ANCIENT SOLUTIONS OF THE RICCI FLOW ON COMPACT HOMOGENEOUS SPACES

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ABSTRACT

ANCIENT SOLUTIONS OF THE RICCI FLOW ON COMPACT HOMOGENEOUS SPACES

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We investigate the behaviour of the Ricci flow for homogeneous metrics on spheres and on general compact homogeneous spaces. In particular we complete the classification of ancient homogeneous solutions on spheres and discover a new 1-parameter family of ancient solutions. These solutions can be described in terms of shrinking the fibers of the Hopf fibration $S^1 \to S^{4n+3} \to \mathbb{CP}^{2n+1}$ while varying the metric on the $\mathbb{CP}^{2n+1}$ base. Precisely one solution collapses along the backwards flow to the Fubini-Study metric while the rest collapse to Ziller's second Einstein metric on $\mathbb{CP}^{2n+1}$. We then proceed to determine a general criterion for the existence of collapsed ancient solutions on compact homogeneous spaces. In particular, we show that whenever G/H is the total space of a homogeneous fibration $T^n \to G/H \to G/K$ where T^n is a maximal torus in a compact complement of H in $N_G(H)$, then for every Einstein metric on the base G/K there exists a family of ancient solutions on G/H which collapse to the given Einstein metric under the backwards flow. This construction generalizes all previously known examples of collapsed homogeneous ancient solutions in the literature, and also leads to many new families of examples.

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CHAPTER 1

Introduction

Hamilton's Ricci flow is given by the geometric PDE

$$\frac{\partial}{\partial t} g_t = -2\mathrm{Ric}(g_t).$$

Up to a time-dependent family of diffeomorphisms, the Ricci flow is equivalent to a parabolic PDE, and so similar to the heat equation, the Ricci flow has regularizing properties for Riemannian metrics, making it useful for proving classification-type theorems in geometry. In general, the hope is that one can run the Ricci flow to simplify the metric of a Riemannian manifold so that one can eventually conclude information about its topology. It was used, for example, to prove that simply connected 3-manifolds with positive Ricci curvature are spheres [25], as well as simply connected n-manifolds with positive curvature operators [14] and those with quarter-pinched metrics [17].

Typically the Ricci flow develops singularities in finite time, and hence for many applications one must perform surgeries along the flow. That is, one must excise neighborhoods of the singularities and then reattach suitable components. The Ricci flow with surgery was used in Perelman's celebrated proof of the Poincaré conjecture [33], and was also used in recent work by S. Brendle to classify manifolds with positive isotropic curvature in dimensions $n \geq 12$ [16]. Of course, in order to perform surgeries it is important to understand the geometry near a singularity.

The only way a finite-time singularity can develop is if there is a sequence of points $x_i \in M^n$ and times $t_i \to T$ such that $K_i := |\mathrm{Rm}(g(t_i))(x_i, t_i)| \to \infty$, where T is the singular time and $|\mathrm{Rm}(g(t_i))(x_i, t_i)|$ is the norm of the curvature tensor of $g(t_i)$ evaluated at x_i [23]. As a consequence of Hamilton's compactness theorem, one can always pick such a sequence of points and times so that the sequence of parabolically-rescaled solutions

$$g_i(t) := K_i g\left(t_i + \frac{t}{K_i}\right)$$

subconverges (in a suitable sense) to a limit solution $g_\infty(t)$, called a singularity model, on a complete n-dimensional manifold M^n_∞ defined on the time interval $(-\infty, 0]$ (e.g. cite CLN). Solutions defined on $(-\infty, 0]$ are called ancient, and hence, by the above, ancient solutions are important for understanding the Ricci flow near its singular times.

Similar to the heat equation, the backwards Ricci flow is generally ill-posed and hence ancient solutions are rare. On a homogeneous space $M^n = \mathsf{G}/\mathsf{H}$ however, the Ricci flow is equivalent to an ODE and in particular the backwards flow exists for short time intervals. In fact if G/H is compact, then the normalized Ricci flow, which preserves volume and is equivalent to the Ricci flow up to rescaling and reparametrization, is the gradient flow for the scalar curvature functional on the space of homogeneous metrics and fixed points of the flow are precisely the homogeneous Einstein metrics. The global behaviour of the scalar curvature functional on the space of homogeneous metrics was studied in [13] and is closely related to the structure of the set of intermediate subgroups between H and G.

From now on we will let $M^n = \mathsf{G}/\mathsf{H}$ be a compact homogeneous space. Since fixed points of the normalized flow are homogeneous Einstein metrics, it turns out that non-collapsed ancient solutions exist whenever there M admits an unstable G-invariant Einstein metric (see [12]). On the other hand, collapsed solutions have been constructed on a case by case basis and up til now there have been few general existence theorems.

We begin this thesis by studying the behaviour of the Ricci flow on homogeneous spheres as in [44]. Besides the left-invariant metrics on S^3, homogeneous metrics on spheres can be described in terms of the Hopf fibrations [48]

$$S^1 \to S^{2n+1} \to \mathbb{CP}^n \qquad S^7 \to S^{15} \to S^8 \qquad S^3 \to S^{4n+3} \to \mathbb{HP}^n. \tag{1.0.1}$$

Associated to each fibration, there is a 2-parameter family of homogeneous metrics $g_{t,s}$ which can be obtained by starting with the round metric and scaling the fiber by t and the base by s. The behaviour of the Ricci flow for these metrics was studied in [21] and [6], and we indicate their behaviour in Chapter 2.1. See also [26] and [22] for the case of left-invariant metrics on S^3. There exists, however, a larger class of homogeneous metrics associated to the fibration $S^3 \to S^{4n+3} \to \mathbb{HP}^n$ by allowing the metric on S^3 to be an arbitrary left-invariant metric. It can be shown that, up to isometry, this left-invariant metric can be diagonalized with eigenvalues $x \leq y \leq z$. Hence, we obtain a 4-parameter family of metrics, which we denote by $g_{x,y,z,s}$. This is the only family of homogeneous metrics on spheres for which the Ricci flow has not yet been studied, and is the main object of the Chapter 2 and [44].

First we determine the forwards behaviour of solutions. Then we classify the ancient solutions and exhibit a new one-parameter family. The main theorem of Chapter 2 is the following.

Theorem A. *Let g_t be an $\mathrm{Sp}(n+1)$-invariant solution of the Ricci flow or the normalized Ricci flow on S^{4n+3} with initial condition $g_0 = g_{x,y,z,s}$, where $x \leq y \leq z$. Then g_t is ancient if and only if $x \leq y = z \leq s$.*

These (non-isometric) ancient solutions all have positive sectional curvature and a larger isometry group, namely $\mathrm{Sp}(n+1)\mathrm{Sp}(1)$, $\mathrm{Sp}(n+1)\,\mathrm{U}(1)$, or $\mathrm{U}(2n+2)$ (see Lemma 2.3.1). Two ancient solutions converge, under the backwards flow, to Jensen's second Einstein metric and are non-collapsed (see [6]). All the remaining ones are collapsed and can be viewed as shrinking the fiber of the Hopf fibration $S^1 \to S^{4n+3} \to \mathbb{CP}^{2n+1}$ and simultaneously letting the metric on the base vary. One solution parametrizes the well known Berger metrics and, under a rescaling of the backwards flow, converges in the Gromov-Hausdorff topology to the Fubini-Study metric on $\mathbb{CP}^{2n+1}$. The rest of the solutions are new. Under a rescaling of the backwards flow, these solutions converge to Ziller's second homogeneous Einstein metric on $\mathbb{CP}^{2n+1}$ [48]. Similar to the ancient solutions found in [12], the limit solitons do not depend continuously on the initial metric. Figure 1.1 illustrates the behaviour of the backwards flow for the volume-normalized solutions, which can be obtained by setting $x = (y^2 s^{4n})^{-1}$. The y-axis represents the ratio y/s and the x-axis represents the value of s. See also Figure 2.1 and Figure 2.2 for the graph of the scalar curvature functional on the set of volume 1 metrics. The reader may also compare the behaviour of the Ricci flow on S^{4n+3} with that of the Ricci iteration studied in [20].

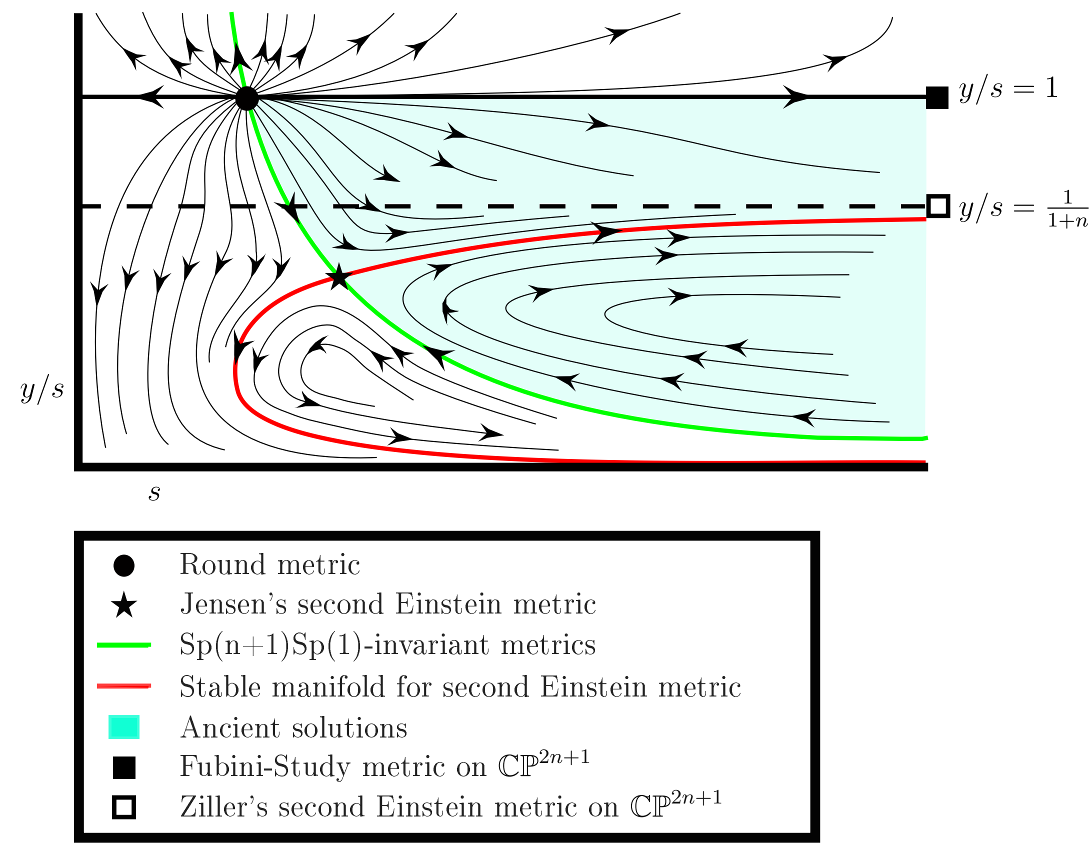

Figure 1.1: Backwards flow of $\mathrm{Sp}(n+1)\,\mathrm{U}(1)$-invariant metrics

It is interesting to note that the ancient solutions above are in some sense attracted to Ziller's second Einstein metric on the $\mathbb{CP}^{2n+1}$ base. Indeed, Ziller's second Einstein metric is a local minimum among homogeneous metrics on $\mathbb{CP}^{2n+1}$ and the Fubini-Study metric is a local maximum, hence the single ancient solution converging to the Fubini-Study metric and the family of solutions converging to Ziller's second Einstein metric. The formula for the scalar curvature of a homogeneous fibration with shrinking fibers (see 2.1.4) indicates that as the flat fiber shrinks, the scalar curvature is dominated by the scalar curvature of the metric on the base. Hence when the fiber is small, one might expect that along the backwards Ricci flow the fiber will continue to shrink while the metric on the base evolves approximately according to the negative gradient flow of the scalar curvature functional restricted to homogeneous metrics on the base.

It is thus natural to hypothesize that a collapsed ancient solution should exist whenever G/H is the total space of a homogeneous torus fibration

$$\mathsf{T}^n = \mathsf{K}/\mathsf{H} \to \mathsf{G}/\mathsf{H} \to \mathsf{G}/\mathsf{K}$$

and the base K/H admits a homogeneous Einstein metric. Furthermore, as in the case of spheres, the dimension of the family of ancient solutions collapsing to the given Einstein metric should be controlled by the coindex.

Chapter 3 consists of joint work with Francesco Pediconi from [39] where we confirm this hypothesis. In particular we show the following.

Theorem B. *Let $\mathsf{H} \subsetneq \mathsf{TH} \subsetneq \mathsf{G}$ be compact, connected Lie groups, where T is a maximal torus of a compact complement of H in the normalizer $\mathsf{N}_{\mathsf{G}}(\mathsf{H})$ with $d = \dim(\mathsf{T}) \geq 1$. For any G-invariant, unit volume Einstein metric $\overline{g}$ on $N = \mathsf{G}/\mathsf{TH}$ of coindex q, there exists a $\left(\frac{d(d+1)}{2} + q - 1\right)$-parameter family of collapsed ancient solutions to the homogeneous Ricci flow on $M = \mathsf{G}/\mathsf{H}$ that, under a suitable rescaling, shrink the fibers of $\mathsf{T} \to M \to N$ and converge to $(N, \overline{g})$ in the Gromov-Hausdorff topology as $t \to -\infty$.*

Here, the coindex of the Einstein metric $\overline{g}$ is defined as its coindex as a critical point of the scalar curvature functional on the space $M^{\mathsf{G}}_{N,1}$ of unit volume G-invariant metrics on N (see Section 3.1.2). We mention that all the ancient solutions obtained by means of Theorem B are of *submersion type* with respect to the homogeneous torus fibration $\mathsf{T} \to M \to N$, and we expect this property to be true in general (see e.g. [44]). Furthermore we stress that, along the solutions obtained by our theorem, the metric on the fibers T and the base N will in general change. Notice also that Theorem B holds true even when $\overline{g}$ has coindex $q = 0$, i.e. it is a local maximum of scalar curvature on $M^{\mathsf{G}}_{N,1}$ (compare with [12, Lemma 5.4]).

We illustrate Theorem B with a series of examples.

Corollary A.

a) On $\mathsf{SU}(3)$, $\mathsf{SU}(3)/\mathsf{S}^1$, $\mathsf{SU}(4)$, $\mathsf{SU}(4)/\mathsf{S}^1$, $\mathsf{SU}(4)/\mathsf{T}^2$, G_2 *and* $\mathsf{G}_2/\mathsf{S}^1$ *there exists a 3, respectively, 1, 7, 4, 2, 3 and 1-parameter family of ancient solutions to the Ricci flow collapsing, under a suitable rescaling, to a Kähler-Einstein metric on a full flag manifold.*

b) On $\mathsf{SU}(n)/\mathsf{T}^{n-1-k}$, *with* $n \geq 3$ *and* $1 \leq k \leq n-1$, *there exists a* $\left(\frac{k(k+1)}{2} + n - 2\right)$-*parameter family of ancient solutions to the Ricci flow collapsing to the normal Einstein metric on* $\mathsf{SU}(n)/\mathsf{T}^{n-1}$.

c) On $\mathsf{SO}(4)$ *and* $\mathsf{SO}(4)/\mathsf{S}^1$ *there exists a 3, respectively, 1-parameter family of ancient solutions to the Ricci flow collapsing to the normal Einstein metric on* $\mathsf{SO}(4)/\mathsf{T}^2$.

Notice that the the circles $\mathsf{S}^1 \subset \mathsf{SO}(4)$ in Corollary A can be chosen with arbitrary slope and that the manifolds $\mathsf{SO}(4)/\mathsf{S}^1$ are diffeomorphic to $S^3 \times S^2$ (see e.g. [47]). In particular we obtain infinitely many families of ancient solutions on $S^3 \times S^2$ that are homogeneous under inequivalent group actions. Moreover, let us also observe that in [9] it was shown that under the assumptions of Theorem B most homogeneous spaces G/TH admit homogeneous Einstein metrics with large coindex plus nullity (see also [46, 13, 11]), to which Theorem B can be applied.

We remark that Theorem B allows us to reconstruct all known examples of collapsed homogeneous ancient solutions to the Ricci flow, such as those in [44, 21, 12, 6, 32], which we will now discuss. In [6] and [32], the authors construct ancient solutions which consist of submersion metrics $F \to M \to B$ where one assumes either that F, M, B admit Einstein metrics, F is a torus, or B is a product of Kähler-Einstein metrics. In these constructions the metric on the base remains fixed, or in the latter case stays within the set of products of Kähler-Einstein metrics. On the contrary, along the ancient solutions provided by Theorem B, the induced metric on the base N varies and does not necessarily remain Einstein, as opposed to the solutions constructed in [6] and those constructed in [32]. Furthermore in our situation M need not admit an invariant Einstein metric and $\overline{g}$ need not be Kähler-Einstein.

This thesis is organized as follows. Chapter 2 is based on the paper [44]. In section 2.1 we discuss known properties of the Ricci flow on homogeneous spaces and describe the relevant homogeneous metrics on spheres in more detail. In section 2.2 we relate the existence-time of the Ricci flow on any compact homogeneous space with that of the normalized flow. Here we prove that being ancient for the Ricci flow is equivalent to being ancient for the normalized flow, and that the normalized flow develops a finite-time singularity unless it converges to an Einstein metric. In section 2.3 we study the dynamics of the forwards Ricci flow for $\mathrm{Sp}(n+1)$-invariant metrics. In section 2.4 we study the backwards flow and prove Theorem A. Chapter 3 is based on the paper [39] which is joint work with Francesco Pediconi. In section 3.1, we recall some facts about compact homogeneous spaces, toral H-subalgebras and ancient solutions to the Ricci flow. In section 3.2, we introduce the fundamental tools for proving Theorem B, namely the space of generalized submersion metrics and the projected Ricci flow. In section 3.3 we prove Theorem B. In section 3.4 we construct examples of collapsed ancient solutions and prove Corollary A. Both Chapter 2 and Chapter 3 retain the original text from [44] and [39].

CHAPTER 2

Ricci flow on spheres

2.1. Preliminaries

In the remainder of the paper we alternate between the notation g_t and $g(t)$ for a solution of the Ricci flow, and between $\widetilde{g}_t$ and $\widetilde{g}(t)$ for a solution of the normalized flow, whenever convenient. All manifolds and homogeneous spaces are assumed to be compact.

We denote by $\mathcal{M}$ the space of Riemannian metrics on the manifold M^n and $\mathcal{M}^G \subset \mathcal{M}$ the space of G-invariant metrics, where G is a Lie group acting on M. Likewise, we denote the space of volume-1 metrics on M by $\mathcal{M}_1$ and the space of G-invariant volume-1 metrics by $\mathcal{M}_1^G \subset \mathcal{M}^G$.

The space $\mathcal{M}$ can be endowed with the L^2 inner product, which assigns to any symmetric 2-tensor field h (viewed as a tangent vector to a metric $g \in \mathcal{M}$) the length

$$||h||_g^2 = \int_M g(h,h) d\mu_g,$$

where $d\mu_g$ is the volume element for g. From now one we denote the corresponding metric on $\mathcal{M}$ by d_{L^2}.

We denote by $\mathbf{S}$ the total scalar curvature functional on $\mathcal{M}$:

$$\mathbf{S}(g) = \int_M S(g) d\mu_g,$$

where $S(g)$ is the scalar curvature of g.

It is well known that, restricted to $\mathcal{M}_1$, the L^2 gradient of $\mathbf{S}$ is given by the negative traceless Ricci tensor $\nabla \mathbf{S} = -\mathrm{Ric}^0(g) = -(\mathrm{Ric}(g) - \frac{S(g)}{n} g)$ (e.g. [7], p.120). In particular, Einstein metrics are the critical points of $\mathbf{S}$.

Let $M = G/H$ be a homogeneous space where $H \leq G$ are compact Lie groups with Lie algebras $\mathfrak{h} \subset \mathfrak{g}$. Since G is compact we can fix an Ad_G-invariant inner product Q on $\mathfrak{g}$. Let $\mathfrak{p}$ be the Q-orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$ so that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$. Then via action fields there is an isomorphism $\mathfrak{p} \simeq T_pM : X \mapsto \frac{d}{dt}|_{t=0} \exp(tX) \cdot p$ where $p \in M$ is the identity coset. Moreover, there is a one-to-one correspondence between Ad_H-invariant inner products on $\mathfrak{p}$ and G-invariant metrics on G/H, and hence $\mathcal{M}^G$ is a finite-dimensional manifold.

Note also that for homogeneous metrics, scalar curvature and Ricci curvature are constant, so, in particular, on $\mathcal{M}_1^G$, the functional $\mathbf{S}$ just assigns to each metric its scalar curvature at a point. From now on, we restrict to $\mathcal{M}_1^G$ and identify $\mathbf{S}(g)$ with $S(g)$, so that the L^2 gradient of S is $\nabla S(g) = -\mathrm{Ric}^0(g)$.

Recall that a solution of the Ricci flow is a smooth family of metrics $g_t \in \mathcal{M}$ satisfying

$$\frac{\partial}{\partial t} g_t = -2\mathrm{Ric}(g_t).$$

Since the Ricci tensor is diffeomorphism invariant, isometry groups are preserved under the Ricci flow. That is, if $g_0 \in \mathcal{M}^G$, then $g_t \in \mathcal{M}^G$ for all t.

The normalized Ricci flow, which we denote by $\widetilde{g}_t$, is given by the equation

$$\frac{\partial}{\partial t} \widetilde{g}_t = 2 \left(-\mathrm{Ric}(\widetilde{g}_t) + \frac{\mathbf{S}(\widetilde{g}_t)}{n \mathrm{Vol}_{\widetilde{g}_t}(M)} \widetilde{g}_t \right).$$

The normalized Ricci flow preserves volume and can be obtained from the Ricci flow as follows. Let $g(t)$ be a solution of the Ricci flow and let $S(t) := S(g(t))$. The corresponding solution of the normalized flow is given by $\widetilde{g}(f(t)) = r(t)g(t)$ where

$$r(t) = \exp\left(\frac{2}{n}\int_0^t S(\tau)d\tau\right), \quad f'(t) = \exp\left(\frac{2}{n}\int_0^t S(\tau)d\tau\right),$$

and $f(0) = 0$ (see for instance [23]). Hence we can restrict the normalized flow to $\mathcal{M}_1^G$ where it becomes an ODE given by

$$\frac{\partial}{\partial t}\widetilde{g}_t = 2\left(-\mathrm{Ric}(\widetilde{g}_t) + \frac{S(\widetilde{g}_t)}{n}\widetilde{g}_t\right) = 2\nabla S(\widetilde{g}_t).$$

In particular, the normalized Ricci flow coincides, up to the factor 2, with the L^2 gradient flow for S. For the remainder of the paper, we denote by $(T_{\min}, T_{\max})$ the maximal time interval on which the Ricci flow exists, as well as $(\widetilde{T}_{\min}, \widetilde{T}_{\max})$ for the normalized flow.

In [13], the authors studied the global behaviour of S on $\mathcal{M}_1^G$ with the goal of determining sufficient conditions for the existence of a G-invariant Einstein metric. In particular, they proved that, for any fixed $\epsilon > 0$, the functional S satisfies the Palais-Smale compactness condition on the set $(\mathcal{M}_1^G)_\epsilon = \{g \in \mathcal{M}_1^G \ : \ S(g) \geq \epsilon\}$. That is, every sequence of metrics $\{g_i\}_{i=1}^\infty$ in $(\mathcal{M}_1^G)_\epsilon$ with $|S(g_i)|$ bounded and $|\nabla S(g_i)| = |\mathrm{Ric}^0(g_i)| \to 0$ has a convergent subsequence, which, in particular, converges to an Einstein metric. As a consequence, the set of G-invariant Einstein metrics has only finitely many components, and each of them is compact. They also noted that this result is optimal in the sense that it is impossible to have a convergent sequence of metrics in $\mathcal{M}_1^G$ with $S(g_i) < 0$ and $|\mathrm{Ric}^0| \to 0$ since the limit would have to be an Einstein metric of negative scalar curvature, or would have to be Ricci flat. The first possibility is ruled out by Bochner's theorem, and the second can only occur if M is flat by Alekseevsky-Kimel'fel'd [1]. On the other hand, there may exist sequences $\{g_i\}_{i=1}^\infty$ with $S(g_i) > 0$, $S(g_i) \to 0$ and $|\mathrm{Ric}^0(g_i)| \to 0$, so-called 0-Palais Smale sequences, which do not converge unless M is a torus.

From now on, we will assume that M is a compact homogeneous space which is not a torus. By Theorem 1 and Theorem 2 in [10], a homogeneous solution g_t to the Ricci flow on M develops a Type-1 singularity in finite time. Recall that a finite-time singularity of the Ricci flow is said to be Type-1 if the curvature tensor blows up at most linearly, that is, there exists some $C > 0$ so that $|\mathrm{Rm}(g_t)|(T_{\max} - t) \leq C$ for t near $T_{\max} < \infty$, where $|\mathrm{Rm}|$ is the norm of the curvature tensor. By [30], this implies that the scalar curvature goes to $+\infty$ near the singular time. In particular, by starting the flow at a later time, we can assume

$S(g_0) > 0$.

Since $\frac{\partial}{\partial t}\widetilde{g}_t = 2\nabla S(\widetilde{g}_t) = -2\mathrm{Ric}^0(\widetilde{g}_t)$, we have

$$d_{L^2}(\widetilde{g}_t, \widetilde{g}_0) \leq 2\int_0^t |\mathrm{Ric}^0(\widetilde{g}_s)|ds \leq 2t^{1/2}\left(\int_0^t |\mathrm{Ric}^0(\widetilde{g}_s)|^2 ds\right)^{1/2}, \tag{2.1.1}$$

and

$$S(\widetilde{g}_t) - S(\widetilde{g}_0) = 2\int_0^t |\mathrm{Ric}^0(\widetilde{g}_s)|^2 ds,$$

Thus, there are two possibilities for solutions $\widetilde{g}_t$ of the normalized Ricci flow on $\mathcal{M}_1^G$. The first possibility is that $S(\widetilde{g}_t) \to \infty$ as $t \to \widetilde{T}_{\max}$, and the second is that $S(\widetilde{g}_t) \leq C$ for all $t \in (0, \widetilde{T}_{\max})$, in which case (2.1.1) implies that $\widetilde{T}_{\max} = \infty$. Similarly, (2.1.1) implies that if S is bounded from below then $\widetilde{T}_{\min} = -\infty$. In the case that $S(\widetilde{g}_t) \leq C$, Palais-Smale further implies that $\widetilde{g}_t$ converges to an Einstein metric as $t \to \infty$.

We will also examine solutions of the Ricci flow as $t \to -\infty$. As remarked above, a lower bound on S already implies that $\widetilde{g}_t$ is ancient. If g_t is an ancient solution of the Ricci flow then it is an easy consequence of the maximum principle applied to the evolution equation for S,

$$\frac{\partial}{\partial t}S(g_t) = \Delta S(g_t) + 2|\mathrm{Ric}(g_t)|^2, \tag{2.1.2}$$

that g_t either has positive scalar curvature for all time, or is Ricci flat (e.g. [23] p. 102). Since G/H is not a torus, the latter is ruled out. Hence, the corresponding solution $\widetilde{g}_t$ of the normalized flow has $S(\widetilde{g}_t) > 0$ for all t, which implies that $\widetilde{T}_{\min} = -\infty$ and $|\mathrm{Ric}^0(\widetilde{g}_t)| \to 0$ as $t \to -\infty$. Thus there are two possibilities for the corresponding solution $\widetilde{g}_t$ of the normalized flow. The first possibility is that $S(\widetilde{g}_t) > \epsilon > 0$ for all time, in which case it follows from Palais-Smale and the fact that the gradient flow of S is analytic that $\widetilde{g}_t$ converges to an Einstein metric as $t \to -\infty$ (see e.g. Theorem 4.2 in [12]). The second possibility is that $S(\widetilde{g}_t) \to 0$, i.e., $\widetilde{g}_t$ is 0-Palais-Smale. 0-Palais-Smale sequences were studied in [13] where the authors showed that if one exists, then there exists an intermediate subgroup $H \leq K \leq G$

such that K/H is a torus. In [36], F. Pediconi further proved that every divergent sequence of metrics in $\mathcal{M}_1^G$ with $|\mathrm{Rm}|$ bounded has a subsequence which asymptotically approaches a submersion metric for a torus fibration with shrinking fibers as in (2.1.3). By the Gap Theorem for compact homogeneous spaces, $|\mathrm{Rm}(g)| \leq C|\mathrm{Ric}(g)|$ (see [Bö2] Theorem 4), and hence 0-Palais Smale sequences are special examples of divergent sequences of metrics with bounded curvature.

Let us recall the definition of submersion metrics. If $\mathfrak{h} \subset \mathfrak{k} \subset \mathfrak{g}$ is an intermediate Lie subalgebra with $\mathfrak{k} = \mathrm{Lie}(K)$, then we can further decompose $\mathfrak{p} = \mathfrak{p}_{\mathfrak{k}} + \mathfrak{p}_{\mathfrak{k}}^{\perp}$ where $\mathfrak{p}_{\mathfrak{k}} = \mathfrak{p} \cap \mathfrak{k}$. We say that an Ad_H-invariant inner product g on $\mathfrak{p}$ is a $\mathfrak{k}$-submersion metric provided that $\mathfrak{p} = \mathfrak{p}_{\mathfrak{k}} + \mathfrak{p}_{\mathfrak{k}}^{\perp}$ is orthogonal with respect to g and that the restriction of g to $\mathfrak{p}_{\mathfrak{k}}^{\perp}$ is Ad_K-invariant.

Note that in this language, $T_e(K/H) = \mathfrak{p}_{\mathfrak{k}}$ and $T_e(G/K) = \mathfrak{p}_{\mathfrak{k}}^{\perp}$. The orthogonality assumption and Ad_K-invariance imply that the homogeneous fibration

$$K/H \to G/H \to G/K \tag{2.1.3}$$

is a Riemannian submersion, where the induced metric on G/K is G-invariant (see [7] p. 257). As in [7], such a submersion, in addition, has totally geodesic fibers.

If we start with a $\mathfrak{k}$-submersion metric, scale the metric on the fiber by t, and normalize volume to be 1, we obtain a "divergent" path of metrics in $\mathcal{M}_1^G$, whose scalar curvature is given by the formula

$$t^{f/n}\left(\frac{1}{t}S(K/H) + S(G/K) - t||A||^2\right), \tag{2.1.4}$$

where $f = \dim(K/H)$, $S(K/H)$ is the scalar curvature of g restricted to K/H, $S(G/K)$ is the scalar curvature of the metric induced by g on G/K, and $||A||$ is the norm of the O'Neill tensor computed with respect to g (see [7] p. 253). Hence if K/H is not a torus (and g is chosen so that $S(K/H) > 0$), then $S \to \infty$ as $t \to 0$. On the other hand, if K/H is a torus, then $S \to 0$ as $t \to 0$ (see [46], [13]). Conversely, the existence of a path with $S \to \infty$, or a path with $S \to 0$ and $|\mathrm{Ric}^0| \to 0$, implies the existence of such subgroups ([46],[13]).

We now describe the Ricci flow on the two-parameter families of homogeneous metrics on spheres as studied in [21] and [6]. When the volume is normalized, they become one-parameter families, and the normalized flow can be understood in terms of the gradient flow for the single-variable function $S(t)$ on $\mathcal{M}_1^G$, where t is the scale of the fiber in the Hopf fibration (1.0.1). Figure 2.1 depicts the graph of S as a function of t.

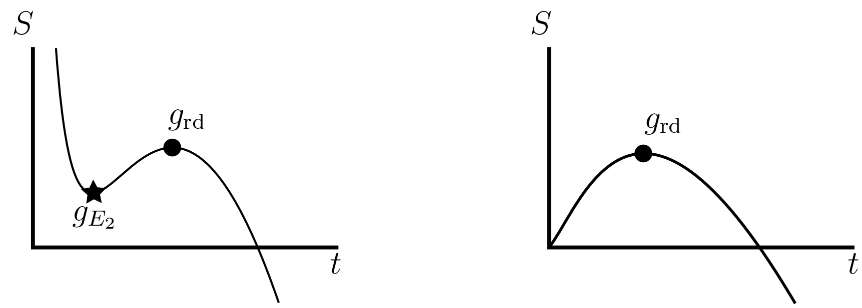

Figure 2.1: Scalar curvature of two-parameter families of metrics as fiber is scaled by t. The graph on the left describes Spin(9) and $\mathrm{Sp}(n+1) \times \mathrm{Sp}(1)$-invariant metrics, and the graph on the right describes $\mathrm{U}(n+1)$-invariant metrics.

For the graph on the left, there are exactly two G-invariant Einstein metrics, the round metric, which we denote by g_{rd}, and a second Einstein metric (see [28],[15]), which we denote in each case by g_{E_2} (although these are not isometric for different G). It follows from our remarks above that there are exactly two ancient solutions $\widetilde{g}_t$, both converging to g_{E_2} under the backwards flow.

For the graph on the right, every solution converges to g_{rd}. There is one solution $\widetilde{g}_s$ with $S \to 0$ as the S^1 fiber shrinks to a point under the backwards flow, and hence by (2.1.1) this solution is ancient. In [6], it was shown in both cases that these conclusions also hold for the Ricci flow, although the arguments are more involved. Note also that in Theorem 2.2.2, we prove that a solution to the Ricci flow is ancient if and only if the corresponding solution for the normalized flow is also ancient.

For the case of left-invariant metrics on S^3, in [26] the authors showed that the normalized flow always converges to the round metric. In [22], the authors further proved that the only ancient solutions for the Ricci flow are the Berger spheres, i.e., metrics satisfying $x \leq y = z$ where x, y and z are the eigenvalues of the metric.

The only remaining family of homogeneous metrics on spheres are the $\mathrm{Sp}(n+1)$-invariant metrics, which we now describe.

2.1.1. $\mathrm{Sp}(n+1)$**-invariant metrics on spheres**

We view S^{4n+3} as the unit sphere in $\mathbb{H}^{n+1} = \mathbb{R}^{4n+4}$ with the standard Euclidean inner product. The group of quaternionic-linear isometries $\mathrm{Sp}(n+1)$ acts transitively on S^{4n+3} with stabilizer $\mathrm{Sp}(n)$ at the point $p = (1, 0, ..., 0)$, so that $S^{4n+3} \simeq \mathrm{Sp}(n+1)/\mathrm{Sp}(n)$. With respect to the $\mathrm{Ad}_{\mathrm{Sp}(n+1)}$-invariant inner product Q on $\mathfrak{sp}(n+1)$, $Q(A, B) = -\frac{1}{2}\operatorname{Re}(\operatorname{trace}(AB))$, we have the orthogonal decompositions $\mathfrak{sp}(n+1) = \mathfrak{sp}(n) \oplus \mathfrak{p}$ and $\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_1$, where $\mathfrak{p}_0 \simeq \mathfrak{sp}(1) = \langle i, j, k \rangle$ is the Lie algebra embedded diagonally as

$$\left(\begin{array}{c|c} \mathfrak{sp}(1) & 0 \\ \hline 0 & 0 \end{array}\right),$$

and $\mathfrak{p}_1 \simeq \mathbb{H}^n$ via

$$X \mapsto \left(\begin{array}{c|c} 0 & -\overline{X}^t \\ \hline X & 0 \end{array}\right).$$

Notice that i, j, k have length $\frac{1}{2}$ and are Q-orthogonal. The representation of $\mathrm{Ad}_{\mathrm{Sp}(n)}$ on $\mathfrak{p}$ is trivial on $\mathfrak{p}_0$ and acts by usual matrix multiplication on $\mathfrak{p}_1$. Hence, by Schur's lemma any $\mathrm{Ad}_{\mathrm{Sp}(n)}$-invariant inner product on $\mathfrak{p}$ is of the form $\sigma + s\langle \cdot, \cdot \rangle_{\mathbb{H}^n}$ where $\langle \cdot, \cdot \rangle_{\mathbb{H}^n}$ is the Euclidean inner product on $\mathbb{H}^n$ and σ is any inner product on $\mathfrak{p}_0 \simeq \mathbb{R}^3$. As observed in [48], via right translations, the normalizer $N(\mathrm{Sp}(n))/\mathrm{Sp}(n) = \mathrm{SO}(3)$ acts by diffeomorphisms on G/H and induces the usual linear action of $\mathrm{SO}(3)$ on $\mathfrak{p}_0 = \mathbb{R}^3$. Moreover, this linear action is by Q-isometries, so we can diagonalize σ with respect to the Q-orthogonal basis $\langle i, j, k \rangle$.

Henceforth, we write an Sp$(n+1)$-invariant metric on S^{4n+3} as

$$g = x\langle,\rangle|_{(i)} + y\langle,\rangle|_{(j)} + z\langle,\rangle|_{(k)} + s\langle,\rangle|_{\mathbb{H}^n}, \tag{2.1.5}$$

where $\langle,\rangle$ is the standard metric on $\mathbb{H}^{n+1} = \mathbb{R}^{4n+4}$ and $(i), (j)$, and (k) are the subspaces spanned by i, j, and k respectively. Furthermore, we can use the action of SO(3) to switch the signs of $i, j, k \in T_pS^{4n+3}$, two at a time. These are only isometries if the metric is of the form (2.1.5). Since isometries are always preserved under the Ricci flow, metrics of the form (2.1.5) are preserved as well.

In order to study the Ricci flow, it will often be convenient to consider the Ricci endomorphism. We denote the Ricci endomorphism by ric and the Ricci curvature tensor by Ric, i.e., $\mathrm{Ric}(X,Y) = g(\mathrm{ric}(X), Y)$.

For the above metrics, the Ricci endomorphism decomposes as

$$\mathrm{ric} = r_i \,\mathrm{Id}\,|_{(i)} + r_j \,\mathrm{Id}\,|_{(j)} + r_k \,\mathrm{Id}\,|_{(k)} + r_h \,\mathrm{Id}\,|_{\mathbb{H}^n},$$

where

$$\begin{aligned} r_i &= 2\left(\frac{x^2 - y^2 - z^2}{xyz}\right) + \frac{4}{x} + \frac{4nx}{s^2} \\ r_j &= 2\left(\frac{y^2 - x^2 - z^2}{xyz}\right) + \frac{4}{y} + \frac{4ny}{s^2} \\ r_k &= 2\left(\frac{z^2 - x^2 - y^2}{xyz}\right) + \frac{4}{z} + \frac{4nz}{s^2} \\ r_h &= -2\left(\frac{x + y + z}{s^2}\right) + \frac{4n+8}{s} \end{aligned} \tag{2.1.6}$$

(see [48]). Thus, the scalar curvature is given by the formula

$$S = \frac{4}{x} + \frac{4}{y} + \frac{4}{z} + \frac{16n(n+2)}{s} - 4n\left(\frac{x+y+z}{s^2}\right) - 2\left(\frac{x^2+y^2+z^2}{xyz}\right). \tag{2.1.7}$$

As in [48], for any $T \in \mathbb{H}^n$ we have the sectional curvatures $K(i,T) = \frac{x}{s^2}$, $K(j,T) = \frac{y}{s^2}$, and $K(k,T) = \frac{z}{s^2}$. From this and the fact that an isometry preserves eigenspaces of the Ricci tensor, it is not difficult to see that there are no further isometries among metrics of the form (2.1.5), besides permuting the variables x, y and z using the normalizer $N(H)$.

Our work is closely related to the examples of homogeneous Einstein metrics on spheres and projective spaces. These were classified by Ziller in [48] and can be obtained by scaling the fibers in the Hopf fibrations

$$S^1 \to S^{4n+3} \to \mathbb{CP}^{2n+1} \qquad S^3 \to S^{4n+3} \to \mathbb{HP}^n \qquad S^7 \to S^{15} \to S^8.$$

In [48] it was shown that the only $\mathrm{Sp}(n+1)$-invariant Einstein metrics on S^{4n+3} are, up to scaling, the round metric g_{rd}, given by $x = y = z = s = 1$, and Jensen's second Einstein metric g_{E_2}, given by $x = y = z = 1$ and $s = 2n+3$.

If we view $\langle i \rangle = \mathfrak{u}(1)$ as tangent to the Hopf action, then $\mathrm{Sp}(n+1)$-invariant metrics on $\mathbb{CP}^{2n+1}$ are precisely the metrics induced by $\mathfrak{u}(1)$-submersion metrics on S^{4n+3}. Since $\mathrm{U}(1) \subset N(H)$ acts by fixing i and rotating the j, k plane, $\mathfrak{u}(1)$-submersion metrics on S^{4n+3} satisfy $y = z$ and are hence of the form

$$g = x\langle , \rangle|_{(i)} + y\langle , \rangle|_{(j)} + y\langle , \rangle|_{(k)} + s\langle , \rangle|_{\mathbb{H}^n}. \tag{2.1.8}$$

On $\mathbb{CP}^{2n+1}$, these induce metrics of the form

$$y\langle , \rangle|_{(j)} + y\langle , \rangle|_{(k)} + s\langle , \rangle|_{\mathbb{H}^n},$$

and Ziller showed that the only two Einstein metrics in this family are given by $y = s$ (the Fubini-Study metric), which we denote by $g^{\mathrm{FS}}_{\mathbb{CP}^{2n+1}}$, and $y/s = 1/(n+1)$, which we denote by $g^2_{\mathbb{CP}^{2n+1}}$. Note that metrics of this form can be obtained by scaling the fibers and base of the Hopf fibration $S^2 \to \mathbb{CP}^{2n+1} \to \mathbb{HP}^n$ as in [48].

2.2. General Results

In this section, we discuss some results that hold for all compact homogeneous spaces, relating properties of solutions of the Ricci flow to those of the normalized flow. We first prove that, as in the case of the Ricci flow, the normalized flow develops a singularity in finite time, unless it converges to an Einstein metric. Our proof is similar to the proof of Theorem 4.1 in [10].

Theorem 2.2.1. *Let $\widetilde{g}_t$ be a solution to the normalized Ricci flow on a compact homogeneous space. Then $\widetilde{T}_{max} = \infty$ if and only if $\widetilde{g}_t$ converges to an Einstein metric. Furthermore, if $\widetilde{T}_{max} < \infty$ then $S(\widetilde{g}_t) \to \infty$ as $t \to \widetilde{T}_{max}$.*

Proof. Böhm showed that on a compact homogeneous space that is not a torus, the Ricci flow develops a Type-1 singularity in finite time [10], and hence, by results of Naber [34], Enders, Müller, Topping [24], Petersen and Wylie [42], along any sequence of times $t_i \to T_{\max}$, a parabolic sequence of rescaled solutions

$$g_i(t) := S(g(t_i))g\left(t_i + \frac{t}{S(g(t_i))}\right)$$

subconverges to a soliton $g_\infty(t)$ on $E^k_\infty \times \mathbb{R}^{n-k}$, where E^k_∞ is a compact homogeneous Einstein manifold and $\mathbb{R}^{n-k}$ is endowed with the flat metric. Furthermore, the dimension of the Euclidean factor depends only on the initial metric, not on the sequence t_i [10]. As we will see, the presence of a Euclidean factor in the limit determines whether or not the normalized flow converges to an Einstein metric. We then use the dimension of the Euclidean factor to control the growth of S near the extinction time.

Since the normalized flow is the L^2 gradient flow for S, we can derive an evolution equation for S under the normalized flow:

$$\frac{\partial}{\partial t} S(\widetilde{g}_t) = \langle \nabla S, \nabla S \rangle_{L^2} = \left\langle -2 \left(\mathrm{Ric}(\widetilde{g}_t) - \frac{S(\widetilde{g}_t)}{n} \widetilde{g}_t \right), -2 \left(\mathrm{Ric}(\widetilde{g}_t) - \frac{S(\widetilde{g}_t)}{n} \widetilde{g}_t) \right) \right\rangle_{L^2}$$
$$= 4 \left(|\mathrm{Ric}(\widetilde{g}_t)|^2 - \frac{S^2(\widetilde{g}_t)}{n} \right).$$

Let $\overline{g}(t) = S(g(t))g(t)$. We claim that the eigenvalues of $\mathrm{Ric}(\overline{g}(t))$ all converge to $r_\infty = \frac{1}{k}$ or 0. If not, there would exist a $\delta > 0$ and a sequence of times t_i such that $\mathrm{Ric}(\overline{g}(t_i)) = \mathrm{Ric}(g_i(0))$ has an eigenvalue in $(-\infty, -\delta) \cup (\delta, r_\infty - \delta) \cup (r_\infty + \delta, \infty)$. But then the same would be true for any subsequence of $g_i(0)$ and hence also for the limit soliton, which is a contradiction since $S(g_\infty(0)) = \lim_{i\to\infty} S(\overline{g}(t_i)) = 1$ and the dimension of the Einstein factor depends only on the initial metric.

Let $r_1(t), ..., r_k(t)$ be the eigenvalues of $\mathrm{Ric}(\overline{g}(t))$ which converge to r_∞. Then for t sufficiently close to $T_{\max}$,

$$\begin{aligned} |\mathrm{Ric}(g(t))|^2 - \frac{S^2(g(t))}{n} &= S^2(g(t)) \left(|\mathrm{Ric}(\overline{g}(t))|^2 - \frac{1}{n} \right) \\ &\geq S^2(g(t)) \left(\sum_{i=1}^k r_i^2(t) - \epsilon - \frac{1}{n} \right) \\ &\geq S^2(g(t)) \left(k r_\infty^2 - 2\epsilon - \frac{1}{n} \right) = S^2(g(t)) \left(\frac{1}{k} - \epsilon - \frac{1}{n} \right) \end{aligned}$$

Since both sides of the above inequality scale the same way, the same is true for $\widetilde{g}_t$. Hence if t is sufficiently large and $k < n$, $\frac{\partial S(\widetilde{g}_t)}{\partial t} \geq CS^2(\widetilde{g}_t)$, which implies that $S(\widetilde{g}_t) \to \infty$ in finite time.

On the other hand, if the dimension of the Euclidean factor in the limit is zero, then $\overline{g}(t)$ converges to a G-invariant Einstein metric, and hence the volume-normalized solution converges to an Einstein metric and $\widetilde{T}_{\max} = \infty$.

□

For the main example of our paper, we would also like to classify the ancient solutions for the Ricci flow on $\mathcal{M}_1^G$. Before doing so we first show that on homogeneous spaces, being ancient for the Ricci flow is equivalent to being ancient for the normalized flow.

Theorem 2.2.2. *A solution g_t of the Ricci flow on a compact homogeneous space is ancient if and only if the corresponding solution $\widetilde{g}_t$ of the normalized flow is also ancient. Furthermore, if $\widetilde{g}_t$ is ancient and does not converge to an Einstein metric as $t \to -\infty$, then $S(\widetilde{g}_t) \to 0$ and $|Ric^0(\widetilde{g}_t)| \to 0$ as $t \to -\infty$ and hence $\widetilde{g}_t$ is a 0-Palais Smale solution.*

Proof. If g_t is ancient then since $S(\widetilde{g}_t) > 0$ for all t (see Section 2), and since $\widetilde{g}_t$ is the gradient flow for S, $\widetilde{g}_t$ is ancient as well.

In order to prove that if $\widetilde{g}_t$ is ancient then g_t is also ancient, we first prove that an ancient solution of the normalized flow on a compact homogeneous space has positive scalar curvature. Since a theorem of Lafuente (see [30]) states that a homogeneous solution of the Ricci flow with finite backwards singular-time must have $S(g_t) \to -\infty$ as $t \to T_{\min}$, it follows that g_t must be ancient as well.

Now, suppose we have an ancient solution of the normalized flow with $S(\widetilde{g}_0) \leq 0$. Then, since G/H is not a torus, and hence is not Ricci flat, $S(\widetilde{g}_t) < 0$ for all $t < 0$. By Bochner's theorem, the Ricci tensor of a compact homogeneous space has at least one positive eigenvalue. In particular, if $\{r_i\}_{i=1}^n$ are the eigenvalues of Ric and $r_1, ..., r_{n-k}$ are all the positive eigenvalues (where $k \leq n-1$), then by Cauchy-Schwarz and the fact that $S < 0$,

$$|S|^2 \leq \left| \sum_{i=n-k+1}^{n} r_i \right|^2 \leq k \sum_{i=n-k+1}^{n} r_i^2 \leq k|\mathrm{Ric}|^2.$$

For the backwards normalized flow, the evolution equation for S is $\frac{\partial S}{\partial t} = 4\left(\frac{S^2}{n} - |\mathrm{Ric}|^2\right)$, and hence

$$\frac{\partial S}{\partial t} = 4\left(\frac{S^2}{n} - |\mathrm{Ric}|^2\right) \leq 4\left(\frac{k}{n}|\mathrm{Ric}|^2 - |\mathrm{Ric}|^2\right) \leq \frac{4(k-n)}{n}|\mathrm{Ric}|^2 \leq \frac{4(k-n)}{n^2}S^2.$$

Thus $\frac{\partial S}{\partial t} = -CS^2$ for some $C > 0$ and hence $S(t) \to -\infty$ in finite time, contradicting the assumption that $\widetilde{g}_t$ is ancient. $\square$

2.3. Ricci flow on Spheres

We now study the Ricci flow of $\mathrm{Sp}(n+1)$-invariant metrics on spheres. Recall that metrics of the form (2.1.5) are preserved under the Ricci flow. We can thus view x, y, z, and s as functions of time.

Recall also that the normalized Ricci flow on $\mathcal{M}_1^G$ is given by the ODE

$$\frac{\partial}{\partial t}\widetilde{g}_t = -2\left(\mathrm{Ric} - \frac{S(\widetilde{g}_t)}{\dim(M)}\widetilde{g}_t\right)$$

and hence satisfies

$$\begin{aligned} x' &= -2x\left(r_i - \frac{S}{4n+3}\right) \\ y' &= -2y\left(r_j - \frac{S}{4n+3}\right) \\ z' &= -2z\left(r_j - \frac{S}{4n+3}\right) \\ s' &= -2s\left(r_h - \frac{S}{4n+3}\right), \end{aligned} \tag{2.3.1}$$

where r_i, r_j, r_k, r_h and S are as in (2.1.6) and (2.1.7). Since the normalized Ricci flow preserves volume, we can restrict these ODE's to $\mathcal{M}_1^G$, the space of volume-1 metrics, i.e. those satisfying $xyzs^{4n} = 1$. We parametrize these metrics by setting $s = \frac{1}{(xyz)^{\frac{1}{4n}}}$. Hence the normalized Ricci flow is equivalent to an ODE in $\mathbb{R}^3_{>0}$. For later convenience, we include the formula for scalar curvature in the above coordinates:

$$S = \frac{4}{x} + \frac{4}{y} + \frac{4}{z} - \frac{2z}{xy} - \frac{2y}{xz} - \frac{2x}{yz} + 16n(n+2)(xyz)^{\frac{1}{4n}} - 4n(x+y+z)(xyz)^{\frac{1}{2n}}. \tag{2.3.2}$$

Lemma 2.3.1. *The metrics where two of the variables agree, i.e. $x = y$, $y = z$, or $x = z$ are precisely those which are* $\mathrm{Sp}(n+1) \times U(1)$*-invariant, and the metrics where $x = y = z$ are* *precisely those which are* $\mathrm{Sp}(n+1) \times \mathrm{Sp}(1)$*-invariant, and hence these metrics are preserved by the Ricci flow. The metrics where* $y = z = s$ *are* $\mathrm{U}(2n+2)$*-invariant, and hence also preserved by the Ricci flow.*

Proof. If $K \subset \mathrm{Sp}(1)$, we view the action of $\mathrm{Sp}(n+1) \times K$ on S^{4n+3} as $(g,k) \cdot p = gpk^{-1}$. Note that since $p = (1, 0, ..., 0)$ is totally real, $kpk^{-1} = p$ for all $k \in K$. Invariant metrics under this larger group can then be viewed as the subset of G-invariant metrics that are also invariant under the adjoint action of $K \subset \mathrm{Sp}(1)$ on $\mathfrak{p}_0 = \mathfrak{sp}(1)$. If $K = \mathrm{U}(1) = \{e^{i\theta}\}_{\theta \in [0,2\pi)}$ for example, then a metric is invariant if and only if rotation in the j, k plane is an isometry, and hence if and only if $y = z$. If $K = \mathrm{Sp}(1)$ then a metric is invariant if and only if its restriction to $\mathfrak{sp}(1)$ is a multiple of the bi-invariant metric and hence if and only if $x = y = z$.

The action of $\mathrm{U}(2n+2)$ on S^{4n+3} is by isometries if and only if the adjoint action of the stabilizer $\mathrm{U}(2n+1)$ acting on $\langle p, ip \rangle^{\perp} = (jp) \oplus (kp) \oplus \mathbb{H}^n$ is by isometries. But this is the case if and only if the metric on $(jp) \oplus (kp) \oplus \mathbb{H}^n$ is a multiple of the Euclidean metric, i.e., if $y = z = s$. ☐

Lemma 2.3.2. *The only fixed points of the normalized flow are the round metric, where* $x = y = z = 1$ *and Jensen's second Einstein metric* $x = y = z = (2n+3)^{-\frac{4n}{4n+3}}$*. The round metric is a stable node and Jensen's second Einstein metric is a saddle node. The tangent space of the unstable manifold at Jensen's metric is given by* $x = y = z$*, and the tangent space of the stable manifold is given by* $x + y + z = 3(2n+3)^{-\frac{4n}{4n+3}}$*.*

Proof. By symmetry, the Hessian of the system when $x = y = z$ must be of the form

$$\begin{bmatrix} a & b & b \\ b & a & b \\ b & b & a \end{bmatrix},$$

which has eigenvectors $(1,1,1), (2,-1,-1)$, and $(-1,2,-1)$ with corresponding eigenvalues

$a+2b$, $a-b$, and $a-b$. A direct calculation shows that at $x=y=z=(2n+3)^{-\frac{4n}{4n+3}}$,

$$a = -8\left(2n^2+7n+5\right)(2n+3)^{-\frac{4n+6}{4n+3}}$$
$$b = 16(n+1)(n+2)(2n+3)^{-\frac{4n+6}{4n+3}}.$$

Thus $a-b<0$ and $a+2b>0$.

At the round metric, a direct calculation shows

$$a=-8(1+n) \quad \text{and} \quad b=0,$$

and hence the round metric is a stable node.

□

By symmetry in the three variables, it suffices to understand the Ricci flow on the set

$$\Omega = \{(x,y,z) \in \mathbb{R}^3 : 0 < x \leq y \leq z\},$$

which is preserved by the Ricci flow since the boundary consists of invariant sets (see Lemma 2.3.1).

As in Section 1, there are two possibilities for the long time behavior of $\widetilde{g}_t$. Either $\widetilde{g}_t$ converges to an Einstein metric or $S(\widetilde{g}_t) \to \infty$ as $t \to \widetilde{T}_{\max} < \infty$. If $S(\widetilde{g}_t) \to \infty$, then since the Ricci flow preserves metrics of the form (2.1.5), we can apply Theorem 4.6 in [10] (see also Remark 5.4 on p. 557), which in this case implies that $S(\widetilde{g}_t)\widetilde{g}_t$ converges to an isometric product $S^3 \times \mathbb{R}^{4n}$ where S^3 and $\mathbb{R}^{4n}$ are endowed with the round metric and flat metric, respectively. We offer an elementary proof along with a monotonicity lemma that is useful for our classification of ancient solutions.

Theorem 2.3.3. *Any* $\mathrm{Sp}(n+1)$*-invariant solution to the normalized Ricci flow on* S^{4n+3}

either converges to the round metric, Jensen's second Einstein metric, or $S \to \infty$ in such a way that $S(\widetilde{g}_t)\widetilde{g}_t$ converges in the pointed C^∞ topology to $S^3 \times \mathbb{R}^{4n}$, where S^3 and $\mathbb{R}^{4n}$ are endowed with $6g_{rd}$ and the flat metric respectively. Furthermore, in the last case, $x, y, z \to 0$ in such a way that x/z and y/z monotonically converge to 1.

Proof. The structure of our proof is as follows. First, we will show that the ratios x/z and y/z are monotonic along any solution of the normalized flow. Then we will see that $S \to \infty$ only if all the variables go to zero. Lastly, we will show that when all the variables tend to zero, their ratios tend to 1.

Lemma 2.3.4. *For any solution of the normalized Ricci flow in Ω, the ratios y/z and x/z are non-decreasing. In particular, if $x \to 0$ as $t \to \widetilde{T}_{max}$, then $z \to 0$ and $y \to 0$ as well.*

Proof. The derivative of (y/z) is given by

$$\left(\frac{y}{z}\right)' = \frac{y'z - z'y}{z^2} = (yz)\frac{-r_j + r_k}{z^2}. \tag{2.3.3}$$

In particular, $(y/z)'$ has the same sign as

$$r_k - r_j = \frac{-4(y-z)\left(n(xyz)^{\frac{1}{2n}+1} - x + y + z\right)}{xyz}. \tag{2.3.4}$$

Since we assumed $x \le y \le z$, we see that $(y/z)' \ge 0$, with equality if and only if $y = z$. Similarly, x/z is also non-decreasing.

□

Assume from now on that $\widetilde{g}_t$ does not converge to an Einstein metric, i.e., that $S(\widetilde{g}_t) \to \infty$. Since S is continuous on Ω the only way for $S \to \infty$ is if $\widetilde{g}_t$ approaches the boundary of Ω, that is if $x \to 0$, or if $z \to \infty$.

If $z \to \infty$ and x remains bounded away from zero, then the only way we can have $S \to \infty$

in (2.3.2) is if $(xyz) \to \infty$. However, if (xyz) is sufficiently large then $(xyz)^{\frac{1}{2n}} > (xyz)^{\frac{1}{4n}}$, in which case the $-4nz(xyz)^{\frac{1}{2n}}$ term dominates all the positive terms, and thus we can assume that $x \to 0$ and hence also that $y \to 0$ and $z \to 0$ as well by Lemma 2.3.4.

Since the ratios x/z and y/z are less than or equal to 1 and non-decreasing they each converge to some finite positive constant. Let $\lim_{t\to\widetilde{T}_{\max}} x/z = C$, $\lim_{t\to\widetilde{T}_{\max}} y/z = D$. Since the ratio x/z is scale invariant, the limit is the same for the Ricci flow and the normalized flow. Suppose for the moment that for the Ricci flow, $\lim_{t\to T_{\max}} x'/z'$ and $\lim_{t\to T_{\max}} y'/z'$ exist, and hence that

$$\lim_{t\to T_{\max}} \frac{x}{z} = \lim_{t\to T_{\max}} \frac{x'}{z'} = \lim_{t\to T_{\max}} \frac{x}{z}\frac{r_i}{r_k}$$

Since $\lim_{t\to T_{\max}} \frac{x}{z}$ is some positive constant, this implies $\lim_{t\to T_{\max}} \frac{r_i}{r_k} = 1$. The same reasoning implies $\lim_{t\to T_{\max}} \frac{r_j}{r_k} = 1$ as well. A quick computation shows that as $s \to \infty$ the ratio r_i/r_k tends to $-(x+y-z)/(x-y-z)$, and hence $(-x-y+z)/(x-y-z) \to 1$ as $t \to T_{\max}$, and hence also as $t \to \widetilde{T}_{\max}$. From this and the corresponding limit for the other quotient it follows that $x/z \to 1$ and $y/z \to 1$, and hence also $x/y \to 1$.

Now we show the limits $\lim_{t\to T_{\max}} x'/z'$ and $\lim_{t\to T_{\max}} y'/z'$ exist. We remark, that for the Ricci flow as well $\lim_{t\to T_{\max}} x/s = \lim_{t\to T_{\max}} z/s = 0$ since this is true for the normalized flow.

From (2.1.6),

$$\begin{aligned} xr_i &= 2\left(\frac{x^2-y^2-z^2}{yz}\right) + 4 + 4n\frac{x^2}{s^2} \to 2C^2D^{-1}D - 2D - 2D^{-1} + 4 \\ zr_k &= 2\left(\frac{z^2-x^2-y^2}{xy}\right) + 4 + 4n\frac{z^2}{s^2} \to 2C^{-1}D^{-1} - 2CD^{-1} - 2C^{-1}D + 4, \end{aligned}$$

and hence $\lim_{t\to T_{\max}} \frac{x'}{z'} = \lim_{t\to T_{\max}} \frac{x}{z}\frac{r_i}{r_k}$ exists. The calculation for the other ratios is similar.

One can see from the formula (2.1.7) that $S(\widetilde{g}_t) \to \infty$ at a rate of $6/x$ and hence if $x(t), y(t), z(t), s(t)$ are the eigenvalues of $S(\widetilde{g}_t)\widetilde{g}_t$, then $x(t), y(t), z(t) \to 6$ and $s(t) \to \infty$.

To see directly that $S(\widetilde{g}_t)\widetilde{g}_t$ converges in the pointed C^∞ topology to a standard metric on $S^3 \times \mathbb{R}^{4n}$, let $\epsilon > 0$ be any lower bound for the injectivity radius of the Fubini-Study metric on $\mathbb{HP}^n$. Then, since $s(t) \to \infty$, the collection of open sets $\{S^3 \times B_{\epsilon s(t)}(0)\}_{t\in[0,\widetilde{T}_{\max})}$ exhausts $S^3 \times \mathbb{R}^{4n}$. Let $f_t : S^3 \times B_{\epsilon s(t)}(0) \to S^{4n+3}$ be defined by $f_t(g, v) = g \cdot \gamma_{\frac{v}{s(t)}}(1)$ where $\gamma_{\frac{v}{s(t)}}(r)$ is a geodesic in S^{4n+3} beginning at p with horizontal initial velocity $\frac{v}{s(t)} \in \mathbb{H}^n$. Then, since the fibers of $S^3 \to S^{4n+3} \to \mathbb{HP}^n$ are embedded, f_t is a diffeomorphism onto its image. Moreover, it is not difficult to see using Jacobi fields that $f_t^*(S(\widetilde{g}_t)\widetilde{g}_t) \to 6g_{\mathrm{rd}} + \langle,\rangle_{\mathbb{R}^{4n}}$.

□

We can say slightly more about the qualitative behaviour of the Ricci flow on $\mathcal{M}_1^G$. Recall that by definition of the stable manifold, all metrics in it converge to g_{E_2}. Recall also that stable manifolds are always smooth manifolds (see e.g. [18] p.122).

Theorem 2.3.5. *The stable manifold for the second Einstein metric separates the space of metrics into two connected, invariant components, namely into the set of metrics which converge to the round metric and the set of metrics where $S \to \infty$ under the normalized flow.*

Proof. First we prove that the set of metrics in $\mathcal{M}_1^G$ with $S \to \infty$ under the normalized flow is open. Recall that the normalized flow is the L^2 gradient flow for S on $\mathcal{M}_1^G$. By [13], the set of Einstein metrics in $\mathcal{M}_1^G$ is compact, and hence has bounded scalar curvature, say, by α. Now let $\widetilde{g}_t$ be a solution of the normalized flow with $S(\widetilde{g}_t) \to \infty$. Then there exists a time t_0 such that $S(\widetilde{g}_{t_0}) > \alpha$. By continuous dependence on initial conditions, there is an open set U around $\widetilde{g}_0$ so that for every metric $h \in U$, the solution $\widetilde{h}_t$ of the normalized flow with $\widetilde{h}_0 = h$ satisfies $S(\widetilde{h}_{t_0}) > \alpha$. But by Palais-Smale, if the scalar curvature of a solution $\widetilde{h}_t$ surpasses α, then in fact $S(\widetilde{h}_t) \to \infty$.

Now, recall that any solution $\widetilde{g}_t$ either converges to an Einstein metric, or has $S(\widetilde{g}_t) \to \infty$ in finite time. Let $\gamma(t)$ be a path in $\mathcal{M}_1^G$ with $\gamma(0)$ converging to the round metric and $\gamma(1)$ a metric with $S \to \infty$ under the normalized flow. For each $t \in [0,1]$ define $F(t) \in \mathbb{R} \cup \{\infty\}$ so that $[0, F(t))$ is the maximal interval of existence of the normalized flow with initial condition $\gamma(t)$. Note that $F(0) = \infty$ and $F(1)$ is finite by Theorem 2.2.1. Let $t_0 = \inf(\{t \in [0,1] : F(t) = \infty\})$. Then we claim $\gamma(t_0)$ must lie in the stable manifold of the second Einstein metric. On the one hand, $\gamma(t_0)$ cannot converge to the round metric, since, as an attractor, the set of metrics converging to the round metric is open. On the other hand, since the set of metrics with finite extinction time for the normalized flow is open, $F(\gamma(t_0)) = \infty$ (finite extinction time is equivalent to $S(\widetilde{g}_t) \to \infty$). In particular, the solution of the normalized flow with initial condition $\gamma(t_0)$ must converge to an Einstein metric which is not the round metric, and hence must converge to the second Einstein metric. □

2.4. Ancient Solutions

We now turn to classifying the ancient solutions for the Ricci flow in $\mathcal{M}^G$. Recall that given an ancient solution g_t, there are two possibilities as $t \to -\infty$ for the corresponding normalized solution $\widetilde{g}_t$ in $\mathcal{M}_1^G$. Either $\widetilde{g}_t$ converges to an Einstein metric or $S(\widetilde{g}_t) \to 0$ and $|\mathrm{Ric}^0(\widetilde{g}_t)| \to 0$, i.e., $\widetilde{g}_t$ is 0-Palais-Smale. In [36], Pediconi proved that a 0-Palais-Smale sequence asymptotically approaches a submersion metric for a homogeneous fibration $K/H \to G/H \to G/K$ where K is some intermediate subgroup with K/H a torus. We will use this result, together with our monotonicity result, to argue that any such solution is actually a submersion metric *for all time* with respect to the Hopf fibration $\mathrm{U}(1) \to S^{4n+3} \to \mathbb{CP}^{2n+1}$. Besides these 0-Palais-Smale solutions, there are two more ancient solutions which converge to g_{E_2} as $t \to -\infty$. These arise by starting with the round metric and scaling the fibers and base of the Hopf fibration $S^3 \to S^{4n+3} \to \mathbb{HP}^n$ (see Section 1).

By Lemma 2.3.1 we can assume, up to isometry, that our solutions $\widetilde{g}_t$ satisfy $x \le y \le z$ for all t.

Lemma 2.4.1. *Let $\widetilde{g}_t$ be an ancient solution for the normalized flow with $x \le y \le z$ and $S(\widetilde{g}_t) \to 0$ as $t \to -\infty$. Then for all $t \in (-\infty, T_{max})$, $y = z$. In particular, $\widetilde{g}_t$ is invariant under the larger group* $\mathrm{U}(1)\mathrm{Sp}(n+1)$.

Proof. Since $\lim_{t\to-\infty} |\mathrm{Ric}^0(\widetilde{g}_t)| = 0$, it follows that for any sequence of times $t_i \to -\infty$, the sequence of metrics $\widetilde{g}(t_i)$ must diverge in $\mathcal{M}_1^G$, otherwise there would exist a subsequence converging to a flat metric, contrary to our assumption.

Each metric in $\mathcal{M}_1^G$ can be written uniquely in the form

$$g_{v,s} := e^{sv_1}\langle,\rangle_{(i)} + e^{sv_2}\langle,\rangle_{(j)} + e^{sv_3}\langle,\rangle_{(k)} + e^{sv_4}\langle,\rangle_{\mathbb{H}^n}$$

where $v_1^2 + v_2^2 + v_3^2 + v_4^2 = 1$ and $v_1 + v_2 + v_3 + v_4 = 0$.

Define the sequences $v^{(i)} \in S^3$, $s^{(i)} \in \mathbb{R}$ by $\widetilde{g}(t_i) = g_{v^{(i)},s^{(i)}}$. Then, since S^3 is compact, there exists a subsequence $v^{(i)} \to v^{(\infty)}$ and $s^{(i)} \to \infty$. By Theorem 4.1 in [36], $v^{(\infty)}$ is a so-called submersion direction for some toral H-subalgebra $\mathfrak{k}$, that is, a subalgebra $\mathfrak{k} = \mathrm{Lie}(K)$ where K is connected, $H \subset K \subset G$, and the quotient K/H is a torus. Moreover, $\mathfrak{k} \cap \mathfrak{p}$ is generated by the Ad_H-irreducible summands of $\mathfrak{p}$ corresponding to the most shrinking eigenvalue. By Proposition 3.10 in [36], if v is a submersion direction for an H-subalgebra $\mathfrak{k}$, then $g_{v,s}$ is a $\mathfrak{k}$-submersion metric for all $s \in \mathbb{R}$ and moving along the path $\gamma_v(s) = g_{v,s}$ is equivalent to shrinking the fibers of the homogeneous fibration $K/H \to G/H \to G/K$.

In our case $v^{(\infty)}$ is a submersion direction for some toral $\mathrm{Sp}(n)$-subalgebra. On the other hand, the only toral subalgebras containing $\mathfrak{sp}(n)$ are isomorphic to $\mathfrak{sp}(n) \oplus \mathfrak{u}(1)$, where $\mathfrak{u}(1)$ is the Lie algebra of some circle subgroup of $\mathrm{Sp}(1)$ (the only Lie subgroups of $\mathrm{Sp}(n+1)$ containing $\mathrm{Sp}(n)$ are isomorphic to $\mathrm{Sp}(n)$, $\mathrm{U}(1)\mathrm{Sp}(n)$ and $\mathrm{Sp}(1)\mathrm{Sp}(n)$). Since we assumed $x \le y \le z$, it follows that x is the most shrinking eigenvalue, and hence $\mathfrak{k} = \mathfrak{sp}(n) \oplus (i)$.

Let $(v_1^{(\infty)}, v_2^{(\infty)}, v_3^{(\infty)}, v_4^{(\infty)})$ be the components of $v^{(\infty)}$. Since $\gamma_{v^{(\infty)}}(s)$ is invariant under the larger isometry group $\mathrm{U}(1)\mathrm{Sp}(n+1)$ where $\mathrm{U}(1) = \{e^{i\theta}\}_{\theta\in[0,2\pi)} \subset \mathrm{Sp}(1)$, we can conclude that $v_2^{(\infty)} = v_3^{(\infty)}$ and $v_1^{(\infty)} < 0$. Moreover, since $Q([j,k],i) = 1 \neq 0$, Theorem 4.1 in [36] further implies that $y(t_i)/z(t_i) \to 1$ as $t_i \to -\infty$.

On the other hand, by Lemma 2.3.4, along the backwards flow y/z is non-increasing. Hence

$\lim_{t\to-\infty} y/z$ exists and equals 1. But again, since y/z is non-increasing and $y/z \le 1$, this is only possible if $y = z$ for all t. $\square$

Hence for the purpose of classifying ancient solutions, it suffices to consider metrics in $\mathcal{M}_1^G$ of the form

$$\frac{1}{y^2 s^{4n}}\langle,\rangle_{(i)} + y\langle,\rangle_{(j)} + y\langle,\rangle_{(k)} + s\langle,\rangle_{\mathbb{H}^n}. \tag{2.4.1}$$

Moreover, referring to the above proof, since $v_1^{(\infty)} < 0$, we can assume that $\frac{1}{y^2 s^{4n}} \to 0$ for ancient solutions that do not converge to an Einstein metric as $t \to -\infty$. Notice that in this section our normalization differs from the one in Section 3. For metrics of the form (2.4.1), the scalar curvature is given by

$$S = \frac{16n^2}{s} - \frac{8ny}{s^2} - \frac{1}{s^{4n}y^2}\left(\frac{4n}{s^2} + \frac{2}{y^2}\right) + \frac{32n}{s} + \frac{8}{y}. \tag{2.4.2}$$

We prove the following classification result.

Theorem 2.4.2. *Let $\widetilde{g}_0 = g_{x,y,z,s}$ with $x \le y \le z$. Then $\widetilde{g}_t$ is ancient if and only if $x \le y = z \le s$.*

Note that metrics with $y = \frac{1}{y^2 s^{4n}}$ are precisely the ones invariant under the larger group of isometries $\mathrm{Sp}(1)\mathrm{Sp}(n+1)$, and hence these converge to g_{E_2} as $t \to -\infty$. Metrics with $y = s$ are invariant under the group $\mathrm{U}(2n+2)$ by Lemma 2.3.1 and hence are preserved. These two solutions were shown to be ancient in [6]. We begin with a lemma.

Lemma 2.4.3. *For any ancient solution with $\lim_{t\to-\infty} S(\widetilde{g}_t) = 0$, the ratio y/s remains bounded as $t \to -\infty$. Moreover, if $\lim_{t\to-\infty} y/s$ exists and is non-zero, then the only possibilities are $\lim_{t\to-\infty} y/s = 1$ or $\frac{1}{1+n}$.*

Proof. We can bound (2.4.2) above by

$$S < \frac{8n}{s}\left(2n + 4 - \frac{y}{s}\right) + \frac{8}{y} \tag{2.4.3}$$

If $y/s > M$ then we can further bound (2.4.3) above by

$$S < \left(8n(2n + 4 - M) + \frac{8}{M}\right)\frac{1}{s}$$

which is negative if M is sufficiently large. But ancient solutions of the Ricci flow (and hence also of the normalized flow) have non-negative scalar curvature, and hence y/s must be bounded.

Now we examine the possible limits for y/s as $t \to -\infty$. Since $\frac{1}{y^2 s^{4n}} \to 0$ and y/s is bounded as $t \to -\infty$, $s \to \infty$ as well.

Suppose that $\lim_{t\to-\infty} \frac{y}{s} = C > 0$. Then since $s \to \infty$, $y \to \infty$ as well. For the Ricci flow,

$$\lim_{t\to-\infty} \frac{y'}{s'} = \lim_{t\to-\infty} \frac{y}{s}\frac{r_j}{r_h}.$$

From (2.1.6) we have

$$yr_j = -\frac{2x}{y} + 4 + \frac{4ny^2}{s^2} \quad \text{and} \quad sr_h = 8 + 4n - \frac{2x}{s} - \frac{4y}{s}.$$

Since $x/y \to 0$ and $x/s \to 0$ under the normalized flow, the same is true for the Ricci flow. Thus, since $y, s \to \infty$ and since we assumed $\lim_{t\to-\infty} \frac{y}{s} = C$, both of the above quantities tend to finite limits, and hence $\lim_{t\to-\infty} \frac{y'}{s'}$ exists. But then also

$$C = \lim_{t\to-\infty} \frac{y}{s} = \lim_{t\to-\infty} \frac{y'}{s'} = \frac{4 + 4nC^2}{8 + 4n - 4C}.$$

Solving the above equation yields $C = 1$ or $C = \frac{1}{1+n}$. Since the ratio y/s is scale-invariant, the same holds for the normalized flow.

∎

Note that these two ratios correspond to the two homogeneous Einstein metrics $g^{\mathrm{FS}}_{\mathbb{CP}^{2n+1}}$ and $g^{2}_{\mathbb{CP}^{2n+1}}$ on the base $\mathbb{CP}^{2n+1}$ (see Section 2 and [48]).

Lemma 2.4.4. *Solutions with $y/s > 1$ are not ancient.*

Proof. We will show that if $y/s > 1$ then $(y/s)' > 0$ under the backwards flow. But since y/s is bounded for any ancient solution, y/s would converge to a finite limit greater than 1, which would contradict the previous lemma.

The derivative of y/s under the backwards flow is

$$(ys)\frac{r_j - r_h}{s^2},$$

which has the same sign as

$$r_j - r_h = -\frac{8+4n}{s} - \frac{2}{y^4 s^{4n}} + \frac{2}{y^2 s^{2+4n}} + \frac{4}{y} + \frac{(4+4n)y}{s^2} \tag{2.4.4}$$

$$= \frac{2(y-s)\left(s + y + 2s^{4n}y^3((1+n)y - s)\right)}{y^4 s^{2+4n}}. \tag{2.4.5}$$

which is positive since $y > s$. ∎

Now to conclude the proof of Theorem 2.4.2 we only need to show that the remaining solutions are ancient.

Lemma 2.4.5. *Solutions satisfying $\frac{1}{y^2 s^{4n}} \leq y \leq s$ are ancient for the normalized flow. Furthermore, along such a solution $y, s \to \infty$ and either $y = s$ or $y/s \to \frac{1}{1+n}$.*

Proof. We already saw that metrics satisfying $y = \frac{1}{y^2 s^{4n}}$ or $y = s$ are preserved by the Ricci flow and are ancient. Hence, solutions which begin in the set $\Gamma = \{\frac{1}{y^2 s^{4n}} \leq y \leq s\}$ remain in that set, and we can assume from now on $\frac{1}{y^2 s^{4n}} < y < s$.

Now, we prove that $s \to \infty$ for any solution $\widetilde{g}_t$ in the interior of Γ. Assume $\widetilde{g}_t$ is not ancient. Then one of the variables must go to 0 or ∞ as $t \to \widetilde{T}_{\min}$. We will show that in each possible scenario, $s \to \infty$. If $s \to 0$ then since $y/s < 1$, $y \to 0$ as well, but this contradicts $y^3 > \frac{1}{s^{4n}}$. If $y \to \infty$, then $y/s < 1$ implies $s \to \infty$ as well. If $y \to 0$ then $y^3 > \frac{1}{s^{4n}}$ implies $s \to \infty$.

Next, we look at the derivative of y/s under the backwards flow, which, as before, has the same sign as (2.4.5), except now $y - s < 0$, since $\widetilde{g}_t$ is in the interior of Γ. Hence it has the same sign as

$$-s - y + 2s^{4n}y^3(s - (1+n)y). \tag{2.4.6}$$

Since $s \to \infty$, we see further, that for fixed y/s, and for large enough s, (2.4.6) is positive if $y/s < \frac{1}{1+n}$ and negative if $y/s \geq \frac{1}{1+n}$. Hence y/s does not return to the same value infinitely many times. But this implies $y/s \to \frac{1}{1+n}$, for if y/s crosses $\frac{1}{1+n}$, then one can argue that y/s is eventually contained in any neighborhood of $\frac{1}{1+n}$. If $y/s > \frac{1}{1+n}$ for all time, then y/s must converge to $\inf_{t \in (\widetilde{T}_{\min}, \widetilde{T}_{\max})}\{(y/s)(t)\}$, and hence by Lemma 2.4.3, must converge to $\frac{1}{1+n}$, and similarly if $y/s < \frac{1}{1+n}$ for all time. Thus $y/s \to \frac{1}{1+n}$ and both $y, s \to \infty$.

From the formula for the scalar curvature (2.4.2), it follows that $S \to 0$, and, in particular, S is bounded from below, and thus the solution is ancient.

□

Proposition 2.4.1. *Solutions satisfying $\frac{1}{y^2 s^{4n}} < y \leq s$ are collapsed. Moreover, if $\frac{1}{y^2 s^{4n}} < y < s$ then a rescaling of $\widetilde{g}_t$ converges in the Gromov-Hausdorff sense to $g^2_{\mathbb{CP}^{2n+1}}$ as $t \to -\infty$, and if $y = s$ then a rescaling of $\widetilde{g}_t$ converges to $g^{FS}_{\mathbb{CP}^{2n+1}}$ as $t \to \infty$.*

Proof. By Lemma 2.4.5, such solutions satisfy $y, s \to \infty$, and $\lim_{t \to -\infty} y/s = 1$ or $\lim_{t \to -\infty} y/s = \frac{1}{1+n}$. From equations (2.1.6), it follows that the eigenvalues of the Ricci tensor decay at a rate of $O(\frac{1}{y})$ as $t \to -\infty$, and hence $|\mathrm{Ric}(\widetilde{g}_t)|$ decays at a rate of $O(\frac{1}{y})$. By the Gap theorem [12], this implies $|\mathrm{Rm}(\widetilde{g}_t)|$ decays at a rate of $O(\frac{1}{y})$ as $t \to -\infty$ as well. Thus the length of i goes to zero for the curvature normalized solution $|\mathrm{Rm}(\widetilde{g}_t)|\widetilde{g}_t$. Since the U(1) fibers are totally-geodesic, this implies the injectivity radius tends to zero, and hence $\widetilde{g}_t$ is collapsed. In the proof of Lemma 2.4.5, we showed that if $y < s$ then $\frac{y}{s} \to \frac{1}{1+n}$. In particular, for every solution in our 1-parameter family (besides the one with $y = s$), the metric on the base tends to the second Einstein metric on $\mathbb{CP}^{2n+1}$ (see [48]).

□

Remark 2.4.6. By Theorem B in [45], metrics with $x \leq y = z \leq s$ have positive sectional curvature. This fact and Lemma 2.3.1 already imply these solutions are ancient.

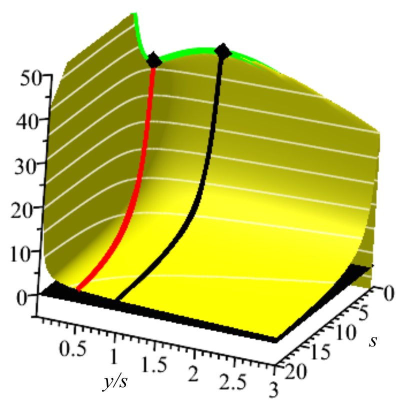

Figure 2.2: The graph of S over metrics with $y = z$ when $n = 1$ (compare with Figure 1.1). The green line represents the $\mathrm{Sp}(n+1)\mathrm{Sp}(1)$-invariant metrics, and the black line represents the line $y = s$, or the $\mathrm{U}(2n+2)$-invariant metrics. The diamonds represent the round metric, which is a local maximum, and Jensen's second Einstein metric, which is a saddle point. Ancient solutions with $y < s$ asymptotically approach the red line, which represents the stable manifold for Jensen's second Einstein metric.

CHAPTER 3

Ancient solutions on compact homogeneous spaces

3.1. Preliminaries on compact homogeneous spaces

3.1.1. The space of invariant metrics

Let $M = \mathsf{G}/\mathsf{H}$ be a compact, connected and almost-effective m-dimensional homogeneous space, where G is a compact Lie group and H a closed subgroup. Furthermore assume that M is not a torus. Notice that neither G nor H are assumed to be connected.

Fix an $\mathrm{Ad}(\mathsf{G})$-invariant Euclidean inner product Q on the Lie algebra $\mathfrak{g} = \mathrm{Lie}(\mathsf{G})$ and denote by $\mathfrak{m}$ the Q-orthogonal complement of $\mathfrak{h} = \mathrm{Lie}(\mathsf{H})$ in $\mathfrak{g}$. By means of the canonical identification $\mathfrak{m} \simeq T_{e\mathsf{H}}M$ given by the evaluation map

$$V \mapsto V^*_{e\mathsf{H}} = \tfrac{\mathrm{d}}{\mathrm{d}s} \exp(sV)\mathsf{H}\big|_{s=0} ,$$

we identify any G-invariant tensor field on M with the corresponding $\mathrm{Ad}(\mathsf{H})$-invariant tensor on $\mathfrak{m}$. The restriction $Q_{\mathfrak{m}} = Q|_{\mathfrak{m}\otimes\mathfrak{m}}$ defines a *normal* G-invariant Riemannian metric on M.

We denote by $\mathcal{M}^{\mathsf{G}}_M$ the set of G-invariant Riemannian metrics on M, which is identified with the linear space of $Q_{\mathfrak{m}}$-symmetric, $\mathrm{Ad}(\mathsf{H})$-invariant, positive-definite endomorphisms of $\mathfrak{m}$, i.e.

$$\mathcal{M}^{\mathsf{G}}_M = \mathrm{Sym}_+(\mathfrak{m}, Q_{\mathfrak{m}})^{\mathrm{Ad}(\mathsf{H})} , \tag{3.1.1}$$

by means of the correspondence

$$g \mapsto P_g , \quad Q_{\mathfrak{m}}(P_g.V_1, V_2) = g(V_1, V_2) \quad \text{for any } V_1, V_2 \in \mathfrak{m} . \tag{3.1.2}$$

From now on, we will always identify a metric with the associated endomorphism via (3.1.2).

We recall that (3.1.1) provides the set $\mathcal{M}^{\mathsf{G}}_M$ with a structure of finite-dimensional smooth manifold. Moreover, the natural L^2-metric defined by

$$\langle B_1, B_2\rangle_P = \det(P)^{\frac{1}{2}}\,\mathrm{Tr}(P^{-1}.B_1.P^{-1}.B_2) \quad \text{for any } B_1, B_2 \in T_P M^{\mathsf{G}}_M = \mathrm{Sym}(\mathfrak{m}, Q_{\mathfrak{m}})^{\mathrm{Ad}(\mathsf{H})}$$

turns M^{G}_M into a Riemannian symmetric space of non-compact type and the subset

$$M^{\mathsf{G}}_{M,1} = \{P \in \mathrm{Sym}_+(\mathfrak{m}, Q_{\mathfrak{m}})^{\mathrm{Ad}(\mathsf{H})} : \det(P) = 1\}$$

of unit volume G-invariant Riemannian metrics into a totally geodesic submanifold.

For any Riemannian metric $P \in M^{\mathsf{G}}_M$, we consider the $\mathrm{Ad}(\mathsf{H})$-invariant map

$$S_M(P) : \mathfrak{m} \to \mathrm{End}(\mathfrak{m})$$

defined by (see [29, Thm 3.3, Ch X])

$$\begin{aligned} -2Q_{\mathfrak{m}}(S_M(P)(V_1).V_2, V_3) &= Q_{\mathfrak{m}}([V_1, V_2]_{\mathfrak{m}}, V_3) \\ &\quad + Q_{\mathfrak{m}}([P^{-1}.V_3, V_1]_{\mathfrak{m}}, P.V_2) + Q_{\mathfrak{m}}([P^{-1}.V_3, V_2]_{\mathfrak{m}}, P.V_1)\ . \end{aligned} \tag{3.1.3}$$

Here, the symbol $[V_1, V_2]_{\mathfrak{m}}$ denotes the Q-orthogonal projection of $[V_1, V_2]$ on $\mathfrak{m}$. The map $S_M(P)$ corresponds to the G-invariant $(1, 2)$-tensor field on M given by the difference between the canonical Ambrose-Singer connection and the Levi-Civita connection (see e.g. [38]). It is worth mentioning that this tensor encodes all the geometric information about the metric P. Indeed, following [29, Thm 2.3, Ch X], the *Riemannian curvature tensor* $\mathrm{Rm}_M(P)$ of P is explicitly expressed in terms of $S_M(P)$ by

$$\mathrm{Rm}_M(P)(V_1, V_2) = \mathrm{ad}([V_1, V_2]_{\mathfrak{h}}) - [S_M(P)(V_1), S_M(P)(V_2)] - S_M(P)([V_1, V_2]_{\mathfrak{m}})\ , \tag{3.1.4}$$

where again the $[V_1, V_2]_{\mathfrak{h}}$ denotes the Q-orthogonal projection of $[V_1, V_2]$ on $\mathfrak{h}$. Consequently, the *Ricci curvature* $\mathrm{Ric}_M(P)$ of P is

$$Q_{\mathfrak{m}}(\mathrm{Ric}_M(P).V_1, V_2) = \mathrm{Tr}\big(\mathrm{Rm}_M(P)(V_1, \cdot\,).V_2\,\big) \tag{3.1.5}$$

and the *scalar curvature* $\mathrm{scal}_M(P)$ of P

$$\mathrm{scal}_M(P) = \mathrm{Tr}(P^{-1}.\mathrm{Ric}_M(P))\ . \tag{3.1.6}$$

Notice that, according to (3.1.5), we denote by Ric_M the endomorphism obtained by raising an index of the Ricci bilinear form by means of the background metric Q. Therefore, the standard "Ricci endomorphism" corresponds in our notation to $P^{-1}.\mathrm{Ric}_M(P)$.

We also denote by

$$\mathrm{Ric}^0_M(P) = \mathrm{Ric}_M(P) - \frac{\mathrm{scal}_M(P)}{m}P \tag{3.1.7}$$

the *traceless Ricci curvature of* P and we recall that P is said to be *Einstein* if $\mathrm{Ric}^0_M(P) = 0$.

We finally mention that Einstein metrics are the critical points of the *normalized scalar curvature functional*

$$\widetilde{\mathrm{scal}}_M : \mathcal{M}^{\mathsf{G}}_M \to \mathbb{R}\ , \quad \widetilde{\mathrm{scal}}_M(P) = \det(P)^{\frac{1}{m}}\,\mathrm{scal}_M(P)\ .$$

Indeed, following [7, Ch 4] the differential of $\widetilde{\mathrm{scal}}_M$ at $P \in \mathcal{M}^{\mathsf{G}}_M$ in the direction of $B \in T_P\mathcal{M}^{\mathsf{G}}_M$ is

$$\mathrm{d}\,\widetilde{\mathrm{scal}}_M|_P(B) = -\det(P)^{\frac{2-m}{2m}}\langle \mathrm{Ric}^0_M(P), B\rangle_P \tag{3.1.8}$$

and so $\mathrm{d}\,\widetilde{\mathrm{scal}}_M|_P = 0$ if and only if P is Einstein.

3.1.2. Homogeneous torus bundles and the coindex of Einstein metrics

Let us consider a *toral* H-*subalgebra* $\mathfrak{k}$ *of* $\mathfrak{g}$, that is, an $\mathrm{Ad}(\mathsf{H})$-invariant Lie subalgebra of $\mathfrak{g}$ which lies properly between $\mathfrak{h}$ and $\mathfrak{g}$ such that $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{h}$. Then, if we denote by K^o the connected Lie subgroup of G with Lie algebra equal to $\mathfrak{k}$, it turns out that the subgroup K

generated by H and K^o is a (not necessarily closed) Lie subgroup of G and $\mathsf{T} = \mathsf{K}/\mathsf{H}$ is a (immersed) torus in M. This gives rise to a (locally defined) homogeneous torus fibration

$$\mathsf{T} = \mathsf{K}/\mathsf{H} \to M = \mathsf{G}/\mathsf{H} \to N = \mathsf{G}/\mathsf{K} \ . \tag{3.1.9}$$

For more details on this construction, see e.g. [9, Sect 4], [36, Sect 3] and [37, Prop 6.1].

At the Lie algebra level, we get the Q-orthogonal decomposition

$$\mathfrak{g} = \underbrace{\mathfrak{h} + \overbrace{\mathfrak{t}}^{\mathfrak{m}}}_{\mathfrak{k}}\overbrace{+\mathfrak{n}}\ , \qquad \text{with} \quad \mathfrak{t} = \mathrm{Lie}(\mathsf{T}) \ , \ \ \mathfrak{n} \simeq T_{e\mathsf{K}}N \ . \tag{3.1.10}$$

We recall that a metric $P \in \mathcal{M}^{\mathsf{G}}_M$ is called $\mathfrak{k}$-*submersion metric* if it preserves the decomposition $\mathfrak{m} = \mathfrak{t} + \mathfrak{n}$ and its restriction to the subspace $\mathfrak{n}$ is $\mathrm{Ad}(\mathsf{K})$-invariant. We denote by $\mathcal{M}^{\mathsf{G}}_M(\mathfrak{k})$ the subset of all the $\mathfrak{k}$-submersion metrics and observe that it naturally splits as

$$\mathcal{M}^{\mathsf{G}}_M(\mathfrak{k}) = \mathrm{Sym}_+(\mathfrak{t}, Q_{\mathfrak{t}})^{\mathrm{Ad}(\mathsf{H})} \oplus \mathrm{Sym}_+(\mathfrak{n}, Q_{\mathfrak{n}})^{\mathrm{Ad}(\mathsf{K})} \ , \quad P = P_{\mathfrak{t}} \oplus P_{\mathfrak{n}} \tag{3.1.11}$$

where $Q_{\mathfrak{t}} = Q|_{\mathfrak{t}\otimes\mathfrak{t}}$ and $Q_{\mathfrak{n}} = Q|_{\mathfrak{n}\otimes\mathfrak{n}}$. Notice that any $P \in \mathcal{M}^{\mathsf{G}}_M(\mathfrak{k})$ turns the (locally) homogeneous torus fibration (3.1.9) into a Riemannian submersion with totally geodesic fibers (see e.g. [36, Sect 3.2]). Notice that all the metrics in $\mathcal{M}^{\mathsf{G}}_M(\mathfrak{k})$ are invariant under the action of the larger group $\mathsf{G}\times\mathsf{T}$, which acts on $M = \mathsf{G}/\mathsf{H}$ via $(a, n) \cdot b\mathsf{H} = abn^{-1}\mathsf{H}$ with isotropy at the origin $\mathsf{H}\Delta\mathsf{T} = \{(hn, n) : h \in \mathsf{H}, n \in \mathsf{T}\}$.

Let us consider now a *maximal toral* H-*subalgebra* of $\mathfrak{g}$, i.e. a toral H-subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ such that $\mathsf{T} = \mathsf{K}/\mathsf{H}$ is a maximal torus of a compact complement of H^o in $\mathsf{N}_{\mathsf{G}}(\mathsf{H}^o)^o$. Here, we denote by H^o the identity component of H and by $\mathsf{N}_{\mathsf{G}}(\mathsf{H}^o)^o$ the identity component of the normalizer of H^o in G. Notice that this condition implies that K is closed in G and hence $N = \mathsf{G}/\mathsf{K}$ is a compact homogeneous space. Moreover, it also implies the following

Lemma 3.1.1. *The complement* $\mathfrak{n}$ *in* (3.1.10) *does not contain any* $\mathrm{Ad}(\mathsf{K})$-*invariant submodule on which* $\mathrm{Ad}(\mathsf{K}^o)$ *acts trivially.*

Proof. Let $\widetilde{\mathfrak{n}} \subset \mathfrak{n}$ be an $\mathrm{Ad}(\mathsf{K})$-invariant submodule such that $\mathrm{Ad}(\mathsf{K}^{\mathrm{o}}).X = \{X\}$ for any $X \in \widetilde{\mathfrak{n}}$. Then, this implies that $\widetilde{\mathfrak{k}} = \mathfrak{k} + \widetilde{\mathfrak{n}}$ is a toral H-subalgebra of $\mathfrak{g}$. Since $\mathfrak{k}$ is assumed to be maximal, it follows that $\widetilde{\mathfrak{k}} = \mathfrak{k}$ and so $\widetilde{\mathfrak{n}} \subset \mathfrak{k}$. Since $\mathfrak{k}$ and $\mathfrak{n}$ are Q-orthogonal, we get $\widetilde{\mathfrak{n}} = \{0\}$. □

Let now $\overline{P}_{\mathfrak{n}} \in M^{\mathsf{G}}_{N,1}$ be a unit volume Einstein metric on N. Then $\mathrm{Ric}^0_N(\overline{P}_{\mathfrak{n}}) = 0$ and so, by (3.1.8), it follows that

$$\mathrm{Hess}\big(\mathrm{scal}_N|_{M^{\mathsf{G}}_{N,1}}\big)\big|_{\overline{P}_{\mathfrak{n}}}(B_1, B_2) = -\big\langle \mathrm{d}\big(\mathrm{Ric}^0_N|_{M^{\mathsf{G}}_{N,1}}\big)\big|_{\overline{P}_{\mathfrak{n}}}(B_1), B_2\big\rangle_{\overline{P}_{\mathfrak{n}}} \tag{3.1.12}$$

for any $B_1, B_2 \in T_{\overline{P}_{\mathfrak{n}}} M^{\mathsf{G}}_{N,1}$. Therefore, in virtue of (3.1.12) and [31, Def 3.14], we recall the following notion of coindex for invariant Einstein metrics on N.

Definition 3.1.2. The *coindex of a unit volume Einstein metric* $\overline{P}_{\mathfrak{n}} \in M^{\mathsf{G}}_{N,1}$ is its coindex as a critical point of the restricted scalar curvature functional $\mathrm{scal}_N|_{M^{\mathsf{G}}_{N,1}}$, i.e. the number of negative eigenvalue of the linear map $\mathrm{d}\big(\mathrm{Ric}^0_N|_{M^{\mathsf{G}}_{N,1}}\big)\big|_{\overline{P}_{\mathfrak{n}}}$.

We refer to [31, 27] for a detailed treatment on stability and non-degeneracy of invariant Einstein metrics on homogeneous spaces.

3.1.3. Ancient solutions to the Ricci flow

We recall that a solution to the *Ricci flow* on M is a smooth 1-parameter family of metrics that evolve in the direction of their Ricci tensors. By diffeomorphism invariance of the Ricci tensor, isometries are preserved by the Ricci flow, and hence one can restrict it to a dynamical system on the space of G-invariant metrics M^{G}_M, i.e.

$$P'(t) = -2\mathrm{Ric}_M(P(t)) \ , \quad P(0) = P_{\mathrm{o}} \ .$$

If $P_{\mathrm{o}} \in M^{\mathsf{G}}_{M,1}$, then the *normalized Ricci flow* on M starting at P_{o} takes the form

$$\widetilde{P}'(t) = -2\mathrm{Ric}^0_M(\widetilde{P}(t)) \ , \quad \widetilde{P}(0) = P_{\mathrm{o}}$$

where the traceless Ricci tensor has been defined in (3.1.7). It is well known that the normalized Ricci flow preserves the submanifold $M^{\mathsf{G}}_{M,1}$ and that it is equivalent to the Ricci flow up to rescaling and time reparametrization. Moreover, by (3.1.8), the normalized Ricci flow coincides, up to a positive constant, with the L^2-gradient flow of the restricted scalar curvature functional on $M^{\mathsf{G}}_{M,1}$.

In [13], the authors studied the global behaviour of the restricted scalar curvature functional on $M^{\mathsf{G}}_{M,1}$ in order to prove the existence of Einstein metrics using variational techniques. In particular, the authors proved that for any $\epsilon > 0$, the scalar curvature functional satisfies the *Palais-Smale compactness condition* on the set

$$(M^{\mathsf{G}}_{M,1})_\epsilon = \{P \in M^{\mathsf{G}}_{M,1} : \mathrm{scal}_M(P) > \epsilon\} \ ,$$

that is, if $(P^{(n)}) \subset M^{\mathsf{G}}_{M,1}$ is a sequence with

$$\mathrm{scal}_M(P^{(n)}) \to \epsilon \quad \text{ and } \quad \big\langle \mathrm{Ric}^0_M(P^{(n)}), \mathrm{Ric}^0_M(P^{(n)}) \big\rangle_{P^{(n)}} \to 0 \quad \text{ as } n \to +\infty \ ,$$

M M scal then there exists a subsequence of $(P^{(n)})$ converging in the $\mathcal{C}^\infty$-topology to an Einstein metric $P^{(\infty)} \in M^{\mathsf{G}}_{M,1}$, as $n \to +\infty$, with $\mathrm{scal}_M(P^{(\infty)}) = \epsilon$. In general, the Palais-Smale compactness condition does not hold on the full space $M^{\mathsf{G}}_{M,1}$ due to the existence of the so called *0-Palais-Smale sequences*, that are $(P^{(n)}) \subset M^{\mathsf{G}}_{M,1}$ such that $\mathrm{scal}_M(P^{(n)}) \to 0$ and $\big\langle \mathrm{Ric}^0_M(P^{(n)}), \mathrm{Ric}^0_M(P^{(n)}) \big\rangle_{P^{(n)}} \to 0$ as $n \to +\infty$. Notice that such sequences cannot admit convergent subsequences since M is not a torus. In fact, the limit of any convergent subsequence would be a Ricci-flat, and hence flat (see [2]), G-invariant metric. By [13, Thm 2.1], the existence of such a solution implies that $\mathsf{G}^{\mathrm{o}}/\mathsf{H}^{\mathrm{o}}$ is the total space of a homogeneous torus bundle, where G^{o} (resp. H^{o}) denotes the identity component of G (resp. H). More precisely, since 0-Palais-Smale sequences have bounded sectional curvature by the Gap theorem [12], by [36] we know that the sum of the eigenspaces associated to the shrinking eigenvalues of any 0-Palais-Smale sequence converges to a reductive complement of $\mathfrak{h}$ into a toral H-subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ and that such sequences collapse along the fibers of the induced (locally) homogeneous torus fibration (3.1.9) while asymptotically approaching, in a precise sense, a $\mathfrak{k}$-submersion metric.

Now let $P(t)$ be the solution to the Ricci flow starting from $P_o \in \mathcal{M}^{\mathsf{G}}_{M,1}$ and $\widetilde{P}(t)$ the corresponding solution to the normalized Ricci flow. We recall that $P(t)$ (resp. $\widetilde{P}(t)$) is said to be *ancient* if it exists on the time interval $(-\infty, 0]$. It is a well known consequence of the maximum principle that if $P(t)$ is ancient, then it must have monotonic non-negative scalar curvature (see e.g. [23, p. 102]). Since the two flows are equivalent up to rescaling and time reparametrization, the same is true for the solution $\widetilde{P}(t)$. Furthermore, by [44], $P(t)$ is ancient if and only if $\widetilde{P}(t)$ is ancient. In particular there are exactly two possibilities for the behaviour of the normalized Ricci flow as $t \to -\infty$.

The first possibility is that there exists an $\epsilon > 0$ such that $\mathrm{scal}_M(\widetilde{P}(t)) > \epsilon$ for any $t \leq 0$, in which case $\widetilde{P}(t)$ (and hence $P(t)$) is *non-collapsed* and, by [12, Thm 5.2], $\widetilde{P}(t)$ converges to an Einstein metric as $t \to -\infty$. Since the traceless Ricci tensor is the negative L^2-gradient of the functional $\mathrm{scal}_M|_{\mathcal{M}^{\mathsf{G}}_{M,1}}$, such ancient solutions are known to exist whenever M admits a G-*unstable*, G-invariant Einstein metric (see e.g. [3, 12]). The second possibility is that $\mathrm{scal}_M(\widetilde{P}(t)) \to 0$ as $t \to -\infty$. In this case, one can always find a sequence of times $t^{(n)} \to -\infty$ such that $P(t^{(n)})$ is a 0-Palais-Smale sequence and so $\widetilde{P}(t)$ (and hence $P(t)$) is *collapsed*. Indeed, for the sake of the reader, we recall the following

Remark 3.1.1. A 1-parameter family $\{P(t)\}_{t\in I}$ of G-invariant metrics, $I \subset \mathbb{R}$ an interval, is said to be *non-collapsed* if there exists $\delta > 0$ such that

$$\mathrm{inj}(P(t))\big(|\mathrm{Rm}_M(P(t))|_{P(t)}\big)^{\frac{1}{2}} \geq \delta \quad \text{for any } t \in I \ ,$$

where $\mathrm{inj}(P)$ denotes the injectivity radius of the metric P at the origin $e\mathsf{H}$ and $|\cdot|_P$ denotes the norm on $\mathfrak{m}$, and hence on the tensor space over $\mathfrak{m}$, induced by P. Accordingly, $\{P(t)\}_{t\in I}$ is said to be *collapsed* if it is not non-collapsed, i.e. if there exists a sequence $(t^{(n)}) \subset I$ such that

$$\mathrm{inj}(P(t^{(n)}))\big(|\mathrm{Rm}_M(P(t^{(n)}))|_{P(t^{(n)})}\big)^{\frac{1}{2}} \to 0 \quad \text{as } n \to +\infty \ .$$

These properties are invariant under time-dependent rescaling and time reparametrization.

We also give a proof of the following known result (see e.g. [12, Rem 5.3]).

Proposition 3.1.3. *Let $P(t)$ be an ancient solution to the homogeneous Ricci flow on $M = \mathsf{G}/\mathsf{H}$ starting from $P_{\mathrm{o}} \in \mathcal{M}^{\mathsf{G}}_{M,1}$ and $\widetilde{P}(t)$ the corresponding solution to the normalized Ricci flow. Then, $P(t)$ is collapsed if and only if $\mathrm{scal}_M(\widetilde{P}(t)) \to 0$ as $t \to -\infty$.*

Proof. By [12], there exists a constant $C > 0$, depending on the starting metric P_{o}, such that

$$-t|\mathrm{Rm}_M(P(t))|_{P(t)} < C \quad \text{for any } t \leq 0 \ .$$

In particular, by means of Remark 3.1.1 and the Cheeger-Gromov compactness theorem, it follows that $P(t)$ is non-collapsed if and only if for any sequence $t^{(n)} \to -\infty$ there exists a subsequence $(t^{(n_i)}) \subset (t^{(n)})$ such that $P(t^{(n_i)})$ converges in the $\mathcal{C}^{1,\alpha}$-topology to a limit metric on M as $i \to +\infty$. Moreover, the limit metric is necessarily invariant with respect to the same action of G on M and the convergence is G-equivariant. As in the proof of [13, Thm 1.1], we notice that G-equivariant convergence actually takes place in the $\mathcal{C}^{\infty}$-topology (see also [38]). Moreover, since $\widetilde{P}(t)$ coincides, up to time reparametrization, to the volume-normalized family $\det(P(t))^{-\frac{1}{m}}P(t)$, it follows that: $P(t)$ is non-collapsed if and only if for any sequence $t^{(n)} \to -\infty$ there exists a subsequence $(t^{(n_i)}) \subset (t^{(n)})$ such that $\widetilde{P}(t^{(n_i)})$ converges in the $\mathcal{C}^{\infty}$-topology to a limit G-invariant metric in $\mathcal{M}^{\mathsf{G}}_{M,1}$ as $i \to +\infty$. This concludes the proof. □

Notice that, as a byproduct of Proposition 3.1.3 and [36], M admits a collapsed ancient solution to the Ricci flow only if it is the total space of a homogenous torus bundle (see also [12, Rem 5.3]).

3.2. The projected Ricci flow

In this section, we introduce two important tools that will be fundamental for the proof of our main results, namely the space of *generalized submersion metrics* and the *projected Ricci tensor*. In the following, we consider a compact homogeneous space $M = \mathsf{G}/\mathsf{H}$ and we use the same notation introduced in Section 3.1.

3.2.1. The space of generalized submersion metrics

Consider a maximal toral H-subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ and the associated homogeneous torus fibration (3.1.9). We introduce the space of *generalized* $\mathfrak{k}$*-submersion metrics on* M as

$$\widehat{\mathcal{M}^{\mathsf{G}}_{M}(\mathfrak{k})} = \mathrm{Sym}(\mathfrak{t}, Q_{\mathfrak{t}})^{\mathrm{Ad}(\mathsf{H})} \oplus \mathrm{Sym}_{+}(\mathfrak{n}, Q_{\mathfrak{n}})^{\mathrm{Ad}(\mathsf{K})} \ , \tag{3.2.1}$$

i.e. we allow the metric on $\mathfrak{t}$ to be degenerate, and we prove the following crucial result.

Proposition 3.2.1. *The Ricci curvature* Ric_M *can be extended analytically to the space* $\widehat{\mathcal{M}^{\mathsf{G}}_{M}(\mathfrak{k})}$ *of generalized* $\mathfrak{k}$*-submersion metrics on* M*.*

Proof. We write $P = P_{\mathfrak{t}} \oplus P_{\mathfrak{n}}$ for any $P \in \mathcal{M}^{\mathsf{G}}_{M}(\mathfrak{k})$ and we observe that $[\mathrm{ad}(T), P_{\mathfrak{n}}](X) = 0$ for any $T \in \mathfrak{t}$, $X \in \mathfrak{n}$. Hence, a straightforward computation shows that the tensor $S_M(P)$ defined by (3.1.3) is explicitly given by

$$\begin{aligned}
S_M(P)(T).\widetilde{T} &= 0 \ , \\
S_M(P)(T).Y &= -\,\mathrm{ad}(T).Y + \tfrac{1}{2} P_{\mathfrak{n}}^{-1}.\,\mathrm{ad}(P_{\mathfrak{t}}.T).Y \ , \\
S_M(P)(X).\widetilde{T} &= -\tfrac{1}{2} P_{\mathfrak{n}}^{-1}.\,\mathrm{ad}(X).P_{\mathfrak{t}}.\widetilde{T} \ , \\
S_M(P)(X).Y &= -\tfrac{1}{2}\pi_{\mathfrak{m}}.\,\mathrm{ad}(X).Y - \tfrac{1}{2} P_{\mathfrak{n}}^{-1}.\pi_{\mathfrak{n}}.(\mathrm{ad}(X).P_{\mathfrak{n}} - \mathrm{ad}(P_{\mathfrak{n}}.X)).Y \ ,
\end{aligned} \tag{3.2.2}$$

where $X, Y \in \mathfrak{n}$ and $T, \widetilde{T} \in \mathfrak{t}$. Here, we denote by $\pi_{\mathfrak{m}} : \mathfrak{g} \to \mathfrak{m}$ and $\pi_{\mathfrak{n}} : \mathfrak{g} \to \mathfrak{n}$ the Q-orthogonal projections onto $\mathfrak{m}$ and $\mathfrak{n}$, respectively. In particular, (3.2.2) implies that $S_M(P)$ can be defined for any generalized metric $P \in \widehat{\mathcal{M}^{\mathsf{G}}_{M}(\mathfrak{k})}$ and that it depends analytically on

P. Therefore, formulas (3.1.4) and (3.1.5) can be used to define $\mathrm{Rm}_M(P)$ and $\mathrm{Ric}_M(P)$ for any $P \in \widehat{M^{\mathsf{G}}_M(\mathfrak{k})}$. □

Moreover, by using Schur's Lemma, we get

Lemma 3.2.1. *For any $P \in \widehat{M^{\mathsf{G}}_M(\mathfrak{k})}$, it holds that*

$$Ric_M(P) \in \mathrm{Sym}(\mathfrak{t}, Q_{\mathfrak{t}})^{\mathrm{Ad}(\mathsf{H})} \oplus \mathrm{Sym}(\mathfrak{n}, Q_{\mathfrak{n}})^{\mathrm{Ad}(\mathsf{K})} \ . \tag{3.2.3}$$

Proof. Notice that, by hypothesis, the submodule $\mathfrak{t}$ is $\mathrm{Ad}(\mathsf{K})$-invariant and the representation $\mathrm{Ad}(\mathsf{K}^o)|_{\mathfrak{t}}$ is trivial. Moreover, by Lemma 3.1.1, $\mathfrak{n}$ does not contain any $\mathrm{Ad}(\mathsf{K})$-invariant submodule on which $\mathrm{Ad}(\mathsf{K}^o)$ acts trivially. Fix now $P \in \widehat{M^{\mathsf{G}}_M(\mathfrak{k})}$ and notice that, since $\mathsf{K} = \mathsf{H}\mathsf{K}^o$, both P and the decomposition (3.1.10) are $\mathrm{Ad}(\mathsf{K})$-invariant. By (3.1.3) it follows that $S_M(P)$ is $\mathrm{Ad}(\mathsf{K})$-invariant and so $\mathrm{Ric}_M(P)$ is $\mathrm{Ad}(\mathsf{K})$-invariant as well. Therefore, the claim follows from Schur's Lemma. □

We are going to use (3.2.2) to compute the differential of the tensor S_M defined in (3.1.3). In order to do this, fix a generalized metric $P \in \widehat{M^{\mathsf{G}}_M(\mathfrak{k})}$ and a tangent direction $B \in T_P\widehat{M^{\mathsf{G}}_M(\mathfrak{k})}$. Since

$$\tfrac{\mathrm{d}}{\mathrm{d}s}(P_{\mathfrak{n}} + sB_{\mathfrak{n}})^{-1}\big|_{s=0} = -P_{\mathfrak{n}}^{-1}.B_{\mathfrak{n}}.P_{\mathfrak{n}}^{-1} \ , \tag{3.2.4}$$

it follows that the differential $\mathrm{d}S_M|_P(B)$ at P in the direction of B is given by

$$\begin{aligned}
&\mathrm{d}S_M|_P(B)(T).\widetilde{T} = 0 \ , \\
&\mathrm{d}S_M|_P(B)(T).Y = -\tfrac{1}{2}P_{\mathfrak{n}}^{-1}.B_{\mathfrak{n}}.P_{\mathfrak{n}}^{-1}.\,\mathrm{ad}(P_{\mathfrak{t}}.T).Y + \tfrac{1}{2}P_{\mathfrak{n}}^{-1}.\,\mathrm{ad}(B_{\mathfrak{t}}.T).Y \ , \\
&\mathrm{d}S_M|_P(B)(X).\widetilde{T} = \frac{1}{2}P_{\mathfrak{n}}^{-1}.B_{\mathfrak{n}}.P_{\mathfrak{n}}^{-1}.\,\mathrm{ad}(X).P_{\mathfrak{t}}.\widetilde{T} - \tfrac{1}{2}P_{\mathfrak{n}}^{-1}.\,\mathrm{ad}(X).B_{\mathfrak{t}}.\widetilde{T} \ , \\
&\mathrm{d}S_M|_P(B)(X).Y = \tfrac{1}{2}P_{\mathfrak{n}}^{-1}.B_{\mathfrak{n}}.P_{\mathfrak{n}}^{-1}.\pi_{\mathfrak{n}}.(\mathrm{ad}(X).P_{\mathfrak{n}} - \mathrm{ad}(P_{\mathfrak{n}}.X)).Y \\
&\qquad\qquad - \tfrac{1}{2}P_{\mathfrak{n}}^{-1}.\pi_{\mathfrak{n}}.(\mathrm{ad}(X).B_{\mathfrak{n}} - \mathrm{ad}(B_{\mathfrak{n}}.X)).Y \ ,
\end{aligned} \tag{3.2.5}$$

where $X, Y \in \mathfrak{n}$ and $T, \widetilde{T} \in \mathfrak{t}$. Moreover, by differentiating (3.1.4) and (3.1.5) at P in the direction of B, we get

$$\begin{aligned}
\mathrm{dRm}_M|_P(B)(V_1,V_2) &= -[\mathrm{d}S_M|_P(B)(V_1), S_M(P)(V_2)] - [S_M(P)(V_1), \mathrm{d}S_M|_P(B)(V_2)] \\
&\quad - \mathrm{d}S_M|_P(B)([V_1,V_2]_{\mathfrak{m}}) \ , \\
Q(\mathrm{dRic}_M|_P(B).V_1,V_2) &= \mathrm{Tr}(\mathfrak{m} \ni Z \mapsto \mathrm{dRm}_M|_P(B)(V_1,Z).V_2) \ .
\end{aligned} \tag{3.2.6}$$

Therefore, we obtain the following.

Proposition 3.2.2. *Fix a metric on the base space $P_{\mathfrak{n}} \in M_N^{\mathsf{G}}$. Then, the extended Ricci curvature satisfies*

$$\begin{aligned}
Ric_M(0 \oplus P_{\mathfrak{n}}) &= 0 \oplus Ric_N(P_{\mathfrak{n}}) \ , \\
\mathrm{d}Ric_M|_{0\oplus P_{\mathfrak{n}}}(0 \oplus B_{\mathfrak{n}}) &= 0 \oplus \mathrm{d}Ric_N|_{P_{\mathfrak{n}}}(B_{\mathfrak{n}})
\end{aligned} \tag{3.2.7}$$

for any horizontal direction $B_{\mathfrak{n}} \in \mathrm{Sym}(\mathfrak{n}, Q_{\mathfrak{n}})^{\mathrm{Ad}(\mathsf{K})}$, and

$$\mathrm{d}Ric_M|_{0\oplus P_{\mathfrak{n}}}(B_{\mathfrak{t}} \oplus 0).T = 0 \tag{3.2.8}$$

for any vertical direction $B_{\mathfrak{t}} \in \mathrm{Sym}(\mathfrak{t}, Q_{\mathfrak{t}})^{\mathrm{Ad}(\mathsf{H})}$ and for any $T \in \mathfrak{t}$.

Proof. Fix $B_{\mathfrak{n}} \in \mathrm{Sym}(\mathfrak{n}, Q_{\mathfrak{n}})^{\mathrm{Ad}(\mathsf{K})}$ and let $X, Y, Z \in \mathfrak{n}$, $T, \widetilde{T} \in \mathfrak{t}$. Then, from (3.2.5), it follows that the operators $S_M(0 \oplus P_{\mathfrak{n}})$ and $\mathrm{d}S_M|_{0\oplus P_{\mathfrak{n}}}(0 \oplus B_{\mathfrak{n}})$ satisfy

$$\begin{aligned}
&S_M(0 \oplus P_{\mathfrak{n}})(T).\widetilde{T} = 0 \ , \quad S_M(0 \oplus P_{\mathfrak{n}})(T).Y = -\mathrm{ad}(T).Y \ , \quad S_M(0 \oplus P_{\mathfrak{n}})(X).\widetilde{T} = 0 \ , \\
&S_M(0 \oplus P_{\mathfrak{n}})(X).Y = -\tfrac{1}{2}\pi_{\mathfrak{m}}.\mathrm{ad}(X).Y - \tfrac{1}{2}P_{\mathfrak{n}}^{-1}.\pi_{\mathfrak{n}}.(\mathrm{ad}(X).P_{\mathfrak{n}} - \mathrm{ad}(P_{\mathfrak{n}}.X)).Y
\end{aligned} \tag{3.2.9}$$

and

$$\begin{aligned}
&\mathrm{d}S_M|_{0\oplus P_{\mathfrak{n}}}(0 \oplus B_{\mathfrak{n}})(T).\widetilde{T} = 0 \ , \quad \mathrm{d}S_M|_{0\oplus P_{\mathfrak{n}}}(0 \oplus B_{\mathfrak{n}})(T).Y = 0 \ , \\
&\mathrm{d}S_M|_{0\oplus P_{\mathfrak{n}}}(0 \oplus B_{\mathfrak{n}})(X).\widetilde{T} = 0 \ , \\
&\mathrm{d}S_M|_{0\oplus P_{\mathfrak{n}}}(0 \oplus B_{\mathfrak{n}})(X).Y = \tfrac{1}{2}P_{\mathfrak{n}}^{-1}.B_{\mathfrak{n}}.P_{\mathfrak{n}}^{-1}.\pi_{\mathfrak{n}}.(\mathrm{ad}(X).P_{\mathfrak{n}} - \mathrm{ad}(P_{\mathfrak{n}}.X)).Y \\
&\qquad - \tfrac{1}{2}P_{\mathfrak{n}}^{-1}.\pi_{\mathfrak{n}}.(\mathrm{ad}(X).B_{\mathfrak{n}} - \mathrm{ad}(B_{\mathfrak{n}}.X)).Y \ .
\end{aligned} \tag{3.2.10}$$

On the other hand, by using (3.1.3) and (3.2.4), it follows that the operators $S_N(P_{\mathfrak{n}})$ and $\mathrm{d}S_N|_{P_{\mathfrak{n}}}(B_{\mathfrak{n}})$ satisfy

$$\begin{aligned} S_N(P_{\mathfrak{n}})(X).Y &= -\tfrac{1}{2}\pi_{\mathfrak{n}}.\,\mathrm{ad}(X).Y - \tfrac{1}{2}P_{\mathfrak{n}}^{-1}.\pi_{\mathfrak{n}}.(\mathrm{ad}(X).P_{\mathfrak{n}} - \mathrm{ad}(P_{\mathfrak{n}}.X)).Y \ , \\ \mathrm{d}S_N|_{P_{\mathfrak{n}}}(B_{\mathfrak{n}})(X).Y &= \tfrac{1}{2}P_{\mathfrak{n}}^{-1}.B_{\mathfrak{n}}.P_{\mathfrak{n}}^{-1}.\pi_{\mathfrak{n}}.(\mathrm{ad}(X).P_{\mathfrak{n}} - \mathrm{ad}(P_{\mathfrak{n}}.X)).Y \\ &\quad - \tfrac{1}{2}P_{\mathfrak{n}}^{-1}.\pi_{\mathfrak{n}}.(\mathrm{ad}(X).B_{\mathfrak{n}} - \mathrm{ad}(B_{\mathfrak{n}}.X)).Y \ . \end{aligned} \tag{3.2.11}$$

A straightforward computation based on (3.1.4), (3.2.6), (3.2.9) and (3.2.10) shows that

$$\mathrm{Rm}_M(0 \oplus P_{\mathfrak{n}})(T, \cdot).\widetilde{T} = \mathrm{dRm}_M|_{0\oplus P_{\mathfrak{n}}}(0 \oplus B_{\mathfrak{n}})(T, \cdot).\widetilde{T} = 0$$

and so, by using (3.1.5) and (3.2.6), we get

$$\mathrm{Ric}_M(0 \oplus P_{\mathfrak{n}})(T) \in \mathfrak{n} \quad \text{ and } \quad \mathrm{dRic}_M|_{0\oplus P_{\mathfrak{n}}}(0 \oplus B_{\mathfrak{n}})(T) \in \mathfrak{n} \ .$$

Therefore, (3.2.3) implies that

$$\mathrm{Ric}_M(0 \oplus P_{\mathfrak{n}})(T) = \mathrm{dRic}_M|_{0\oplus P_{\mathfrak{n}}}(0 \oplus B_{\mathfrak{n}})(T) = 0 \ . \tag{3.2.12}$$

Again, using (3.1.4), (3.2.6), (3.2.9) and (3.2.10) one can directly check that

$$\mathrm{Rm}_M(0 \oplus P_{\mathfrak{n}})(X, \cdot).\widetilde{T} = \mathrm{dRm}_M|_{0\oplus P_{\mathfrak{n}}}(0 \oplus B_{\mathfrak{n}})(X, \cdot).\widetilde{T} = 0$$

and so (3.1.5) and (3.2.6) imply that

$$\mathrm{Ric}_M(0 \oplus P_{\mathfrak{n}})(X) \in \mathfrak{n} \quad \text{ and } \quad \mathrm{dRic}_M|_{0\oplus P_{\mathfrak{n}}}(0 \oplus B_{\mathfrak{n}})(X) \in \mathfrak{n} \ . \tag{3.2.13}$$

Finally, another direct computation based on (3.1.4), (3.2.6), (3.2.9), (3.2.10) and (3.2.11) shows that

$$\begin{aligned} \pi_{\mathfrak{n}}(\mathrm{Rm}_M(0 \oplus P_{\mathfrak{n}})(X,Y).Z) &= \mathrm{Rm}_N(P_{\mathfrak{n}})(X,Y).Z \ , \\ \pi_{\mathfrak{n}}(\mathrm{dRm}_M|_{0\oplus P_{\mathfrak{n}}}(0 \oplus B_{\mathfrak{n}})(X,Y).Z) &= \mathrm{dRm}_N|_{P_{\mathfrak{n}}}(B_{\mathfrak{n}})(X,Y).Z \ . \end{aligned} \tag{3.2.14}$$

Notice now that (3.2.7) follows from (3.2.12), (3.2.13) and (3.2.14). In order to prove (3.2.8), fix $B_{\mathfrak{t}} \in \mathrm{Sym}(\mathfrak{t}, Q_{\mathfrak{t}})^{\mathrm{Ad}(\mathsf{H})}$ and observe that, from (3.2.5), it follows that the operator $\mathrm{d}S_M|_{0\oplus P_{\mathfrak{n}}}(B_{\mathfrak{t}} \oplus 0)$ satisfies

$$\begin{aligned}
&\mathrm{d}S_M|_{0\oplus P_{\mathfrak{n}}}(B_{\mathfrak{t}} \oplus 0)(T).\widetilde{T} = 0 \ , \\
&\mathrm{d}S_M|_{0\oplus P_{\mathfrak{n}}}(B_{\mathfrak{t}} \oplus 0)(T).Y = +\tfrac{1}{2} P_{\mathfrak{n}}^{-1}.\,\mathrm{ad}(B_{\mathfrak{t}}.T).Y \ , \\
&\mathrm{d}S_M|_{0\oplus P_{\mathfrak{n}}}(B_{\mathfrak{t}} \oplus 0)(X).\widetilde{T} = -\tfrac{1}{2} P_{\mathfrak{n}}^{-1}.\,\mathrm{ad}(X).B_{\mathfrak{t}}.\widetilde{T} \ , \\
&\mathrm{d}S_M|_{0\oplus P_{\mathfrak{n}}}(B_{\mathfrak{t}} \oplus 0)(X).Y = 0 \ .
\end{aligned} \tag{3.2.15}$$

Again, by using (3.1.4), (3.1.5), (3.2.6), (3.2.9) and (3.2.15), one can show that

$$\mathrm{dRic}_M|_{0\oplus P_{\mathfrak{n}}}(B_{\mathfrak{t}} \oplus 0)(T) \in \mathfrak{n}$$

and so, using (3.2.3), we get (3.2.8). □

3.2.2. The $\overline{P}_{\mathfrak{n}}$-projected Ricci tensor

Fix a unit volume Einstein metric $\overline{P}_{\mathfrak{n}} \in M^{\mathsf{G}}_{N,1}$ on N, i.e. $\mathrm{Ric}_N(\overline{P}_{\mathfrak{n}}) = \lambda\, \overline{P}_{\mathfrak{n}}$ for some $\lambda \in \mathbb{R}$. Since N is compact, Bochner's Theorem implies that λ is non-negative (see [8]). Moreover, since $M = \mathsf{G}/\mathsf{H}$ is not a torus, then also N is not a torus and so $\lambda > 0$.

We introduce the Euclidean inner product $\langle\!\langle \cdot, \cdot \rangle\!\rangle^{(\overline{P}_{\mathfrak{n}})}$ on the linear space $\mathrm{Sym}(\mathfrak{m}, Q_{\mathfrak{m}})^{\mathrm{Ad}(\mathsf{H})}$ defined by

$$\langle\!\langle B_1, B_2 \rangle\!\rangle^{(\overline{P}_{\mathfrak{n}})} = \dim(N)^{-1}\,\mathrm{Tr}\big((\mathrm{Id}_{\mathfrak{t}} \oplus (\overline{P}_{\mathfrak{n}})^{-1}).B_1.(\mathrm{Id}_{\mathfrak{t}} \oplus (\overline{P}_{\mathfrak{n}})^{-1}).B_2\big) \tag{3.2.16}$$

and the $\overline{P}_{\mathfrak{n}}$*-projected Ricci curvature*

$$\begin{aligned}
&\mathcal{R}^{(\overline{P}_{\mathfrak{n}})}_M : \widehat{M^{\mathsf{G}}_M(\mathfrak{k})} \to \mathrm{Sym}(\mathfrak{t}, Q_{\mathfrak{t}})^{\mathrm{Ad}(\mathsf{H})} \oplus \mathrm{Sym}(\mathfrak{n}, Q_{\mathfrak{n}})^{\mathrm{Ad}(\mathsf{K})} \ , \\
&\mathcal{R}^{(\overline{P}_{\mathfrak{n}})}_M(P) = \mathrm{Ric}_M(P) - \frac{\langle\!\langle \mathrm{Ric}_M(P), P \rangle\!\rangle^{(\overline{P}_{\mathfrak{n}})}}{\langle\!\langle P, P \rangle\!\rangle^{(\overline{P}_{\mathfrak{n}})}} P \ .
\end{aligned} \tag{3.2.17}$$

We remark that, for any $P \in \widehat{M^{\mathsf{G}}_M(\mathfrak{k})}$, the image $\mathcal{R}^{(\overline{P}_{\mathfrak{n}})}_M(P)$ lies in

$$\mathrm{Sym}(\mathfrak{t}, Q_{\mathfrak{t}})^{\mathrm{Ad}(\mathsf{H})} \oplus \mathrm{Sym}(\mathfrak{n}, Q_{\mathfrak{n}})^{\mathrm{Ad}(\mathsf{K})}$$

by means of (3.2.3). As a consequence of Proposition 3.2.2, we get the following

Corollary 3.2.2. *The $\overline{P}_{\mathfrak{n}}$-projected Ricci curvature $\mathcal{R}^{(\overline{P}_{\mathfrak{n}})}_M$ satisfies*

$$\begin{aligned} \mathcal{R}^{(\overline{P}_{\mathfrak{n}})}_M(0 \oplus \overline{P}_{\mathfrak{n}}) &= 0 \oplus Ric^0_N(\overline{P}_{\mathfrak{n}}) \ , \\ \mathrm{d}\mathcal{R}^{(\overline{P}_{\mathfrak{n}})}_M\big|_{0\oplus\overline{P}_{\mathfrak{n}}}(0 \oplus B_{\mathfrak{n}}) &= 0 \oplus \mathrm{d}Ric^0_N|_{\overline{P}_{\mathfrak{n}}}(B_{\mathfrak{n}}) \end{aligned} \tag{3.2.18}$$

for any $B_{\mathfrak{n}} \in \mathrm{Sym}(\mathfrak{n}, Q_{\mathfrak{n}})^{\mathrm{Ad}(\mathsf{K})}$, and

$$\mathrm{d}\mathcal{R}^{(\overline{P}_{\mathfrak{n}})}_M\big|_{0\oplus\overline{P}_{\mathfrak{n}}}(B_{\mathfrak{t}} \oplus 0).T = -\lambda B_{\mathfrak{t}}.T \tag{3.2.19}$$

for any $B_{\mathfrak{t}} \in \mathrm{Sym}(\mathfrak{t}, Q_{\mathfrak{t}})^{\mathrm{Ad}(\mathsf{H})}$, $T \in \mathfrak{t}$.

Proof. Notice that (3.2.18) follows from a direct computation based on (3.1.3), (3.1.4), (3.1.5) and (3.2.7). Moreover, from (3.2.7) and (3.2.8), we get

$$\begin{aligned} \mathrm{d}\mathcal{R}^{(\overline{P}_{\mathfrak{n}})}_M|_{0\oplus\overline{P}_{\mathfrak{n}}}(B_{\mathfrak{t}} \oplus 0).T &= -\,\mathrm{d}\Big(\frac{\langle\!\langle \mathrm{Ric}_M(P), P\rangle\!\rangle^{(\overline{P}_{\mathfrak{n}})}}{\langle\!\langle P, P\rangle\!\rangle^{(\overline{P}_{\mathfrak{n}})}} P\Big)\Big|_{0\oplus\overline{P}_{\mathfrak{n}}}(B_{\mathfrak{t}} \oplus 0).T \\ &= -\,\mathrm{d}\Big(\frac{\langle\!\langle \mathrm{Ric}_M(P), P\rangle\!\rangle^{(\overline{P}_{\mathfrak{n}})}}{\langle\!\langle P, P\rangle\!\rangle^{(\overline{P}_{\mathfrak{n}})}}\Big)\Big|_{0\oplus\overline{P}_{\mathfrak{n}}}(B_{\mathfrak{t}} \oplus 0)\cdot(0 \oplus \widetilde{P}_{\mathfrak{n}}).T \\ &\quad -\Big(\frac{\langle\!\langle \mathrm{Ric}_M(0\oplus\overline{P}_{\mathfrak{n}}), 0\oplus\overline{P}_{\mathfrak{n}}\rangle\!\rangle^{(\overline{P}_{\mathfrak{n}})}}{\langle\!\langle 0\oplus\overline{P}_{\mathfrak{n}}, 0\oplus\overline{P}_{\mathfrak{n}}\rangle\!\rangle^{(\overline{P}_{\mathfrak{n}})}}\Big)\cdot(B_{\mathfrak{t}} \oplus 0).T \\ &= 0 - \frac{\mathrm{scal}_N(\overline{P}_{\mathfrak{n}})}{\dim(N)} B_{\mathfrak{t}}.T \\ &= -\lambda B_{\mathfrak{t}}.T \end{aligned}$$

for any $B_{\mathfrak{t}} \in \mathrm{Sym}(\mathfrak{t}, Q_{\mathfrak{t}})^{\mathrm{Ad}(\mathsf{H})}$ and $T \in \mathfrak{t}$, which proves (3.2.19). □

In virtue of Proposition 3.2.1 and (3.2.2), the Ricci flow preserves the subspace $M^{\mathsf{G}}_M(\mathfrak{k})$

of $\mathfrak{k}$-submersion metrics and can be extended to the larger space $\widehat{M^{\mathsf{G}}_M(\mathfrak{k})}$ of generalized $\mathfrak{k}$-submersion metrics. Moreover, since the Ricci curvature Ric_M is scale invariant, we may project the Ricci flow on the unit sphere

$$\Sigma^{(\overline{P}_{\mathfrak{n}})} = \Big\{ P \in \widehat{M^{\mathsf{G}}_M(\mathfrak{k})} : \langle\!\langle P, P \rangle\!\rangle^{(\overline{P}_{\mathfrak{n}})} = 1 \Big\}$$

of $\widehat{M^{\mathsf{G}}_M(\mathfrak{k})}$ with respect to the inner product $\langle\!\langle \cdot, \cdot \rangle\!\rangle^{(\overline{P}_{\mathfrak{n}})}$. Hence up to rescaling, the Ricci flow is equivalent to the flow on $\Sigma^{(\overline{P}_{\mathfrak{n}})}$ defined by

$$P'(t) = -2\mathcal{R}^{(\overline{P}_{\mathfrak{n}})}_M(P(t)) \ , \tag{3.2.20}$$

which we call the $\overline{P}_{\mathfrak{n}}$*-projected Ricci flow.*

3.3. Proof of Theorem B

This section is devoted to the proof of our main result. In the following, we consider a compact homogeneous space $M = \mathsf{G}/\mathsf{H}$, a fixed maximal toral H-subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ and we use the same notation as in Section 3.1 and Section 3.2.

3.3.1. Two preparatory results

Take a sequence $(P^{(n)}) \subset M^{\mathsf{G}}_M(\mathfrak{k})$ of $\mathfrak{k}$-submersion metrics $P^{(n)} = P^{(n)}_{\mathfrak{t}} \oplus P^{(n)}_{\mathfrak{n}}$ such that $P^{(n)}_{\mathfrak{t}} \to 0$ and $P^{(n)}_{\mathfrak{n}} \to P^{(\infty)}_{\mathfrak{n}} \in M^{\mathsf{G}}_N$ as $n \to +\infty$. The first result that we need for proving Theorem B is the following.

Proposition 3.3.1. *The scalar curvature of $P^{(n)}$ converges to the scalar curvature of $P^{(\infty)}_{\mathfrak{n}}$, that is*

$$\mathrm{scal}_M(P^{(n)}) \to \mathrm{scal}_N(P^{(\infty)}_{\mathfrak{n}}) \qquad \text{as } n \to +\infty \ . \tag{3.3.1}$$

Proof. Since the fibers of (3.1.9) are totally geodesic and flat along the sequence, by O'Neill's

Formula (see [7, Eq (9.37)]) we get

$$\mathrm{scal}_M(P^{(n)}) = \mathrm{scal}_N(P^{(n)}_{\mathfrak{n}}) - \left(\left|A^{(n)}\right|_{P^{(n)}}\right)^2 ,$$

where $A^{(n)} : \mathfrak{m} \otimes \mathfrak{m} \to \mathfrak{m}$ is the *O'Neill's tensor* of the metric $P^{(n)}$.

Since the scalar curvature functional is continuous, it follows that $\mathrm{scal}_N(P^{(n)}_{\mathfrak{n}}) \to \mathrm{scal}_N(P^{(\infty)}_{\mathfrak{n}})$. Therefore, in order to prove (3.3.1), it is sufficient to show that $\left|A^{(n)}\right|_{P^{(n)}} \to 0$ as $n \to +\infty$.

For any $n \in \mathbb{N}$, we consider a $Q_{\mathfrak{m}}$-orthogonal, $\mathrm{Ad}(\mathsf{H})$-invariant, irreducible decomposition

$$\mathfrak{m} = \mathfrak{m}^{(n)}_1 + \ldots + \mathfrak{m}^{(n)}_\ell \tag{3.3.2}$$

with respect to which $P^{(n)}$ is diagonal, i.e.

$$P^{(n)} = x^{(n)}_1 \,\mathrm{Id}_{\mathfrak{m}^{(n)}_1} \oplus \ldots \oplus x^{(n)}_\ell \,\mathrm{Id}_{\mathfrak{m}^{(n)}_\ell} \ , \quad x^{(n)}_k > 0 \ \text{ for any } 1 \leq k \leq \ell \ .$$

By hypothesis, we can assume that:

the dimension $m_i = \dim(\mathfrak{m}^{(n)}_i)$ is constant along the sequence for any $1 \leq i \leq \ell$;

the decomposition (3.3.2) converges to a well defined $\mathrm{Ad}(\mathsf{H})$-invariant, irreducible, limit decomposition $\mathfrak{m} = \mathfrak{m}^{(\infty)}_1 + \ldots + \mathfrak{m}^{(\infty)}_\ell$;

there exists $1 \leq r \leq \ell$ such that

$$\mathfrak{t} = \mathfrak{m}^{(n)}_1 + \ldots + \mathfrak{m}^{(n)}_r \ , \quad \mathfrak{n} = \mathfrak{m}^{(n)}_{r+1} + \ldots + \mathfrak{m}^{(n)}_\ell \quad \text{ for any } n \in \mathbb{N} \ ;$$

$P^{(\infty)}_{\mathfrak{n}}$ is diagonal with respect to $\mathfrak{n} = \mathfrak{m}^{(\infty)}_{r+1} + \ldots + \mathfrak{m}^{(\infty)}_\ell$, i.e.

$$P^{(\infty)}_{\mathfrak{n}} = x^{(\infty)}_{r+1} \,\mathrm{Id}_{\mathfrak{m}^{(\infty)}_{r+1}} \oplus \ldots \oplus x^{(n)}_\ell \,\mathrm{Id}_{\mathfrak{m}^{(\infty)}_\ell} \ , \quad x^{(\infty)}_j > 0 \ \text{ for any } r+1 \leq j \leq \ell \ .$$

We consider now a sequence of *adapted bases*, i.e. for any $n \in \mathbb{N}$ we consider a $Q_{\mathfrak{m}}$-orthonormal basis $(e^{(n)}_\alpha)_{1 \leq \alpha \leq m}$ for $\mathfrak{m}$ such that

$$e^{(n)}_1, \dots, e^{(n)}_{m_1} \in \mathfrak{m}^{(n)}_1 \ , \quad e^{(n)}_{m_1+1}, \dots, e^{(n)}_{m_1+m_2} \in \mathfrak{m}^{(n)}_2 \ , \quad \dots \quad , \quad e^{(n)}_{m_1+\dots+m_{\ell-1}+1}, \dots, e^{(n)}_m \in \mathfrak{m}^{(n)}_\ell \ ,$$

and we define the coefficients

$$[ijk]^{(n)} = \sum_{e^{(n)}_\alpha \in \mathfrak{m}^{(n)}_i} \sum_{e^{(n)}_\beta \in \mathfrak{m}^{(n)}_j} \sum_{e^{(n)}_\gamma \in \mathfrak{m}^{(n)}_k} Q\big([e^{(n)}_\alpha, e^{(n)}_\beta], e^{(n)}_\gamma\big)^2 \ . \tag{3.3.3}$$

Notice that $[ijk]^{(n)}$ is symmetric in all its entries and does not depend on the choice of $(e^{(n)}_\alpha)$. Moreover, we can assume that $(e^{(n)}_\alpha)$ converges to a limit adapted basis $(e^{(\infty)}_\alpha)$ for $\mathfrak{m}$ and, as a consequence, $[ijk]^{(n)}$ converges to the coefficient $[ijk]^{(\infty)}$ related to the limit decomposition. For more information about the diagonalization of invariant metrics on compact homogeneous spaces, we refer to [46, 9].

For the sake of shortness, we set

$$A^{(n)}_{ij} = \sum_{e^{(n)}_\alpha \in \mathfrak{m}^{(n)}_i} \sum_{e^{(n)}_\beta \in \mathfrak{m}^{(n)}_j} \big(\big|A^{(n)}\big(e^{(n)}_\alpha, e^{(n)}_\beta\big)\big|_{P^{(n)}}\big)^2 \ .$$

Notice that by [35, Lemma 2] and (3.3.3), it follows that

$$A^{(n)}_{j_1 j_2} = \frac{1}{4} \sum_{1 \leq i \leq r} [i j_1 j_2]^{(n)} x^{(n)}_i \to 0 \quad \text{for any } r+1 \leq j_1, j_2 \leq \ell \ . \tag{3.3.4}$$

Moreover, since O'Neill's tensor is horizontal (see [35, p. 460]), it follows that

$$A^{(n)}_{ik} = 0 \quad \text{for any } 1 \leq i \leq r \ , \ 1 \leq k \leq \ell \ . \tag{3.3.5}$$

Finally, by [35, Cor 1] and [36, Eq (4.5) and (4.7)], we obtain

$$A^{(n)}_{ji} = \tfrac14 \sum_{1\leq k\leq \ell} [ijk]^{(n)} \frac{x^{(n)}_i}{x^{(n)}_j x^{(n)}_k} + \tfrac14 \sum_{1\leq k\leq \ell} [ijk]^{(n)} \Big(\frac{x^{(n)}_j}{x^{(n)}_k} - 1\Big)\Big(-2\frac{x^{(n)}_i}{x^{(n)}_j} + 1 + 3\frac{x^{(n)}_k}{x^{(n)}_j}\Big) \frac{1}{x^{(n)}_i} \quad (3.3.6)$$

for any $1 \leq i \leq r$, $r+1 \leq j \leq \ell$. Since each $P^{(n)}$ is a $\mathfrak{k}$-submersion metric and $\mathfrak{t}$ is abelian, it follows that

$$\begin{aligned} &[i_1 i_2 k]^{(n)} = 0 \quad \text{for any } \ 1 \leq i_1, i_2 \leq r\,, \ 1 \leq k \leq \ell\,, \quad \text{for any } n \in \mathbb{N}\ , \\ &[ij_1j_2]^{(n)} \Big(\frac{x^{(n)}_{j_2}}{x^{(n)}_{j_1}} - 1\Big) = 0 \quad \text{for any } \ 1 \leq i \leq r\,, \ r+1 \leq j_1, j_2 \leq r\,, \quad \text{for any } n \in \mathbb{N}\ . \end{aligned} \quad (3.3.7)$$

Therefore, by (3.3.6) and (3.3.7) we get

$$A^{(n)}_{ji} = \tfrac14 \sum_{r+1\leq j'\leq \ell} [ijj']^{(n)} \frac{x^{(n)}_i}{x^{(n)}_j x^{(n)}_{j'}} \to 0 \quad \text{for any } 1 \leq i \leq r\ ,\ r+1 \leq j \leq \ell \quad (3.3.8)$$

and so the claim follows from (3.3.4), (3.3.5) and (3.3.8). □

Let us denote now by $\mathtt{d}^{(n)}_M$ the Riemannian distance induced by $P^{(n)}$ on M and by $\mathtt{d}^{(n)}_N$ (resp. $\mathtt{d}^{(\infty)}_N$) the Riemannian distance induced by $P^{(n)}_{\mathfrak{n}}$ (resp. $P^{(\infty)}_{\mathfrak{n}}$) on N. We recall that, since N is compact and $P^{(n)}_{\mathfrak{n}} \to P^{(\infty)}_{\mathfrak{n}}$ in the $\mathcal{C}^\infty$-topology, it follows that the metric spaces $(N, \mathtt{d}^{(n)}_N)$ converge to $(N, \mathtt{d}^{(\infty)}_N)$ in the *Gromov-Hausdorff topology* as $n \to +\infty$ (see e.g. [41, p. 415]). For a detailed treatment on Gromov-Hausdorff convergence, we refer to [19, 43].

Proposition 3.3.2. *The sequence of compact metric spaces* $(M, \mathtt{d}^{(n)}_M)$ *converges to* $(N, \mathtt{d}^{(\infty)}_N)$ *in the Gromov-Hausdorff topology as* $n \to +\infty$.

Proof. In order to prove the statement, it is sufficient to show that

$$\big|\mathtt{d}^{(n)}_M(a_0\mathsf{H}, a_1\mathsf{H}) - \mathtt{d}^{(\infty)}_N(a_0\mathsf{K}, a_1\mathsf{K})\big| \xrightarrow{n\to+\infty} 0 \quad \text{uniformly in } a_0, a_1 \in \mathsf{G}\ .$$

Fix $a_0, a_1 \in \mathsf{G}$ and consider for any $n \in \mathbb{N}$ a $\mathtt{d}^{(n)}_N$-geodesic $\gamma^{(n)} : [0,1] \to N$ such that $\gamma^{(n)}(0) =$

$a_0\mathsf{K}$, $\gamma^{(n)}(1) = a_1\mathsf{K}$, which realizes the $\mathsf{d}_N^{(n)}$-distance between $a_0\mathsf{K}$ and $a_1\mathsf{K}$. Consider now the horizontal lift $\gamma^{(n)\uparrow} : [0,1] \to M$ of $\gamma^{(n)}$ to M starting from $a_0\mathsf{H}$ and pick $c^{(n)} \in \mathsf{T}$ such that $\gamma^{(n)\uparrow}(1) = a_1 c^{(n)}\mathsf{H}$. Since $P^{(n)}$ is a $\mathfrak{k}$-submersion metric, it follows that $\mathsf{d}_M^{(n)}(a_0\mathsf{H}, a_1 c^{(n)}\mathsf{H}) = \mathsf{d}_N^{(n)}(a_0\mathsf{K}, a_1\mathsf{K})$. Then, by the reverse triangle inequality, we get

$$\left|\mathsf{d}_M^{(n)}(a_0\mathsf{H}, a_1\mathsf{H}) - \mathsf{d}_N^{(\infty)}(a_0\mathsf{K}, a_1\mathsf{K})\right| \leq \mathsf{d}_M^{(n)}(a_1\mathsf{H}, a_1 c^{(n)}\mathsf{H}) + \left|\mathsf{d}_N^{(n)}(a_0\mathsf{K}, a_1\mathsf{K}) - \mathsf{d}_N^{(\infty)}(a_0\mathsf{K}, a_1\mathsf{K})\right| . \tag{3.3.9}$$

Notice now that both the terms on the right hand side of (3.3.9) converge uniformly to 0 as $n \to +\infty$, and this concludes the proof. □

Let us finally remark that both (3.3.1) and Proposition 3.3.2 hold true for any (not necessarily maximal) toral H-subalgebra $\mathfrak{k}$.

3.3.2. The existence theorem

Consider again a unit volume Einstein metric $\overline{P}_{\mathfrak{n}} \in \mathcal{M}^{\mathsf{G}}_{N,1}$ on N with $\mathrm{Ric}_N(\overline{P}_{\mathfrak{n}}) = \lambda\, \overline{P}_{\mathfrak{n}}$ for some $\lambda > 0$. We also set

$$\nu = \dim\big(\mathrm{Sym}(\mathfrak{t}, Q_{\mathfrak{t}})^{\mathrm{Ad}(\mathsf{H})}\big) .$$

Notice that, if H is connected, then $\mathrm{Ad}(\mathsf{H})|_{\mathfrak{t}}$ is trivial and so $\nu = \frac{d(d+1)}{2}$, where $d = \dim(\mathsf{T})$. However, in the general case it may happen that $1 \leq \nu < \frac{d(d+1)}{2}$.

The main result of this section is the following

Theorem 3.3.1. *If $\overline{P}_{\mathfrak{n}}$ has coindex q, then there exists a $(\nu + q - 1)$-parameter family of ancient solutions to the $\overline{P}_{\mathfrak{n}}$-projected Ricci flow on $\mathcal{M}^{\mathsf{G}}_M(\mathfrak{k})$ which converge to $0 \oplus \overline{P}_{\mathfrak{n}}$ as $t \to -\infty$ and such that the corresponding solutions to the Ricci flow are ancient and collapsed.*

Proof. Let us observe that the $\overline{P}_{\mathfrak{n}}$-projected Ricci tensor (3.2.17) is defined on an open neighborhood of $0 \oplus \overline{P}_{\mathfrak{n}}$ inside $\Sigma^{(\overline{P}_{\mathfrak{n}})}$. Moreover, from (3.2.16) it holds that

$$T_{0\oplus\overline{P}_{\mathfrak{n}}}\Sigma^{(\overline{P}_{\mathfrak{n}})} = \mathrm{Sym}(\mathfrak{t}, Q_{\mathfrak{t}})^{\mathrm{Ad}(\mathsf{H})} \oplus T_{\overline{P}_{\mathfrak{n}}} M^{\mathsf{G}}_{N,1} \tag{3.3.10}$$

and, by (3.2.18) and (3.2.19), it follows that

$$\mathcal{R}^{(\overline{P}_{\mathfrak{n}})}_M(0 \oplus \overline{P}_{\mathfrak{n}}) = 0 \ , \quad \mathrm{d}\mathcal{R}^{(\overline{P}_{\mathfrak{n}})}_M\big|_{0\oplus\overline{P}_{\mathfrak{n}}} = \left(\begin{array}{c|c} -\lambda\,\mathrm{Id}_{\mathrm{Sym}(\mathfrak{t},Q_{\mathfrak{t}})^{\mathrm{Ad}(\mathsf{H})}} & 0 \\ \hline * & \mathrm{dRic}^0_N|_{\overline{P}_{\mathfrak{n}}} \end{array}\right) . \tag{3.3.11}$$

By (3.3.10), (3.3.11) and the Center Manifold Theorem [40, p. 116], it follows that there exists a stable manifold $\widehat{W^{(\overline{P}_{\mathfrak{n}})}}$ for $\mathcal{R}^{(\overline{P}_{\mathfrak{n}})}_M$ at $0\oplus\overline{P}_{\mathfrak{n}}$ of dimension $\dim \widehat{W^{(\overline{P}_{\mathfrak{n}})}} = \nu + q$, where q is the coindex of $\overline{P}_{\mathfrak{n}}$ (see Definition 3.1.2). We remark that $\widehat{W^{(\overline{P}_{\mathfrak{n}})}}$ is a submanifold of $\widehat{M^{\mathsf{G}}_M(\mathfrak{k})}$ and that, being eventually interested in the positive-definite solutions to the Ricci flow, we need to compute the dimension of the manifold $W^{(\overline{P}_{\mathfrak{n}})} = \widehat{W^{(\overline{P}_{\mathfrak{n}})}} \cap M^{\mathsf{G}}_M(\mathfrak{k})$. For this purpose let us observe that, restricting to the sphere $\Sigma^{(\overline{P}_{\mathfrak{n}})}$, the eigenvectors of $\mathrm{d}\mathcal{R}^{(\overline{P}_{\mathfrak{n}})}_M\big|_{0\oplus\overline{P}_{\mathfrak{n}}}$ consist of two families of endomorphisms inside $T_{0\oplus\overline{P}_{\mathfrak{n}}}\Sigma^{(\overline{P}_{\mathfrak{n}})}$, namely:

those coming from the upper left block of (3.3.11), spanned by a basis of the form

$$B_1 = \big((B_{\mathfrak{t}})_i \oplus (B_{\mathfrak{n}})_i\big) \ , \quad 1 \leq i \leq \nu \ ;$$

those coming from the lower right block of (3.3.11), spanned by a basis of the form

$$B_2 = \big(0 \oplus (C_{\mathfrak{n}})_j\big) \ , \quad 1 \leq j \leq p-1 \ ,$$

where p is the number of $\mathrm{Ad}(\mathsf{K})$-invariant, irreducible summands of $\mathfrak{n}$.

We claim that the endomorphisms $(B_{\mathfrak{t}})_i$ must be linearly independent inside $\mathrm{Sym}(\mathfrak{t}, Q_{\mathfrak{t}})^{\mathrm{Ad}(\mathsf{H})}$.

If not, then there is a non-trivial linear combination

$$\sum_i \mu_i\big((B_{\mathfrak{t}})_i \oplus (B_{\mathfrak{n}})_i\big) = 0 \oplus B^\star_{\mathfrak{n}} \quad \text{for some non-zero } B^\star_{\mathfrak{n}} \in T_{\overline{P}_{\mathfrak{n}}} M^{\mathsf{G}}_{N,1} \ .$$

Since $(C_{\mathfrak{n}})_j$ forms a basis for $T_{\overline{P}_{\mathfrak{n}}} M^{\mathsf{G}}_{N,1}$, there is another linear combination

$$\sum_j \widetilde{\mu}_j (C_{\mathfrak{n}})_j = B^\star_{\mathfrak{n}} \ ,$$

but this contradicts the fact that $B_1 \cup B_2$ is a basis for $T_{0\oplus\overline{P}_{\mathfrak{n}}} \Sigma^{(\overline{P}_{\mathfrak{n}})}$. This shows in particular that $W^{(\overline{P}_{\mathfrak{n}})}$ has dimension $\dim W^{(\overline{P}_{\mathfrak{n}})} = \dim \widehat{W^{(\overline{P}_{\mathfrak{n}})}} = \nu + q$.

Let now $P(t) = P_{\mathfrak{t}}(t) \oplus P_{\mathfrak{n}}(t)$ be an ancient solution to the $\overline{P}_{\mathfrak{n}}$-projected Ricci flow lying on $W^{(\overline{P}_{\mathfrak{n}})}$. It remains to prove that the corresponding solution to the Ricci flow is still ancient. Notice that by (3.3.1) it holds that

$$\mathrm{scal}_M(P(t)) \to \lambda \dim(N) \quad \text{as } t \to -\infty \ .$$

Thus for large times $\mathrm{scal}_M(P(t)) > 0$, and hence the same is true for the corresponding solution to the Ricci flow. However, a solution to the Ricci flow whose scalar curvature stays positive is necessarily ancient by [30, Thm 1.1]. Furthermore, as in the proof of Proposition 4.2 in [47], $P(t)$ has bounded curvature and is hence collapsed as the injectivity radius tends to zero. □

Notice now that Theorem B is a direct consequence of Theorem 3.3.1 and Proposition 3.3.2. We finally mention that we do not know, expect in some special cases, whether the solutions found by means of Theorem B are isometric or not.

3.4. Proof of Corollary A

In this section, we produce explicit examples of collapsed homogeneous ancient solutions. As a byproduct, we prove Corollary A. For a detailed study of Einstein equations on generalized flag manifolds, we refer e.g. to [4]. In the following examples, the group G will always be semisimple and so we choose its negative Cartan-Killing form as background metric.

3.4.1. A Kähler-Einstein metric on $\mathsf{SU}(3)/\mathsf{T}^2$

Let $\mathsf{G} = \mathsf{SU}(3)$, $\mathsf{T}^2 = \{\mathrm{diag}(e^{it_1}, e^{it_2}, e^{-i(t_1+t_2)})\}$ its maximal torus and consider the real root spaces decomposition

$$\mathfrak{su}(3) = \mathfrak{t}^2 + \mathfrak{n}_1 + \mathfrak{n}_2 + \mathfrak{n}_3 \ .$$

Then, any G-invariant Riemannian metric $P_{\mathfrak{n}}$ on the flag manifold $N = \mathsf{SU}(3)/\mathsf{T}^2$ takes the form

$$P_{\mathfrak{n}} = \lambda_1 \,\mathrm{Id}_{\mathfrak{n}_1} \oplus \lambda_2 \,\mathrm{Id}_{\mathfrak{n}_2} \oplus \lambda_3 \,\mathrm{Id}_{\mathfrak{n}_3}$$

and its normalized scalar curvature is given by (see e.g. [4, Prop. 4])

$$\widetilde{\mathrm{scal}}_N(P_{\mathfrak{n}}) = (\lambda_1\lambda_2\lambda_3)^{\frac{1}{3}}\Big(\tfrac{1}{\lambda_1} + \tfrac{1}{\lambda_2} + \tfrac{1}{\lambda_3} - \tfrac{1}{6}\big(\tfrac{\lambda_1}{\lambda_2\lambda_3} + \tfrac{\lambda_2}{\lambda_1\lambda_3} + \tfrac{\lambda_3}{\lambda_1\lambda_2}\big)\Big) \ .$$

Take the unit volume Kähler-Einstein metric $P_{\mathfrak{n}}^{\mathrm{KE}}$ corresponding to the values

$$(\lambda_1, \lambda_2, \lambda_3) = \tfrac{1}{3}\big(\tfrac{27}{2}\big)^{\frac{1}{3}}(1, 1, 2)$$

and one can compute that

$$\mathrm{spectrum}\Big(\,\mathrm{Hess}\big(\widetilde{\mathrm{scal}}_N\big)\big|_{P_{\mathfrak{n}}^{\mathrm{KE}}}\Big) = \big\{-\tfrac{1}{3}, 0, \tfrac{4}{3}\big\} \ .$$

Here, the zero eigenvalue corresponds to scaling the metric by a constant and so $P_{\mathfrak{n}}^{\mathrm{KE}}$ has coindex $q = 1$. Now consider the homogeneous fibration

$$\mathsf{T}^2 \to \mathsf{SU}(3) \to \mathsf{SU}(3)/\mathsf{T}^2 \ . \tag{3.4.1}$$

By Theorem 3.3.1, there is a 3-parameter family of ancient solutions to the Ricci flow on $\mathsf{SU}(3)$ which, under the rescaling introduced in Section 3.2, collapse the fibers of (3.4.1) and converge to $0 \oplus P_{\mathfrak{n}}^{\mathrm{KE}}$ as $t \to -\infty$. Similarly if $\mathsf{S}^1_{p,q} = \{(\, e^{ipt}, e^{iqt}, e^{-i(p+q)t})\}$, we get the homogeneous fibration

$$\mathsf{S}^1 = \mathsf{T}^2/\mathsf{S}^1_{p,q} \to \mathsf{SU}(3)/\mathsf{S}^1_{p,q} \to \mathsf{SU}(3)/\mathsf{T}^2 \ , \tag{3.4.2}$$

where $\mathsf{SU}(3)/\mathsf{S}^1_{p,q}$ is an Aloff-Wallach space. By Theorem 3.3.1, there is a 1-parameter family of ancient solutions on $\mathsf{SU}(3)/\mathsf{S}^1_{p,q}$ converging to $0 \oplus P_{\mathfrak{n}}^{\mathrm{KE}}$ as above.

Remark 3.4.1. In [32], Lu and Wang produce a two-parameter family of ancient solutions on $\mathsf{SU}(3)$ and a single ancient solution on $\mathsf{SU}(3)/\mathsf{S}^1_{p,q}$ both collapsing to $P_{\mathfrak{n}}^{\mathrm{KE}}$ as $t \to -\infty$. Our families are slightly larger, which can be explained by the fact that the metric restricted to the base is allowed to vary.

3.4.2. A Kähler-Einstein metric on $\mathsf{SU}(4)/\mathsf{T}^3$

Let $\mathsf{G} = \mathsf{SU}(4)$, $\mathsf{T}^3 = \{\mathrm{diag}(e^{it_1}, e^{it_2}, e^{it_3}, e^{-i(t_1+t_2+t_3)})\}$ its maximal torus and consider the real root spaces decomposition

$$\mathfrak{su}(3) = \mathfrak{t}^2 + \mathfrak{n}_1 + \mathfrak{n}_2 + \mathfrak{n}_3 + \mathfrak{n}_4 + \mathfrak{n}_5 + \mathfrak{n}_6 \ .$$

Then, any G-invariant Riemannian metric $P_{\mathfrak{n}}$ on the flag manifold $N = \mathsf{SU}(4)/\mathsf{T}^3$ takes the form

$$P_{\mathfrak{n}} = \lambda_1 \, \mathrm{Id}_{\mathfrak{n}_1} \oplus \ldots \oplus \lambda_6 \, \mathrm{Id}_{\mathfrak{n}_3}$$

and its normalized scalar curvature is given by (see e.g. [4, Prop. 4])

$$\widetilde{\mathrm{scal}}_N(P_{\mathfrak{n}}) = (\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5\lambda_6)^{\frac{1}{6}} \Big(\tfrac{1}{\lambda_1} + \tfrac{1}{\lambda_2} + \tfrac{1}{\lambda_3} + \tfrac{1}{\lambda_4} + \tfrac{1}{\lambda_5} + \tfrac{1}{\lambda_6} - \tfrac{1}{8} \big(\tfrac{\lambda_1}{\lambda_2\lambda_4} + \tfrac{\lambda_1}{\lambda_3\lambda_5} + \tfrac{\lambda_2}{\lambda_1\lambda_4} + \tfrac{\lambda_2}{\lambda_3\lambda_6}$$
$$+ \tfrac{\lambda_3}{\lambda_1\lambda_5} + \tfrac{\lambda_3}{\lambda_2\lambda_6} + \tfrac{\lambda_4}{\lambda_1\lambda_2} + \tfrac{\lambda_4}{\lambda_5\lambda_6} + \tfrac{\lambda_5}{\lambda_1\lambda_3} + \tfrac{\lambda_5}{\lambda_4\lambda_6} + \tfrac{\lambda_6}{\lambda_2\lambda_3} + \tfrac{\lambda_6}{\lambda_4\lambda_5} \big) \Big) \ .$$

Take the unit volume Kähler-Einstein metric $P_{\mathfrak{n}}^{\mathrm{KE}}$ corresponding to the values

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6) = \tfrac{1}{4}\big(\tfrac{1024}{3}\big)^{\frac{1}{6}}(3, 2, 1, 1, 2, 1).$$

One can compute that the matrix $\mathrm{Hess}\big(\widetilde{\mathrm{scal}}_N\big)\big|_{P_{\mathfrak{n}}^{\mathrm{KE}}}$ has two distinct positive eigenvalues, three negative eigenvalues and one zero eigenvalue, corresponding to the scaling direction. Therefore, $P_{\mathfrak{n}}^{\mathrm{KE}}$ has coindex $q = 2$. By Theorem 3.3.1, on $\mathsf{SU}(4)$ there is a 7-parameter family of ancient solutions to the Ricci flow collapsing, under rescaling, to $0 \oplus P_{\mathfrak{n}}^{\mathrm{KE}}$ as $t \to -\infty$. Similarly on $\mathsf{SU}(4)/\mathsf{S}^1$ there is a 4-parameter family of ancient solutions, and on $\mathsf{SU}(4)/\mathsf{T}^2$ there is a 2-parameter family of ancient solutions.

Notice that, as in the previous example, the construction of Lu and Wang again provides ancient solutions on $\mathsf{SU}(4)$ but their family is two dimensions smaller, due to the fact that in their construction the metric on the base remains fixed.

3.4.3. A Kähler-Einstein metric on $\mathsf{G}_2/\mathsf{T}^2$

Let $\mathsf{G} = \mathsf{G}_2$, T^2 a maximal torus inside G_2 and consider the real root spaces decomposition

$$\mathfrak{su}(3) = \mathfrak{t}^2 + \mathfrak{n}_1 + \mathfrak{n}_2 + \mathfrak{n}_3 + \mathfrak{n}_4 + \mathfrak{n}_5 + \mathfrak{n}_6 \ .$$

Then, any G-invariant Riemannian metric $P_{\mathfrak{n}}$ on the flag manifold $N = \mathsf{G}_2/\mathsf{T}^2$ takes the form

$$P_{\mathfrak{n}} = \lambda_1 \,\mathrm{Id}_{\mathfrak{n}_1} \oplus \ldots \oplus \lambda_6 \,\mathrm{Id}_{\mathfrak{n}_6}$$

and its normalized scalar curvature is given by

$$\widetilde{\mathrm{scal}}_N(P_{\mathfrak{n}}) = (\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5\lambda_6)^{\frac{1}{6}}\Big(\tfrac{1}{\lambda_1} + \tfrac{1}{\lambda_2} + \tfrac{1}{\lambda_3} + \tfrac{1}{\lambda_4} + \tfrac{1}{\lambda_5} + \tfrac{1}{\lambda_6} - \tfrac{1}{6}\big(\tfrac{\lambda_1}{\lambda_3\lambda_4} + \tfrac{\lambda_3}{\lambda_1\lambda_4} + \tfrac{\lambda_4}{\lambda_1\lambda_3}\big)$$
$$- \tfrac{1}{8}\big(\tfrac{\lambda_1}{\lambda_2\lambda_3} + \tfrac{\lambda_1}{\lambda_4\lambda_5} + \tfrac{\lambda_2}{\lambda_1\lambda_3} + \tfrac{\lambda_2}{\lambda_5\lambda_6} + \tfrac{\lambda_3}{\lambda_1\lambda_2} + \tfrac{\lambda_3}{\lambda_4\lambda_6} + \tfrac{\lambda_4}{\lambda_1\lambda_5} + \tfrac{\lambda_4}{\lambda_3\lambda_6} + \tfrac{\lambda_5}{\lambda_1\lambda_4} + \tfrac{\lambda_5}{\lambda_2\lambda_6} + \tfrac{\lambda_6}{\lambda_2\lambda_5} + \tfrac{\lambda_6}{\lambda_3\lambda_4}\big)\Big).$$

For more information about homogeneous Einstein metrics on $N = \mathsf{G}_2/\mathsf{T}^2$, see [5]. Take the unit volume Kähler-Einstein metric $P_{\mathfrak{n}}^{\mathrm{KE}}$ corresponding to the values

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6) = \tfrac{1}{12}(\tfrac{4608}{5})^{\frac{1}{6}}(1, 3, 4, 5, 6, 9)$$

and one can compute that the matrix $\mathrm{Hess}\big(\widetilde{\mathrm{scal}}_N\big)\big|_{P_{\mathfrak{n}}^{\mathrm{KE}}}$ has one positive eigenvalue, four negative eigenvalues and one zero eigenvalue, corresponding to the scaling direction. Therefore, $P_{\mathfrak{n}}^{\mathrm{KE}}$ has coindex $q = 1$. By Theorem 3.3.1, on G_2 there is a 3-parameter family of ancient solutions to the Ricci flow collapsing, under rescaling, to $P_{\mathfrak{n}}^{\mathrm{KE}}$ as $t \to -\infty$. Similarly on $\mathsf{G}_2/\mathsf{S}^1$ there is a 1-parameter family of ancient solutions.

3.4.4. The normal Einstein metric on $\mathsf{SU}(n)/\mathsf{T}^{n-1}$

Let $\mathsf{G} = \mathsf{SU}(n)$, with $n \geq 3$, and $\mathsf{T}^{n-1} \subset \mathsf{SU}(n)$ the diagonally embedded maximal torus. Then for any $1 \leq k \leq n-1$ and any subtorus $\mathsf{T}^{n-1-k} \subset \mathsf{T}^{n-1}$ we have a homogeneous fibration

$$\mathsf{T}^k \to \mathsf{SU}(n)/\mathsf{T}^{n-1-k} \to \mathsf{SU}(n)/\mathsf{T}^{n-1} \ ,$$

where T^k is a complement of T^{n-1-k} in T^{n-1}. By [31], the normal metric on $\mathsf{SU}(n)/\mathsf{T}^{n-1}$ induced by the biinvariant metric on $\mathsf{SU}(n)$ is Einstein with coindex $q = n-1$. Hence by Theorem 3.3.1, there exists a $\big(\frac{k(k+1)}{2} + n - 2\big)$-parameter family of ancient solutions on $\mathsf{SU}(n)/\mathsf{T}^{n-1-k}$ which collapse, under rescaling, to the normal metric on the base as $t \to -\infty$.

3.4.5. The normal Einstein metric on $\mathsf{SO}(4)/\mathsf{T}^2 = S^2 \times S^2$

Let $\mathsf{G} = \mathsf{SO}(4)$, $\mathsf{T}^2 \subset \mathsf{SO}(4)$ be a maximal torus and consider the $\mathrm{Ad}(\mathsf{T}^2)$-irreducible decomposition

$$\mathfrak{so}(4) = \mathfrak{t}^2 + \mathfrak{n}_1 + \mathfrak{n}_2.$$

Then, any G-invariant Riemannian metric $P_{\mathfrak{n}}$ on $N = \mathsf{SO}(4)/\mathsf{T}^2$ takes the form

$$P_{\mathfrak{n}} = \lambda_1 \,\mathrm{Id}_{\mathfrak{n}_1} \oplus \lambda_2 \,\mathrm{Id}_{\mathfrak{n}_2}$$

and its normalized scalar curvature is given by

$$\widetilde{\mathrm{scal}}_N(P_{\mathfrak{n}}) = (\lambda_1\lambda_2)^{\frac{1}{2}}\Big(\tfrac{1}{\lambda_1} + \tfrac{1}{\lambda_2}\Big).$$

The normal metric $P_{\mathfrak{n}}^{\mathrm{E}}$ induced by the biinvariant metric on $\mathsf{SO}(4)$ is Einstein and given by

$$(\lambda_1, \lambda_2) = (1, 1)\ .$$

One can compute that the matrix $\mathrm{Hess}\big(\widetilde{\mathrm{scal}}_N\big)\big|_{P_{\mathfrak{n}}^{\mathrm{E}}}$ has one positive eigenvalue and one zero eigenvalue, corresponding to the scaling direction. Therefore, $P_{\mathfrak{n}}^{\mathrm{E}}$ has coindex $q = 1$. Hence by Theorem 3.3.1 on $\mathsf{SO}(4)$ there is a 3-parameter family of ancient solutions which collapse, under rescaling, to $P_{\mathfrak{n}}^{\mathrm{E}}$ as $t \to -\infty$. Similarly, if $\mathsf{S}^1_{p,q} \subset \mathsf{T}^2$ is a diagonally embedded circle with rational slope $\frac{p}{q}$, then on $\mathsf{SO}(4)/\mathsf{S}^1_{p,q} \simeq S^3 \times S^2$ there is a 1-parameter family of ancient solutions which collapse, under rescaling, to $P_{\mathfrak{n}}^{\mathrm{E}}$ as $t \to -\infty$.

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