Signature PRINTED NAME

PRELIMINARY EXAMINATION, PART I

Friday, April 28, 2023 9:30am-12:00pm

This examination is based on Penn's code of academic integrity

Instructions:

Sign and print your name above.

This part of the examination consists of six problems, each worth ten points. You should work all of the problems. Show all of your work. Try to keep computations well-organized and proofs clear and complete — and justify your assertions. Each problem should be given its own page (or more than one page, if necessary).

If a problem has multiple parts, earlier parts may be useful for later parts. Moreover, if you skip some part, you may still use the result in a later part.

Be sure to write your name both on the exam and on any extra sheets you may submit.

Score (for faculty use only)	
1	
2	
3	
4	
5	
6	
GRADER	

1. Which of the following rings R is a unique factorization domain, respectively a principal ideal domain. Justify your answers!

   (a) $R = \mathbf{Z}[t]$ the polynomial ring in the variable t over the ring of integers $\mathbf{Z}$.

   (b) $R = \mathbf{Z}[i]$ the ring of Gauss integers.

   (c) $R = \mathbf{Z}[\sqrt{-5}]$.

2. (a) Suppose that T is an $n \times n$ matrix over a field F such that $T^2 = I$. Show that if the characteristic of F is not equal to 2, then T may be diagonalized and enumerate the possibilities for the diagonal form of T.

   (b) If F has characteristic 2, give an example of a matrix T such that $T^2 = I$ but T is not diagonalizable.

3. Let V be a complex vector space with inner product $(\ |\ )$ and $||\ ||$ the induced norm. Evaluating $||x+ty||$ for t a real parameter, prove that $||x+y| \leq ||x||+||y||$.

4. *Let $I \subset \mathbf{R}$ be a non-empty interval, $f : I \to \mathbf{R}$ be a continuous function and $A \subset \mathbf{R}$ be a compact subset. Prove or give counter-examples to the following assertions:*

   (a) *If I is open and $A \subset f(I)$, then the preimage $f^{-1}(A)$ is compact.*

   (b) *If I is closed and bounded, then $f^{-1}(A)$ is either empty, or compact.*

5. *Let p and q be distinct primes.*

   (a) *Let $\bar{q} \in \mathbf{Z}/p$ denote the class of q modulo p and let k denote the order of $\bar{q}$ as an element of $(\mathbf{Z}/p)^*$. Prove that no group of order pq^k is simple.*

   (b) *Let G be a group of order pq. Prove G is not simple.*

6. *Let* $I = [0, 1]$ *be the closed unit interval. Prove/disprove/answer the following:*

   (a) I *is not homeomorphic to the closed unit square* $S = I^2$.

   (b) *The closed unit square* $S = I^2$ *is not homeomorphic to the closed unit cube* $Q = I^3$.