Hitchin section

spectral cover smooth at $q=0$

$SL_2(\mathbb{C})$ - Higgs bundle : $\mathrm{Higgs}_G \downarrow H^0(X, K_X^2)$ has section $\mathrm{Higgs}_G \uparrow H^0(X, K_X^2)$ $\quad (K_X^{\frac{1}{2}} \oplus K_X^{-\frac{1}{2}}, \begin{pmatrix} 0 & q \\ 1 & 0 \end{pmatrix}) \uparrow q$

$E = K_X^{\frac{1}{2}} \oplus K_X^{-\frac{1}{2}}$ is well-defined (→ dep on spin structure) stable Higgs bundle (i.e. ϕ inv. subbundles have smaller slope)

$\phi = \begin{pmatrix} 0 & q \\ 1 & 0 \end{pmatrix}$ as $\mathrm{rk}\, E = 2$, $\deg E = 0$, and if $L \subseteq E$, inclusion gives section in $K^{\frac{1}{2}} \otimes L^\vee$ and $K^{-\frac{1}{2}} \otimes L^\vee$

$(E, \phi) \in \Gamma \begin{pmatrix} \mathrm{Hom}(K^{\frac{1}{2}}, K^{\frac{1}{2}}) & \mathrm{Hom}(K^{-\frac{1}{2}}, K^{\frac{1}{2}}) \\ \mathrm{Hom}(K^{\frac{1}{2}}, K^{-\frac{1}{2}}) & \mathrm{Hom}(K^{-\frac{1}{2}}, K^{-\frac{1}{2}}) \end{pmatrix} \otimes K$ so $\deg L \leq 1-g \leq -1$ (or $g-1$ if $g=0$) $\quad$ ($\deg K = -\chi = 2g-2$, $\deg K^{-\frac{1}{2}} = 1-g$, $\deg(K^{-\frac{1}{2}}) \geq \deg L$)

$= \Gamma \begin{pmatrix} K & K^2 \\ 0 & K \end{pmatrix}$ or $L = K^{-\frac{1}{2}}$

$\phi(K^{-\frac{1}{2}}) \subseteq K^{\frac{3}{2}} \subseteq E \otimes K = K^{3/2} \oplus K^{\frac{1}{2}}$ not in $K^{\frac{1}{2}} = K^{-\frac{1}{2}} \otimes K$

so either case slope is getting smaller or L not stable under ϕ.

Note: $p \in H^0(X, K_X)$, then $(\mathcal{O}_X \oplus \mathcal{O}_X, \begin{pmatrix} 0 & p \\ p & 0 \end{pmatrix})$ also stable Higgs bundle: $L \hookrightarrow E$ gives section in $L^\vee$, so $\deg L < 0$ or $L =$ one of $\mathcal{O}_X$ or diagonal $\mathcal{O}_X$. neither fixed by ϕ

Locally, say a spectral curve is given by $Z(\lambda^2 - \mathrm{tr}\,\phi\, \lambda + \det \phi)$, it gives diff. operator

$$P(x, \hbar \tfrac{d}{dx}) = (\hbar \tfrac{d}{dx})^2 - \mathrm{tr}\,\phi(x)(\hbar \tfrac{d}{dx}) + \det \phi(x) \quad \text{on } \varinjlim_n X \times \underbrace{\mathrm{Spec}(\mathbb{C}[[\hbar]]/(\hbar^n))}_{\text{infitesimal NBHD of } 0}$$

$D^\hbar$ = sheaf of $\hbar$-diff. operator on $X[[\hbar]]$ = sheaf glued out of $\mathcal{O}_{U[[\hbar]]}[\hbar \frac{d}{dx}]$

$(E, \nabla^\hbar)$ $\mathbb{C}[[\hbar]]$ - linear

A projective coord. system on X is atlas w. transition function in Mobius $\subseteq$ Holomorphic

e.g.
$$\begin{array}{ccc} \mathbb{H} & \longrightarrow & X \\ \uparrow & & \uparrow \\ \tilde{U}_\alpha & \xrightarrow[\text{biholom}]{} & U_\alpha \text{ contractible NBHD} \end{array}$$

z on $\mathbb{H}$ restrict to coord. on U_α.

, then on $U_\alpha \cap U_\beta$, change of coord is Mobius (constant if in same fund. domain a deck map otherwise)

i.e. $\exists \begin{pmatrix} a_{\alpha\beta} & b_{\alpha\beta} \\ c_{\alpha\beta} & d_{\alpha\beta} \end{pmatrix}$, $z_\alpha = \dfrac{a_{\alpha\beta} z_\beta + b_{\alpha\beta}}{c_{\alpha\beta} z_\beta + d_{\alpha\beta}}$

$$dz_\alpha = \frac{1}{(c_{\alpha\beta} z_\beta + d_{\alpha\beta})^2} dz_\beta$$

Let $\xi_{\alpha\beta} := (c_{\alpha\beta} z_\beta + d_{\alpha\beta})$, then change of coord. from U_α to U_β

for K, it's $\dfrac{dz_\beta}{dz_\alpha} = \xi_{\alpha\beta}^2 = (c z_\beta + d)^2$ → actually $(c(z_\beta(z_\alpha)) + d)^2$, should be viewed as function on $U_\alpha \cap U_\beta$

$K^{\frac{1}{2}}$, it's $\pm \xi_{\alpha\beta}$, $\pm$ dep. on $H^1(X, \mathbb{Z}/2)$ (spin structure) and on $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

T, it's $\dfrac{1}{\xi_{\alpha\beta}^2}$

Nice thing about Mobius transform: Schwarzian derivative $S_z(f)$ is def as $\left(\dfrac{f''}{f'}\right)' - \dfrac{1}{2}\left(\dfrac{f''}{f'}\right)^2 = \left(\dfrac{f''}{(f')^2}\right)'$ and for $f \in$ Mob it vanishes.

For $\psi \in \Gamma(K^{-\frac{1}{2}})$, in proj. coord. we can define $\left(\dfrac{d^2}{dz^2} - g(z)\right)\psi(z) \in \Gamma(K^{\frac{3}{2}})$

as in charts sections are rep by functions ψ_α s.t.

$$\psi_\beta(z_\beta(z_\alpha)) \frac{1}{\sqrt{dz_\beta}} = \psi_\alpha(z_\alpha) \frac{1}{\sqrt{dz_\alpha}} \quad \text{i.e.} \quad \psi_\beta(z_\beta(z_\alpha)) \cdot \xi_{\alpha\beta}^{-1} = \psi_\alpha(z_\alpha)$$

$$g_\beta(z_\beta(z_\alpha))\, \xi_{\alpha\beta}^2 = g_\alpha(z_\alpha) \qquad (g_\beta(z_\beta(z_\alpha))\, dz_\beta^2 = g_\alpha(z_\alpha)\, dz_\alpha^2)$$

ξ is function of z_α

$$\frac{d^2}{dz_\alpha^2}\left[\xi^{-1} \psi_\beta(z_\beta(z_\alpha))\right] = \frac{d}{dz_\alpha}\left[\xi \frac{d}{dz_\beta}\psi_\beta(z_\beta(z_\alpha)) - \xi \xi' \xi^{-2} \psi_\beta(z_\beta(z_\alpha))\right]$$
$$= \xi^3 \frac{d^2}{dz_\beta^2}\psi_\beta(z_\beta(z_\alpha)) + \xi' \frac{d}{dz_\beta}\psi_\beta(z_\beta(z_\alpha))$$
$$- \xi' \frac{d}{dz_\beta}\psi_\beta(z_\beta(z_\alpha)) - \left(\frac{\xi'}{\xi^2}\right)' \psi_\beta(z_\beta(z_\alpha))$$
$$= \xi^3 \frac{d^2}{dz_\beta^2}\psi_\beta(z_\beta(z_\alpha)) - \left(\frac{\xi'}{\xi^2}\right)' \psi_\beta(z_\beta(z_\alpha))$$

, but last term $=0$ if $\xi = \dfrac{dz_\beta}{dz_\alpha}$ is deriv. of Mob. transform.

$$= \xi^3 \frac{d^2}{dz_\beta^2}\psi_\beta(z_\beta(z_\alpha))$$

so it's really a section of $K^{3/2}$.!

Def: An oper on X is diff. operator of order r defined using Projective charts taking sections of $K^{-\frac{r-1}{2}}$ to $K^{\frac{r+1}{2}}$

Spectral curve gives (in projective coordinate chart) differential operator

e.g. λ^2+q - - - - - - - - - - - - $\hbar^2\frac{d^2}{dx^2}+q$

For $\hbar\in\mathbb{C}$, note $\mathrm{Ext}^1(K^{-\frac12},K^{\frac12})=R^1\mathrm{Hom}(K^{-\frac12},K^{\frac12})=R^1\Gamma(K)=H^1(K)$

$\underset{\text{Serre}}{=}H^0(0)^\vee=\mathbb{C}$.

so $\hbar$ determines unique extension VB

$0\to K^{\frac12}\to V_\hbar\to K^{-\frac12}\to 0$ w. transition maps

$g^\hbar_{\alpha\beta}=\begin{pmatrix}\xi_{\alpha\beta} & \hbar\frac{d}{dz_\beta}\xi_{\alpha\beta}\\ 0 & \xi^{-1}_{\alpha\beta}\end{pmatrix}$ i.e. the constant $c_{\alpha\beta}$

Prop: For $\hbar\neq 0$, all $V_\hbar$ are isomorphic as holom. VB. (only need to change trivialization via gauge map $\begin{pmatrix}\sqrt{\hbar} & \\ & \sqrt{\hbar}^{-1}\end{pmatrix}$ to get same transition functions)

$\hbar=0$, it's trivial extension

$\nabla^\hbar$ locally def as $\frac{d}{dz_\alpha}+\frac{1}{\hbar}\begin{pmatrix}0 & q(z_\alpha)dz_\alpha\\ dz_\alpha & 0\end{pmatrix}$ → $\mathfrak{g}$-valued 1-form i.e. section of $K\otimes sl_2(\mathbb{C})$

gives connex (in std sense, not $\hbar$-connex) on $V_\hbar$ called Deligne's $\hbar$-connex or λ-connex.

Thm (Gunning): (curve) X equipped w. cplx structure, then

$H^0(K^2)=$ Moduli of $Sl_2(\mathbb{C})$-opers

$=$ Moduli of projective coordinate systems compatible w. cplx structure.

Donaldson: stable (holomorphic) Higgs bundle of deg 0 (E,ϕ) — E^{top} be the cplx VB underlying holom VB E.

$\leftrightarrow$ (D,ϕ,h) h Hermitian metric on E^{top} → up to choice? determined by metric on curve?

$D=D^{1,0}+D^{0,1}$

$D^{0,1}=\bar\partial_E$

D unitary connex i.e. Chern connex built out of $(\bar\partial_E,h)$

$\phi: E^{top}\to E^{top}\otimes\Omega^1$ $sl_2(\mathbb{C})$-valued 1-form on curve

satisfying Hitchin's eq.

(Hitchin: A connex on $\mathbb{R}^4$, indep of 2 coordinates x_3,x_4, $A=\sum A_i dx^i$, let $\phi=\frac{A_3-iA_4}{2}dz$, it's section of $K\otimes ad(P)$, i.e. Higgs field $A_1dx^1+A_2dx^2$ is connex. $(z=x_1+ix_2)$ Hit. eq is $F_A=*F_A$)

$F_D=[D,D]$

$$\begin{cases}F_D+[\phi,\phi^{t_h}]=0\\ D^{0,1}\phi=0\end{cases}$$

which is equiv. to $F_{D(\zeta)}=0$ for all $\zeta\in\mathbb{C}^\times$ where $D(\zeta):=D+\frac{1}{\zeta}\phi+\zeta\phi^{t_h}$

(expand, ζ^{-2},ζ^2 coef $\Rightarrow\phi$ (1,0) form, $\zeta^{\pm1}$ coef $\Rightarrow\phi$ holom, const term $\Rightarrow F_D+[\ ]=0$)

Models for Higgs / Loc:

$$\mathcal{M}_{Gauge} \overset{\text{Donaldson}}{=\!=\!=} \mathcal{M}_{Dol} \xleftrightarrow[\text{Simpson: diffeomorphic}]{\text{Non-Abelian Hodge}} \mathcal{M}_{DR} \overset{\text{Riem. Hilbert}}{=\!=\!=} \mathcal{M}_{Betti}$$

Moduli of (D, ϕ, h) (HyperKähler)

moduli of stable Higgs (E, ϕ)

moduli of irred. holom. connex (E, ∇) — i.e. no inv. subbundle under ∇

char. variety $\mathrm{Hom}^{irr}(\pi_1(C), sl_r(\mathbb{C}))//SL_r(\mathbb{C})$

$(D, \phi, h) \longmapsto (E^{top}, D(\zeta=1)^{0,1})$ as holom VB

$D(\zeta=1)^{1,0}$ gives holom connex.

Complex, biholo to $\mathcal{M}_{Gauge}$ w. one cplx structure I

complex, biholo to $\mathcal{M}_{Gauge}$ w. another cplx structure J

$SL_r(\mathbb{C})$ case: $X_+ := \begin{pmatrix} 0 & \sqrt{p_1} & 0 & \cdots & 0 \\ & \ddots & \ddots & & \vdots \\ & & \ddots & \ddots & 0 \\ & & & & \sqrt{p_{r-1}} \\ 0 & & & & 0 \end{pmatrix}$ $\quad p_i = i(r-i) \quad q_i \in H^0(K^{i+1})$,

$X_- := X_+^T. \quad H := [X_+, X_-]$

Hitchin section: Choose spin structure, $E = K^{\frac{r-1}{2}} \oplus \cdots \oplus K^{-\frac{r-1}{2}}$,

$\phi := X_- + \sum_{i=1}^{r-1} q_i X_+^i$.

Gaiotto conjecture: (E, ϕ) stable Higgs on Hitchin section! $D(\zeta)$ the corr. connex (flat!)

$D(\zeta, R) := D + \zeta^{-1} R\phi + \zeta R \phi^{\dagger}$, $R \in \mathbb{R}_+$ connexes

(D, ϕ, h) stable $\Rightarrow \forall R$, $(F_D + R^2[\phi, \phi^{\dagger_h}] = 0$, $D^{0,1}\phi = 0$, i.e. $\Rightarrow (D, R\phi, h)$ is Higgs) $\Rightarrow D(\zeta, R)$ flat

then $\lim_{\substack{\zeta/R = \hbar \\ R \to 0, \zeta \to 0}} D(\zeta, R)$ exist for all $\hbar \in \mathbb{C}^\times$

& gives $SL_r(\mathbb{C})$-oper

Dumitrescu-Fredrickson-Kydonakis-Mazzeo-Mulase-Neitzke: true for all simple & simply conn. Lie grps. (not just $sl_r(\mathbb{C})$)

($G = SL_2(\mathbb{C})$, existence: Locally, h metric on $K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}} \Rightarrow$ Hermitian metric $\lambda^2 dz d\bar{z}$ on curve.

In coordinate write down $D(\zeta, R) = d + \frac{R}{\zeta}(X_- + q X_+) dz - \partial \log \lambda H dz$
$+ R\zeta(\bar{q} X_- \lambda^{-2} + \lambda^2 X_+) d\bar{z}$

"prescribe scalar curvature"

$D(\zeta, R)$ flat $\Rightarrow \bar{\partial}\partial(\log \lambda) + R^2(\lambda^{-2} q\bar{q} - \lambda^2) = 0$.

$q = 0 \Rightarrow \Delta \log \lambda = R^2 \lambda^2$ i.e. $\lambda = \frac{1}{R} \frac{i}{z - \bar{z}}$, metric is $\propto \frac{dx^2 + dy^2}{y^2}$ i.e. just hyperbolic. (w. $K = -4R^2$)

$D = d + \frac{1}{\hbar} X_- dz - \partial \log \frac{i}{z - \bar{z}} H dz + \hbar \left(\frac{i}{z - \bar{z}}\right)^2 X_+ d\bar{z}$ indep of R.

$q \neq 0 \Rightarrow \lambda = \frac{i}{(z-\bar{z})R} e^{f_R}$, use implicit function Thm to show f_R analytic &.

$f_R = f_4 R^4 + HOT$.

expand, get $D(\zeta, R) = d + \frac{1}{\hbar}(X_- + q X_+) dz - \partial \log\left(\frac{i}{z-\bar{z}}\right) H dz$

$+ O(R^4) H dz + \hbar \left(\frac{i}{z-\bar{z}}\right)^2 X_+ d\bar{z} + O(R^4) \hbar d\bar{z}$

so one can take limit.

The limit $D(\hbar)$ is gauge eq. to oper $\nabla^\hbar = d + \frac{1}{\hbar}\begin{pmatrix} 0 & q \\ 1 & 0 \end{pmatrix} dz$ w. gauge transform

$g \begin{pmatrix} 1 & \hbar \partial \log\left(\frac{i}{z-\bar{z}}\right) \\ 0 & 1 \end{pmatrix}$, i.e. $\nabla^\hbar = g D(\hbar) g^{-1}$

$SL_r(\mathbb{C})$-oper: V holom VB of deg 0, ∇ irred connex, ($(V,\nabla) \in M_{dR}$ is analogy of Hitchin section).

s.t. $\exists$ global filtration $0 = F_r \hookrightarrow \cdots \hookrightarrow F_0 = V$ in V

sat. Griffith transversality: $\nabla|_{F_{i+1}} : F_{i+1} \to F_i \otimes K \ \forall i$

& $\nabla|_{F_{i+1}}$ induces $\mathcal{O}_C$-linear isom $F_{i+1}/F_{i+2} \cong F_i/F_{i+1} \otimes K$. $\forall i$

For the Hitchin section corr. to $\{q_i\}$, oper is $(V_\hbar, \nabla^\hbar)$,

$V_0 = K^{\frac{r-1}{2}} \oplus \cdots \oplus K^{-\frac{r-1}{2}}$

$V_\hbar$ transition functions $e^{H \cdot \log \xi_{\alpha\beta}} e^{\hbar \frac{d \log \xi_{\alpha\beta}}{d z_\beta} X_+}$

$\nabla^\hbar = d + \frac{1}{\hbar}\phi$

Relation to Gromov-Witten:

$\langle \tau_{d_1}, \cdots \tau_{d_n} \rangle_{g,n} := \int_{\overline{M}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n}$, $\psi_i = c_1(L_i) \in H^2(\overline{M}_{g,n}, \mathbb{Q})$

L_i the LB on $\overline{M}_{g,n}$ given by $\sigma_i^* \omega$. $\sigma_i : \overline{M}_{g,n} \to \overline{M}_{g,n+1}$ ω relative cotangent. of $\overline{M}_{g,n+1} \to \overline{M}_{g,n}$

$S_m(x) := x^{-\frac{3}{2}(m-1)} \cdot 2^{-(m-1)} \sum_{\substack{g \geq 0, n > 0 \\ 2g-2+n = m-1}} \frac{(-1)^n}{n!} \sum_{\sum d_i = 3g-3+n} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_{g,n} \prod_{i=1}^n |2d_i - 1|!!$

let $\psi(x,\hbar) := \exp \sum_{m=0}^\infty \hbar^{m-1} S_m(x)$

then $\left[\left(\hbar \frac{d}{dx}\right)^2 - x\right] \psi = 0$

Q: Where does the ODE come from?

Consider meromophic Higgs field $E = K_{\mathbb{P}^1}^{\frac{1}{2}} \oplus K_{\mathbb{P}^1}^{-\frac{1}{2}}$, $\phi = \begin{pmatrix} & x dx^2 \\ 1 & \end{pmatrix} \in \mathrm{End}\, E \otimes K(5\infty)$

it gives spectral curve $Z\left(\det(\lambda - \pi^*\phi)\right) \subseteq \mathbb{P}(K\oplus \mathcal{O})$

$$H^0(T^*\mathbb{P}^1, \pi^*K^2\otimes \mathcal{O}(5))$$

$\mathbb{P}(\mathcal{O}(-2)\oplus \mathcal{O})$ Hirz surface.

Pick chart on $\mathbb{P}^1$ to trivialize $\mathbb{P}(K\oplus\mathcal{O})$, it's given by equation $x=y^2$

$(y\,dx \in T^*\mathbb{P}^1)$

But very singular at (∞,∞) : let $u=\frac{1}{x}$. $\frac{1}{w}du = y\,dx$ give change of coord.

local expression becomes $w^2 = u^5$

Blow-up twice. Σ become $\tilde{\Sigma}$ which is a $\mathbb{P}^1$ in divisor class

$$\underset{\substack{\uparrow \\ \text{zero section} \\ \text{divisor in } \mathbb{P}(K\oplus\mathcal{O})}}{2B} + 5\,\text{Fiber} - 4E_2 - 2E_1$$

$\tilde{\Sigma} \to Bl(\mathbb{P}(K\oplus\mathcal{O}))$ can be parametrized by $t \mapsto \begin{cases} x = \frac{4}{t^2} \\ y = -\frac{2}{t} \end{cases}$

$\downarrow$

$\mathbb{P}^1$

$$F_{g,n}(t_i) := \frac{(-1)^n}{2^{2g-2+n}} \sum \langle \tau_{d_1}\cdots\tau_{d_n}\rangle \prod |2d_i-1|!!\left(\frac{t_i}{2}\right)^{2d_i+1}$$ should be thought of as

function on $\tilde{\Sigma}^n$ in this given coordinate,

$S_m(x) := \sum_{2g-2+n=m-1} \frac{1}{n!} F_{g,n}(t(x),\dots,t(x))$ gives the formula for $S_m(x)$ before.

($t(x)$ is branch of $\pi:\Sigma\to\mathbb{P}^1$)