Geometric Langlands & Conformal field Theory

1. Free field realization

2. T- duality (in terms of V.O.)

3. Twisted D- module attached to Opers

4. Construct Hecke eigensheave

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Overview:

PART 1 & 2 $\xrightarrow{\text{to proof}}$ $\mathfrak{z}_{-h^\vee}(\mathfrak{g}) \cong \mathrm{Func}\, \mathrm{OP}_{{}^L\mathfrak{g}}(D)$

$\mathfrak{z}(\mathfrak{g}) :=$ $\mathfrak{g}[[t]]$ - invariants in $V_\kappa(\mathfrak{g})$ / center in universal enveloping algebra

$$\mathrm{OP}_{\mathfrak{g}}(D) \simeq \mathrm{Proj}(D) \times \bigoplus_{j=2}^{\ell} \Omega^{\otimes(d_j+1)}(D)$$

$$= \{ \partial_t + p_{-1} + \sum_{j=1}^{\ell} v_j(t) \cdot p_j \}$$

${}^L\mathfrak{g}$:= Langlands dual Lie algebra.

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Overview:

PART 3:

Known $\{\text{center of chiral alg } V_{-h^\vee}(\mathfrak{g})\} \cong \text{Fun } Op_{{}^L\mathfrak{g}}$

twisted D-module on Bun_G. para by ${}^L\mathfrak{g}$-opers

PART 4:

State GLC in terms of space of conformal block

Free field realization

<u>Goal</u> : in $\widehat{sl_2}$ case.

show. $\mathfrak{Z}_{-2}(sl_2) \simeq \mathrm{Fun\,Proj}(D)$

First start wt a inj homomorphism.

$$V_k(sl_2) \longrightarrow \mathcal{F} \otimes \pi_0$$

Heisenberg VOA

$\parallel$

π_0 : chiral alg of free field $\phi(z)$

wt vacuum vector $|0\rangle$

$$\partial_z \phi(z) = \sum_{n \in \mathbb{Z}} b_n \cdot z^{-n-1}$$

φ

$\mathcal{F}$: chiral alg of $\beta\gamma$ system

$$\begin{cases} \text{vacuum vector } |0\rangle \\ \beta_{-1}|0\rangle \mapsto \beta(z) = \sum_n \beta_n z^{-n-1} \\ \gamma_0|0\rangle \mapsto \gamma(z) = \sum_n \gamma_n z^{-n-1} \\ [\beta_n, \gamma_m] = -\delta_{n,-m} \end{cases}$$

Free field realization

finding a inj homomorphism bt Vertex Alg / corresponding alg

Now construct as follows

$$f_\nu : \underset{\text{(Affine VOA)}}{V_k(sl_2)} \longrightarrow \underset{\text{(Tensor of two Heisenberg VA)}}{F \otimes \Pi_0}, \quad \nu = \sqrt{k+2}$$

Explain

$J^+(z) = Y(J^+_{-1} v_k, z)$

$:= \sum_{n\in\mathbb{Z}} J^+_n z^{-n-1}$

where in $\mathcal{U}(\widehat{sl_2})$

$J^+_n = J^+ \otimes t^n$

$$J^+_{-1} v_k \mapsto J^+(z) \longmapsto \beta(z)$$

$$J^0_{-1} v_k \mapsto J^0(z) \longmapsto :\beta(z)\gamma(z): + \frac{\nu i}{2} \partial_z \phi(z)$$

$$J^-_{-1} v_k \mapsto J^-(z) \longmapsto -:\beta(z)\gamma(z)^2: - k\partial_z \gamma(z) - \nu i \cdot \gamma(z) \cdot \partial_z \phi(z)$$

$$\underbrace{\qquad}_{\text{State - field correspondence}} \quad S(z) \longmapsto \frac{1}{4}\hat{b}(z)^2 - \frac{1}{2}\partial_z \hat{b}(z)$$

where $\hat{b}(z) = \nu i \cdot \partial_z \phi(z) = \sum_n \hat{b}_n \cdot z^{-n-1}$

Free field realization

take proper limit when $\nu = -2 = -h^\vee$ of sl_2

embedd. $\mathfrak{z}_{-2}(sl_2) \hookrightarrow |0\rangle \otimes \widehat{\pi}_0 \subseteq \mathcal{F} \otimes \widehat{\pi}_0$

$$\widehat{\pi}_0 = \mathbb{C}[\widehat{b}_n]_{n<0}.$$

Therefore.

$$\widehat{\pi}_0 = \mathbb{C}[\widehat{b}_n]_{n<0} \longleftrightarrow \mathrm{Fun}\,\mathrm{Conn}(D)$$

$$\mathfrak{z}_{-2}(sl_2) \xrightarrow{\text{embedd.}} \mathrm{Im}\left(\mathfrak{z}(sl_2)\right) \hookrightarrow \widehat{\pi}_0$$

Miura Tranform: $\mathrm{Fun}\,\mathrm{Proj}(D) \hookrightarrow \mathrm{Fun}\,\mathrm{Conn}(D)$

$$\mathrm{Im}\left(\mathfrak{z}(sl_2)\right) = \mathrm{Fun}\,\mathrm{Proj}(D)$$

$$\langle S(z) \rangle \longmapsto \left\langle \tfrac{1}{4}\left(\widehat{b}(z)\right)^2_+ - \tfrac{1}{2}\partial_z\left(\widehat{b}(z)\right)_+ \right\rangle$$

Miura Transformation: $\mathrm{Conn}(D) \longrightarrow \mathrm{Proj}(D)$

$$\partial_z + u(z) \longmapsto \left(\partial_z - \tfrac{1}{2}u(z)\right)\left(\partial_z + \tfrac{1}{2}u(z)\right)$$

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Free field realization

Overall. We show:

$$\mathfrak{Z}_{-2}(\mathfrak{sl}_2) = \mathrm{FuncProj}(D)$$

But this relies on the explicit formula of generator $\mathfrak{Z}(\mathfrak{sl}_2)$

not works for general $\mathfrak{g}$

Non critical case. consider screening operator

$$Y(|-\tfrac{i}{\nu}\rangle, z)$$

$$\int V_{-\frac{i}{\nu}}(z)\,dz : \pi_0 \longrightarrow \pi_{-\frac{i}{\nu}}$$

$$V_{-\frac{i}{\nu}}(z) =: e^{-\frac{i}{\nu}\phi(z)} : = T_{-\frac{i}{\nu}} \cdot \exp\left(\frac{i}{\nu}\cdot\sum_{n<0}\frac{ib_n}{n}z^{-n}\right)\exp\left(\frac{i}{\nu}\sum_{n<0}\frac{ib_n}{n}z^{-n}\right)$$

Free field realization

Want to Identify $\mathfrak{Z}(sl_2) = \operatorname{Ker}\left(\int V_{-\frac{1}{\nu}}(z)dz\right)$, then take $\nu \to -2$

More explicitly in noncritical case, i.e. $k \neq -2$

if we identify

$$V_{-\frac{1}{\nu}}(z) = Y\left(|-\tfrac{1}{\nu}\rangle, z\right) = \sum_n |-\tfrac{1}{\nu}\rangle_n \, z^{-n-1}$$ in Heisenberg v.a.

$$\int V_{-\frac{1}{\nu}}(z)dz = |-\tfrac{1}{\nu}\rangle_0$$

Def. $T_\nu(z) = \frac{1}{4} :\tilde{b}(z):^2 + \frac{1}{2}(\nu - \frac{1}{2})\partial_z \tilde{b}(z)$

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Free field realization

Def. $T_\nu(z) = \frac{1}{4} :\tilde{b}(z):^2 + \frac{1}{2}(\nu - \frac{1}{\nu}) \partial_z \tilde{b}(z)$ SEMT

Then we check. $\langle T_\nu(z) \rangle \subseteq \mathrm{Ker}\left(\int V_{-\frac{1}{\nu}}(z) dz \right)$.

$$\left(\frac{1}{4} :\tilde{b}(z):^2 + \frac{\lambda}{2} \partial_z \tilde{b}(z) \right) \cdot V_{-\frac{1}{\nu}}(z)$$

$$= \frac{1}{2} \left[\frac{1}{2} \left(-\frac{1}{\nu} \left(-\frac{1}{\nu} - \lambda \right) \right) \cdot \frac{V_{-\frac{1}{\nu}}(w)}{(z-w)^2} + \frac{\partial_w V_{-\frac{1}{\nu}}(w)}{z-w} + \mathrm{reg} \right]$$

if we take $\lambda = \nu - \frac{1}{\nu}$

singular term $= \partial_w \left(\frac{V_{-\frac{1}{\nu}}(w)}{z-w} \right)$, i.e. conformal dim $=1$ field

Therefore. $\left[T_\nu(z), \int V_{-\frac{1}{\nu}}(w) dw \right] = 0$.

Free field realization

Similarly reason.

One more vertex operator $V_v(z) := :e^{iv\phi(z)}:$ satisfies this property. i.e.

$$\left\{\begin{array}{l}\text{Vertex alg generator by } T_v(z) \\ \text{(chiral)}\end{array}\right\} = \mathrm{Ker}\left(\int V_v(z)\,dz\right)$$
$$= \mathrm{Ker}\left(\int V_{-\frac{1}{v}}(z)\,dz\right)$$

In degenerate/critical case, i.e. $v = -2$

$$\mathrm{Ker}\left(\int V_{-\frac{1}{v}}(z)\,dz\right) \longrightarrow \mathfrak{z}_{-2}(sl_2)$$

$$\mathrm{Ker}\left(\int V_v(z)\,dz\right) \longrightarrow \text{chiral Vir alg} = \mathrm{Fun}\,\mathrm{Proj}(D)$$

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T-Duality & W-alg

This part went to generalize screening op to $\forall\ \mathfrak{g}$.

To show. $\mathfrak{z}_{-h^\vee}(\mathfrak{g}) \cong \mathrm{Fun}\, \mathrm{OP}_{{}^L\mathfrak{g}}(D)$

Free field realization for $\hat{\mathfrak{g}}$

If $\dim(\mathfrak{h}) = l$

l-copies of $\beta\gamma$-system $+$ $\mathfrak{h}^*$-valued free boson field ϕ

Heisenberg vertex alg

$\mathcal{F}(\mathfrak{g}) \otimes \pi_0(\mathfrak{g}) := \left(\bigotimes_{\alpha_i} \mathcal{F}_{\alpha_i} \right) \otimes \pi_0(\mathfrak{g})$, where

$\pi_0(\mathfrak{g}) = \langle \check{\lambda} \cdot \phi(z) \rangle$ s.t. $\check{\lambda}\, \phi(z) \cdot \check{\mu}\, \phi(w) = -k_0(\check{\lambda}, \check{\mu}) \log|z-w| + \mathrm{reg}.$

T-Duality & W-alg

Similarly. we have

1. Magnetic type operator

$$V_{-\underline{\alpha_j}^{\vee}}(z) = :e^{-\frac{i}{\nu}\alpha_j \cdot \phi(z)}: \quad \text{where } \alpha_j \in \mathfrak{h}^* \text{ simple roots}$$

through Killing form $\cong \mathfrak{h}$

2. Eletric type operator

$$V_{\nu\check{\alpha}_j}(z) := :e^{i\nu\check{\alpha}_j \phi(z)}: \quad \text{where } \check{\alpha}_j \in \mathfrak{g} \text{ simple coroots}$$

Still has embedding : $\mathfrak{z}(\mathfrak{g}) \hookrightarrow \left(\bigotimes_{\alpha_i} |0\rangle_{\alpha_i}\right) \otimes \pi_0(\mathfrak{g}) \subseteq \mathcal{F}(\mathfrak{g}) \otimes \pi_0(\mathfrak{g})$

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T-Duality & W-alg

$$\mathrm{Ker}_{\pi_0(\sigma_j)}\left(\int V_{-\frac{\alpha_j}{v}}(z)\,dz\right) = \mathrm{Ker}(\text{along } \alpha_j\text{-direction}) \otimes (\text{orthogonal part}).$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$= \mathrm{Ker}(\text{along } \check{\alpha}_j\text{-direction}) \otimes (\text{orthogonal part})$$

$$= \mathrm{Ker}_{\pi_0(\sigma_j)}\left(\int V_{v\check{\alpha}_j}(z)\,dz\right)$$

"=" use the fact that $\langle \check{\alpha}_j, \alpha_j \rangle = 2$. i.e.

$$(\check{\alpha}_i \cdot \phi(z)) \cdot (\alpha_i \cdot \phi(z)) = -2\log(z-w) + \text{reg}.$$

back to sl_2 case.

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T-Duality & W-alg

chiral W-alg $W_k(\mathfrak{g})$

$$W_k(\mathfrak{g}) = \bigcap_{j=1,\dots,l} \ker_{\pi_0(\mathfrak{g})} \int V_{-\alpha_j/\nu}(z)\,dz$$

Here comes $L\mathfrak{g}$:

$$W_k(L\mathfrak{g}) = \bigcap_{j=1,\dots,l} \ker_{\pi_0(L\mathfrak{g})} \int V_{-L\alpha_j/\nu}(z)\,dz$$

Natural def of $L\mathfrak{g}$

$$= \bigcap_{j=1,\dots,l} \ker_{\pi_0(\mathfrak{g})} \int V_{\nu\alpha_j}(z)\,dz$$

$/\!\!/$

T-Duality & W-alg

In the limit $k \to -h^\vee$, $\check{k} \to +\infty$

quasi-classic limit

$$\begin{tikzcd} W_k(\mathfrak{g}) \arrow[r, phantom, "\cong"] \arrow[d] & W_{\check{k}}({}^L\mathfrak{g}) \arrow[d] \\ \mathfrak{Z}_{-h^\vee}(\mathfrak{g}) \arrow[r, phantom, "\cong"] & \mathrm{Fun}\, \mathrm{OP}_{{}^L\mathfrak{g}}(D) \end{tikzcd}$$

Twisted D-module

Recall Localization functor

$$\Delta: \ (\hat{\mathfrak{g}}, G[[t]])\text{-module} \longrightarrow D'_\kappa\text{-module}$$

$$M \longmapsto \Delta(M)$$

In particular $\quad V_\kappa(\mathfrak{g}) \longmapsto D'_\kappa$

But D'_κ-module of itself is too big

want to find some quotient of $V_\kappa(\mathfrak{g})$ too small!

But if consider irreducible $L_{0,k}$, $\Delta(L_{0,k}) \cong H^0(\mathrm{Bun}_G, \mathcal{L}^{\otimes k *}) \otimes \mathcal{L}^{\otimes k}$

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Twisted D-module (Rep para by opers)

$$\begin{array}{ccc} V_{-h^\vee}(\mathfrak{g}) & & \\ \cup\!\shortmid & \overset{1-1}{} & \\ \mathfrak{z}(\mathfrak{g}) & \longleftrightarrow & \mathrm{End}(V_{-h^\vee}(\mathfrak{g})) : \text{commute w/ } \hat{\mathfrak{g}} \\ & & \Vert \\ A & \longmapsto & (v_{-h^\vee} \to A) \qquad \mathrm{End}_{\hat{\mathfrak{g}}}(V_{-h^\vee}(\mathfrak{g})) \\ f(v_{-h^\vee}) & \longleftarrow\!\shortmid & f \end{array}$$

Since it is annihilated by $\mathfrak{g}[[t]]$

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Twisted D-module. (Rep para by opers)

$$\mathfrak{Z}(\mathfrak{g}) = \mathrm{End}_{\hat{\mathfrak{g}}}(V_{-h^\vee}(\mathfrak{g})) \cong \mathrm{Fun}\, \mathrm{OP}_{L\mathfrak{g}}(D)$$

$\forall x \in \mathrm{OP}_{L\mathfrak{g}}(D)$. $\quad ev_x : \mathrm{Fun}\, \mathrm{OP}_{L\mathfrak{g}}(D) \longrightarrow \mathbb{C}$

$$f \mapsto f(x)$$

$$\begin{array}{lcl} ev_x : & \mathrm{End}_{\hat{\mathfrak{g}}}(V_{-h^\vee}(\mathfrak{g})) & \longrightarrow \mathbb{C} \\ \| & \wr\| & \\ \hat{x} & \mathrm{Fun}\, \mathrm{OP}_{L\mathfrak{g}}(D) & \\ & \wr\| & \\ & \mathfrak{Z}(\mathfrak{g}) & \end{array}$$

$$V_x = V_{-h^\vee}(\mathfrak{g}) \big/ \ker \hat{x} \cdot V_{-h^\vee}(\mathfrak{g})$$

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Twisted D-module. (Rep para by opers)

For instance

$$\mathfrak{g} = sl_2 \quad , \quad Op_{L\mathfrak{g}} = Op_{sl_2}(D) = proj(D)$$

$$x = \partial_t^2 - v(t) \in Proj(D) \quad , \quad v(t) = \sum_{n \leq -2} v_n \cdot t^{-n-2} \quad , \quad v_n \in \mathbb{C}$$

$$End_{\hat{sl_2}}(V_{-2}(sl_2)) = \mathfrak{Z}(sl_2) = \mathbb{C}[S_n]_{n \leq -2}$$

$$ev_x : \quad \mathfrak{Z}(sl_2) \longrightarrow \mathbb{C}$$

$$S_n \longmapsto v_n \in \mathbb{C}$$

Twisted D-module. (Rep para by opers)

Another instructive way to think:

$V_{-h^\vee}(\mathfrak{g}) \longrightarrow \mathrm{OP}_{{}^L\mathfrak{g}}(D)$ as vector bundle

$$\{\text{alg func on } \mathrm{OP}_{{}^L\mathfrak{g}}(D)\} = \mathrm{Fun}\,\mathrm{OP}_{{}^L\mathfrak{g}}(D) = \mathrm{End}_{\check{\mathfrak{g}}}\left(V_{-\check{h}}(\mathfrak{g})\right)$$

$$\begin{array}{c} \downarrow \text{act} \\ V_{-\check{h}}(\mathfrak{g}) \end{array}$$

Observe: Z variety

$$\{\text{Module over alg Fun } Z\} \longleftrightarrow \{\text{quasicoherent sheave over } Z\}$$

$V_{-\check{h}}(\mathfrak{g})$ free module over Fun OP

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Twisted D-module. (Rep para by opers)

V_x is nothing but the fiber of this vector bundle.

$x \in Op_{Log}(D)$.

i.e. skyscraper sheave over point $x \in Op_{Log}(D)$.

21.

Construct Hecke eigensheave.

Start with $V_\chi = V_{-h^\vee}(\mathfrak{g}) \big/ \ker\widehat{\chi} \cdot V_{-h^\vee}(\mathfrak{g})$

Applying localization functor Δ,

$\Delta(V_\chi)$ is a twisted $D'_{-h^\vee}$-module.

by construction it is dual space of conformal block

22.

State GLC using Conformal block.

Setting G. Connected, simply-connected, simple Lie grp

^LG adjoint type

First we can resolve twisted $D'_{-h^\vee}$-module to actually D-module

By some computation $\mathcal{L}^{\otimes(-h^\vee)} \cong K^{\frac{1}{2}}$ canonical line bundle of Bun_G

if $\mathcal{F}$ twisted $D'_{-h^\vee}$-module

$\Downarrow$

$\mathcal{F} \otimes_{\mathcal{O}} K^{-1/2}$ D-module over Bun_G

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State GLC using Conformal block.

Further discussion about $\mathrm{OP}_{{}^L\mathfrak{g}}(X)$

$$\mathrm{OP}_{{}^L\mathfrak{g}}(X) \longrightarrow \mathrm{Loc}_{{}^LG} = \{\mathrm{Bun}_{{}^LG} \text{ with connection}\} \xrightarrow[\text{Connection}]{\text{forget}} \mathrm{Bun}_{{}^LG}$$

Given a ${}^L\mathfrak{g}$-oper χ on X : triple $(\mathcal{F}, \nabla, \mathcal{F}_{B_+})$.

$$\Downarrow$$

we have E_χ : ${}^LG$-bundle with connection

State GLC using conformal block.

$\boxed{\text{Thm}}$

① ${}^LG$-local system E on X $\longleftrightarrow$ Hecke eigensheaf Aut_E

$E_x \longleftrightarrow \mathrm{Aut}_{E_x}$

② $\mathrm{Aut}_{E_x} = \Delta_x(V_{\chi_x}) \otimes K^{-1/2}$

③ $D'_{-h^\vee}$-module $\Delta_x(V_{\chi_x})$ non-zero iff local oper χ_x can extend globally to ${}^L\mathfrak{g}$-oper on X. $\chi \in Op_{{}^L\mathfrak{g}}(X)$

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