j}$ the pullback map on sections induced by α_j^*.

Lemma 5.2.3. *Let $\Gamma' \trianglelefteq \Gamma$ be defined over E and suppose that $\pi : X(\Gamma') \to X(\Gamma)$ is Galois over E. Let F/E be a Galois extension of number fields. Then with the conventions of §5.1, we have*

(a) $S_k^a \subseteq S_k'^a \cap S_k \subseteq S_k^{a-\log[\Gamma':\Gamma]}$, *and*

(b) $S_k^a \subseteq S_{F,k}^a \cap S_k \subseteq S_k^{a-\log[F:E]}$

for all $a \in \mathbb{R}$.

Proof. The first containment for both (a) and (b) is the content of lemma 5.2.2. For the second containment, take any $f \in S_k'^a \cap S_k$ (resp. $s \in S_{F,k}^a \cap S_k$). Then $f = \sum_i g_i$ for $g_i \in S_k'$ with $\lambda_k'(g_i) \geq a$ (resp. $s = \sum_i t_i$ for $t_i \in S_{F,k}$ with $\lambda_{F,k}(t_i) \geq a$). Let $h_i := \sum_j g_i|_{\tau_j}$ (resp. $u_i := \sum_j t_i^{\alpha_j}$). Then we claim that $h_i \in S_k$ with $\lambda_k(h_i) \geq a - \log[\Gamma' : \Gamma]$ (resp. $u_i \in S_k$ with $\lambda_k(u_i) \geq a - \log[F : E]$) for all i, from which the conclusion follows.

First suppose $v = p$ is a finite place. Given any $\mathbb{Z}$-basis $\{b_k\}$ of $\mathscr{M}_k'$, if $g_i = \sum_k \alpha_k b_k$ with $\alpha_k \in \mathbb{Q}$, then $\|g_i\|_{k,p}' = \max\{|\alpha_k|\}_k$. Now since τ_j extends uniquely to $\mathrm{Aut}(X(\Gamma')_{\mathbb{Z}}/X(\Gamma)_{\mathbb{Z}})$, the pullback map $(\cdot)|_{\tau_j} : M_k' \to M_k'$ restricts to an automorphism of the integral sections $\mathscr{M}_k'$ (see the remarks in §3.3). Hence $\{b_k|_{\tau_j}\}$ is also a $\mathbb{Z}$-basis of $\mathscr{M}_k'$, from which we conclude that $\|g_i|_{\tau_j}\|_{k,p}' = \|g_i\|_{k,p}'$ for all τ_j. Hence,

$$\|h_i\|_{k,p} = \|h_i\|_{k,p}' = \left\| \sum_j g_i|_{\tau_j} \right\|_{k,p}' \leq \max_j \|g_i|_{\tau_j}\|_{k,p}' = \|g_i\|_{k,p}'.$$

A similar argument shows that $\|u_i\|_{k,p} \leq \|t_i\|_{F,k,p}$.

Next, suppose $v = \infty$. First, we address (a). We have the base change diagram by $\sigma : E \to \mathbb{C}$

$$\begin{tikzcd} X(\Gamma'_\sigma)_{\mathbb{C}} \arrow[r, "\gamma_j"] \arrow[d] & X(\Gamma'_\sigma)_{\mathbb{C}} \arrow[d] \\ X(\Gamma') \arrow[r, "\tau_j"] & X(\Gamma') \end{tikzcd}$$

where the top horizontal map $\gamma_j \in \mathrm{Aut}(X(\Gamma'_\sigma)_{\mathbb{C}}/X(\Gamma_\sigma)_{\mathbb{C}})$ corresponds to the automorphism of the modular curve $X(\Gamma'_\sigma)_{\mathbb{C}}$ induced by the coset $\gamma_j\Gamma' \in \Gamma/\Gamma'$ for some $\gamma_j \in \Gamma$. Hence for all i and j, we have $(g_i|_{\tau_j})^\sigma = g_i^\sigma|_{\gamma_j}$ (where $|_{\gamma_j}$ denotes the usual slash operator for modular forms, which in this case is simply pre-composition by γ_j). Then we have

$$\begin{aligned}
\|h_i\|^2_{k,\infty} = \|h_i\|'^2_{k,\infty} &= \frac{1}{[E:\mathbb{Q}]} \sum_{\sigma:E\to\mathbb{C}} \|h_i^\sigma \Delta^k\|^2_{\mathrm{Pet}} \\
&\leq \frac{1}{[E:\mathbb{Q}]} \frac{1}{d_{\Gamma'}} \frac{d_{\Gamma'}}{d_\Gamma} \sum_\sigma \int_{\mathcal{F}_{\Gamma'}} \sum_j \left|((g_i|_{\tau_j})^\sigma \Delta^k)(z)\right|^2 (4\pi y)^{12k} \, \frac{dxdy}{y^2} \\
&= \frac{1}{[E:\mathbb{Q}]} \frac{1}{d_{\Gamma'}} \frac{d_{\Gamma'}}{d_\Gamma} \sum_\sigma \int_{\mathcal{F}_{\Gamma'}} \sum_j \left|((g_i^\sigma|_{\gamma_j})\Delta^k)(z)\right|^2 (4\pi y)^{12k} \, \frac{dxdy}{y^2} \\
&= \frac{1}{[E:\mathbb{Q}]} \frac{d_{\Gamma'}}{d_\Gamma} \sum_j \sum_\sigma \|g_i^\sigma \Delta^k\|^2_{\mathrm{Pet}} \\
&= \left(\frac{d_{\Gamma'}}{d_\Gamma}\right)^2 \|g_i\|'^2_{k,\infty}.
\end{aligned}$$

where we use Cauchy-Schwartz inequality in the second line, and the invariance of the Petersson inner product with respect to the slash operator for modular forms in the fourth line. We conclude that

$$\lambda_k(h_i) = -\sum_v \log \|h_i\|_{k,v} \geq -\sum_v \log \|g_i\|'_{k,v} - \log\left(\frac{d_{\Gamma'}}{d_\Gamma}\right) = \lambda'_k(g_i) - \log[\Gamma' : \Gamma].$$

Finally, we address (b). For each $\sigma : E \to \mathbb{C}$, fix a lift $\widetilde{\sigma} : F \to \mathbb{C}$. Then all lifts of σ to F

are given by $\widetilde{\sigma} \circ \alpha_j$ as α_j varies over $\mathrm{Gal}(F/E)$.

$$\begin{aligned}
\|u_i\|_{k,\infty}^2 = \frac{1}{[E:\mathbb{Q}]} \sum_{\sigma: E \to \mathbb{C}} \|u_i^\sigma \Delta^k\|_{\mathrm{Pet}}^2 &= \frac{1}{[E:\mathbb{Q}]} \sum_{\sigma: E \to \mathbb{C}} \|u_i^{\widetilde{\sigma}} \Delta^k\|_{\mathrm{Pet}}^2 \\
&= \frac{1}{[E:\mathbb{Q}]} \sum_{\sigma: E \to \mathbb{C}} \left\| \sum_j (t_i^{\alpha_j})^{\widetilde{\sigma}} \Delta^k \right\|_{\mathrm{Pet}}^2 \\
&\le \frac{[F:E]}{[E:\mathbb{Q}]} \sum_{\sigma: E \to \mathbb{C}} \sum_j \left\| t_i^{\widetilde{\sigma} \circ \alpha_j} \Delta^k \right\|_{\mathrm{Pet}}^2 \\
&= [F:E]^2 \frac{1}{[F:\mathbb{Q}]} \sum_{\tau: F \to \mathbb{C}} \left\| t_i^\tau \Delta^k \right\|_{\mathrm{Pet}}^2 \\
&= [F:E]^2 \|t_i\|_{F,k,\infty}^2.
\end{aligned}$$

Hence, we conclude that

$$\lambda_k(u_i) \ge \lambda_{F,k}(t_i) - \log[F:E],$$

as required. $\square$

5.3. Proof of theorem 2.0.2

Proof of theorem 2.0.2. Let E and E_0 be the fields of constants of $X(\Gamma')$ and $X(\Gamma)$, respectively. Taking global sections of the structure sheaves for the morphism $\pi_{\Gamma',\Gamma} : X(\Gamma') \to X(\Gamma)$ yields an inclusion $E_0 \hookrightarrow E$, and hence an E-morphism $X(\Gamma') \to X(\Gamma) \otimes_{E_0} E$ such that its base change to $\mathbb{C}$ is equal to the natural map $X(\Gamma')_{\mathbb{C}} \to X(\Gamma)_{\mathbb{C}}$ by assumption. Hence, $\Gamma' \subseteq \Gamma$ is defined over E.

Let Γ'' be a finite index subgroup of Γ' with $\Gamma'' \trianglelefteq \Gamma$. Let F be a finite extension of E that is Galois over E_0, and suppose that Γ'' is defined over F, with model $X(\Gamma'')_F$. By extending F if necessary, we may also assume that there exist F-morphisms $X(\Gamma'')_F \to X(\Gamma')_F \to X(\Gamma)_F$ with base change to $\mathbb{C}$ equal to the natural maps $X(\Gamma'')_{\mathbb{C}} \to X(\Gamma')_{\mathbb{C}} \to X(\Gamma)_{\mathbb{C}}$, and that the maps $X(\Gamma'')_F \to X(\Gamma')_F$ and $X(\Gamma'')_F \to X(\Gamma)_F$ are Galois covers of curves. Then repeated application of lemma 5.2.3 gives

$$S^a_{E_0,k} \subseteq S^a_k \cap S_{E_0,k} \subseteq S'^a_k \cap S_{E_0,k} \subseteq S'^a_{F,k} \cap S_{E_0,k}$$
$$\subseteq S''^a_{F,k} \cap S_{E_0,k} \subseteq S^{a-\log[\Gamma'':\Gamma]}_{F,k} \cap S_{E_0,k} \subseteq S^{a-\log[\Gamma'':\Gamma]-\log[F:E_0]}_{E_0,k}.$$

In particular, we have

$$S^a_{E_0,k} \subseteq S'^a_k \cap S_{E_0,k} \subseteq S^{a-\log[\Gamma'':\Gamma]-\log[F:E_0]}_{E_0,k}. \tag{5.3.1}$$

Let $\lambda'_{E_0,k,i}$ denote the successive maxima of $S_{E_0,k}$ with respect to the subspace filtration $S'^a_k \cap S_{E_0,k}$, and let

$$\nu'_{E_0,k} := \frac{1}{\dim_{\mathbb{Q}} S_{E_0,k}} \sum_{i=1}^{\dim_{\mathbb{Q}} S_{E_0,k}} \delta_{\frac{1}{k}\lambda'_{E_0,k,i}}.$$

Equation 5.3.1 gives

$$\lambda_{E_0,k,i} \leq \lambda'_{E_0,k,i} \leq \lambda_{E_0,k,i} + \log[\Gamma'' : \Gamma] + \log[F : E_0]$$

for all $i = 1, \ldots, \dim_{\mathbb{Q}} S_{E_0,k}$. Consequently, $\nu'_{E_0,k} \to \nu_{E_0}$ weakly.

Consider now the short exact sequence

$$0 \to S_{E_0,k} \to S'_k \to W_k := S'_k / S_{E_0,k} \to 0.$$

If we equip W_k with the quotient filtration as on page 16 of (Chen, 2010), then by proposition 1.2.5 of (Chen, 2010), there is a Borel probability measure ω_k on $\mathbb{R}$ such that

$$(\dim_{\mathbb{Q}} S'_k) \cdot \nu'_k = (\dim_{\mathbb{Q}} S_{E_0,k}) \cdot \nu'_{E_0,k} + (\dim_{\mathbb{Q}} W_k) \cdot \omega_k,$$

and hence

$$\omega_k = \frac{\dim_{\mathbb{Q}} S'_k}{\dim_{\mathbb{Q}} W_k} \cdot \nu'_k - \frac{\dim_{\mathbb{Q}} S_{E_0,k}}{\dim_{\mathbb{Q}} W_k} \cdot \nu'_{E_0,k}.$$

Taking the limit as $k \to \infty$, we get that ω_k converges weakly to the Borel probability measure

$$\omega := \frac{[E : E_0][\Gamma : \Gamma']}{[E : E_0][\Gamma : \Gamma'] - 1} \cdot \nu' - \frac{1}{[E : E_0][\Gamma : \Gamma'] - 1} \cdot \nu_{E_0}.$$

Rearranging, we get

$$\nu' = \frac{1}{[E : E_0][\Gamma : \Gamma']} \cdot \nu_{E_0} + \left(1 - \frac{1}{[E : E_0][\Gamma : \Gamma']}\right) \cdot \omega.$$

Finally, since $[E : E_0][\Gamma : \Gamma'] = \deg(\pi_{\Gamma',\Gamma})$, we get the desired conclusion. $\square$

Corollary 5.3.1. *Assume the setup in theorem 2.0.1. Then the support of the limit measure ν is bounded above and unbounded below.*

Proof. That the support of ν is bounded above follows from lemma 4.1.3. The modular curve $X(1)_{\mathbb{C}}$ for $\Gamma(1)$ has a model $X(1)_{\mathbb{Q}}$ over $\mathbb{Q}$ that we identify with $\mathbb{P}^1_{\mathbb{Q}}$ via the j-function. We have the morphism $\pi_\Gamma : X(\Gamma) \to \mathbb{P}^1_{\mathbb{Q}}$ associated to the inclusion $\Gamma \subseteq \Gamma(1)$, and hence we may apply theorem 2.0.2 to get

$$\nu = \frac{1}{\deg(\pi_\Gamma)} \cdot \nu_{\Gamma(1),\mathbb{Q}} + \left(1 - \frac{1}{\deg(\pi_\Gamma)}\right) \cdot \omega,$$

where $\nu_{\Gamma(1),\mathbb{Q}}$ is the limit measure associated to the successive maxima of the spaces of $\mathbb{Q}$-rational cusp forms of level $\Gamma(1)$ and weight $12k$ as in theorem 3.2.2 of (Chinburg et al., 2018), and ω is a Borel probability measure on $\mathbb{R}$. By part (ii) of the same theorem, the support of $\nu_{\Gamma(1),\mathbb{Q}}$ is not bounded below. Consequently, the support of ν is unbounded below as well. $\square$

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