Hitchin: $\mathcal{A} = $ (∞-dim) space of holom. structures on VB V (say rk r, deg d) i.e. $\Omega^{0,1}(M, \mathrm{End}\,V)$ "1-form in End V"

$T^*\mathcal{A}^s = \mathcal{A}^s \times \Omega^0(X, \mathrm{ad}P \otimes K)$
$= \{(\dot{A}, \dot{\Phi}) \mid \dot{\Phi} \in \Omega^{1,0}(\mathrm{Ad}P) = \Omega^0(\mathrm{Ad}P\otimes K),\ \dot{A} \in \Omega^{0,1}(\mathrm{Ad}P)\}$

$\mathcal{A}^s = $ stable structures. an open set in $\mathcal{A}$

$\tilde{G} = \mathrm{Aut}(V)$

$M_X(r,d) = \mathcal{A}^s/\tilde{G} = $ moduli of stable rk r VB

Canonical 1-form $\theta(\dot{A},\dot{\Phi})$ is $\int_X B(\dot{A}\wedge\dot{\Phi})$ i.e. apply killing form (End E is $\mathbb{C}$) ①

e.g. $G = \mathbb{C}^\times$, $M_X(1,d) = \mathrm{Jac}_d(X)$. $T^*M_X(1,0) = \mathrm{Jac}(X) \times H^0(X; K_X)$

i.e. T^* of $\mathrm{Jac}(X)$ $(= \mathcal{O}_{\mathrm{Jac}(X)})$

$p_1, \dots, p_k$ basis of ring of inv. polynomial on $\mathfrak{g}$, get (degree $d_1, \dots, d_k$)

$$h: H^0(X; \mathrm{ad}P \otimes K) \longrightarrow \bigoplus_{i=1}^k H^0(X, K^{d_i})$$

$$\dim H^0(X; K^{d_i}) = \deg K^{d_i} + (1-g) + \dim H^1(X; K^{1-d_i}) = (2g-2)d_i + (1-g) + \begin{cases} 0 & d_i > 1 \\ 1 & d_i = 1\end{cases}$$

$d_i = 0$ thrown away?

Gln case also work!

G semisimple $\Rightarrow$ no d_i is 1 so $= (2d_i - 1)(g-1)$

$$\dim \bigoplus H^0(X, K^{d_i}) = \sum_{i=1}^k (2d_i - 1)\cdot(g-1)$$

Kostant: $\dim G = \sum_{i=1}^k (2d_i - 1)$

They descend from maps $\mathcal{A}^s \times \Omega^0(M; \mathrm{Ad}P\otimes K) \to \bigoplus \Omega^0(M; K^{d_i})$ which Poisson comm.

OTOH, $\dim H^0(X; \mathrm{ad}P \otimes K) = \dim G\,(g-1)$ (call it n) if G semisimple (so $\dim H^1(\mathrm{Ad}P\otimes K) = 0$)

so (p_i) gives map $T^*M_X(r,d) \longrightarrow \bigoplus_{i=1}^k H^0(M; K^{d_i})$ (WLOG 0) which are n Poisson-commute. functions on $2n$-dim symplectic mfd

(T^*M_X is cplx symplectic, and functions actually holomorphic)

if only $a_n \neq 0$, it's cyclic cover.

✓ Case 1: $G = Gl(n, \mathbb{C})$, for $a_i \in H^0(X, K^i)$,

local picture: preimage is $\{(V, \Phi) \mid \det(x - \Phi) = x^n + \dots + a_n\}$

① Note: Lie $\tilde{G} = \Omega^0(X, \mathrm{ad}P)$

$\mathbb{C}_x \times \mathbb{C}_\lambda \supseteq Z(\lambda^n + a_1(x)\lambda^{n-1} + \dots + a_n(x))$

$Z := \mathrm{Tot}(K_X)$ $\quad p^*K_X$ has tautological section λ
$\downarrow p$
X

$\lambda^n + \dots + a_n \in \Gamma(p^* K_X^{\otimes n})$

Action fields: $\psi \in \mathrm{Lie}\,\tilde{G} \mapsto \bar{\partial}_E \psi \in \Omega^{0,1}(\mathrm{Ad}\,P)$ i.e. $\dot{A}$

Moment map: $\mu(E, \Phi) = 0$ iff $\bar{\partial}_E \Phi = 0$

Zero $(\lambda^n + \dots + a_n)$ gives divisor on Z that descent to divisor on $\mathbb{P}(K_X \oplus \mathcal{O})$ i.e. the spectral curve S (see BNR, λ is their x, 1 is their y)

$= \mathbb{P}(\mathcal{O}\oplus T_X)$?

Adjunction in $\mathbb{P}(K_X \oplus \mathcal{O})$: $2g(S) - 2 = S\cdot S + K_{\mathbb{P}(K_X\oplus\mathcal{O})}\cdot S$

$H^*(\mathbb{P}(K_X\oplus\mathcal{O})) = H^*(X)[\xi]/(\xi^2 - c_1(K)\xi)$

$\xi|_{\mathrm{Fiber}} = c_1(\mathcal{O}(1))|_{\mathrm{fiber}}$

$S = nX$, $\quad \mathrm{zero}(x) = \mathbb{P}(\mathcal{O}) \subseteq \mathbb{P}(K\oplus\mathcal{O})$ i.e. 0-section of K?

$K_{\mathbb{P}(K_X\oplus\mathcal{O})} = \mathcal{O}(-2) \otimes p^* K_X$, $\mathcal{O}(1)$ is the divisor class of y

$x \cdot y = 0$, $\quad x \cdot p^*K_X \overset{?}{=} 0$

$X \cdot X = K_X \cdot K_X = -\chi(X) = 2g-2$.

so $K\cdot S = 0$ and $g(S) = n^2(g-1) + 1$

so $S\cdot S = 2n^2(g-1)$

$\lambda|_S \in \Gamma(S, p^*K_X)$ is eigenvalue of $\Phi \in \Gamma(S, \mathrm{End}\,V \otimes p^*K_X)$

Note: S not singular!

Why degree 0? may not be!

$\ker(\lambda \mathrm{Id} - \Phi) \subseteq p^* V$ is generically a LB except when eigenval. repeats, and $\exists !$ LB $L \subseteq \ker(\lambda \mathrm{Id} - \Phi)$ (w. trivial Jord. block) giving pt in $\mathrm{Jac}(S)$

$O_S / I_{p^{-1}(x)} \otimes_{O_S} L = O_{X,(x)}/m_x \otimes_{O_X} L$? as O_X-mod.

For $L \in \mathrm{Jac}(S)$, $(p_* L) = O_S(L)/I_{p^{-1}(x)}$ ← ideal sheaf of $\pi^{-1}(x)$

$\mathrm{Tot}(K) = \mathrm{Sym}^*_{O_X} T$, e.g. $X = \mathrm{Spec}\,\mathbb{C}[t]$, $O = \mathbb{C}[t] = T = K$

$\mathrm{Tot}(K) = \mathrm{Sym}_O O = \mathrm{Spec}\,\mathbb{C}[s,t] = O \oplus O \oplus O \cdots$

Spectral cover $= \mathrm{Spec}\,\mathbb{C}[s,t]/\det(t\mathrm{Id} - \Phi(s))$
$= O \oplus O \oplus \cdots \oplus O$

x not branching pt: $= \bigoplus_{y \in p^{-1}(x)} L_y$

is branching pt: $= \bigoplus_{y \in p^{-1}(x)} J^{k(y)}(L)_y$ $\quad k(y) = $ ram. index at y

trivial Jordan blocks → push. L, get LB; non-trivial: rank missing → ②

V is essentially $p_* L^*$

$J^k(L)_y \to L_y \to 0$ gives $0 \to \bigoplus_{y \in \pi^{-1}(x)} L_y^* \to (p_* L)_x^* \to \bigoplus J^{k(y)}(L)_y^* / L_y^* \to 0$

According to BNR, if S smooth, just push down. $\varphi: E \to E \otimes K$ is from $\pi_*(L) \to \pi_*(\pi^* K \otimes L)$ via taut. section ②

which gives $0 \to$ some locally free sheaf $\to O(p_* L)^* \to S \to 0$

← also locally free. If map is flat, it push loc. free to loc. free.

i.e. the $O(V^*)$ ∈ what is this? holom. section of V^*

S: supported at branch pts (Jet skyscrapers)

The locally free sheaf gives the VB, however may not be abelian.

Case 2. $G = Sp(2m, \mathbb{C})$, i.e. C_m. V rk $2m$ bundle w. symplectic form. ω

$\Phi \in H^0(X, \mathrm{End}\, V \otimes K)$ s.t. $\omega(\Phi v, w) \equiv \omega(v, \Phi w)$

i.e. in $Sp(2m,\mathbb{C})$'s Lie alg. $-JX^TJ = X$

eigenvals appear in pair, $\det(x I - \Phi) = x^{2m} + a_2 x^{2m-2} + \cdots + a_{2m}$

Basis of inv. polynomial: $a_2 \cdots a_{2m}$

S curve of genus $4m^2(g-1)+1$ w. involusion induced from involution $\lambda \mapsto -\lambda$ on $\mathrm{Tot}(K)$

fixed pt of involution $\sigma = $ zero(a_{2m}) (there are $4m(g-1)$ fixed pts)

$\pi_*(L) = Nm(L) \otimes \det \pi_* O = Nm(L) \otimes K^{-\frac{n(n-1)}{2}}$

Norm map: $\mathrm{Pic}(S) \to \mathrm{Pic}(X)$ induced by pushing divisors.

$\sigma \curvearrowright \mathrm{Jac}$, fixed pt is Ab. variety called the Prym variety.

$\dim \mathrm{Jac} = g(S)$. $\dim \mathrm{Prym} = g(S) - g(S/\sigma) = m(2m+1)(g-1)$

$\mathrm{Prym}(S, X) := \ker \mathrm{Pic}(S) \to \mathrm{Pic}(X)$ i.e. those descend to trivial LB on X!

why not $g(S/\sigma)$? ? a surface → very suspicius

dim Jac of that surface

②: Grothendieck RR: $ch(p_* L)\, td(X) = \pi_*\, ch(L)\, td(S)$ so we can compute $\deg(\pi_* L)$
$= n(g-1) + \deg L + (1 - g(S))$

$\deg V^*$ is $\deg(\pi_* L^*) - \deg(\mathrm{Jets})$ ← Ram. divisor is a divisor of a section of $K_S K_M^*$ w. degree $(2g(S)-2) - m(2g-2)$ ← How is it related to Resultant of char polynomial?

so $\deg L = -m(m-1)(g-1) - \deg V^*$

Case 3. $G = SO(2m, \mathbb{C})$, i.e. D_m. V equipped w symbilin-form, $\det(xI - A) = x^{2m} + a_2 x^{2m-2} + \cdots + a_{2m}$

inv. polynomial $= a_2, \cdots, a_{2(m-1)}, p_m$, $p_m = $ Pfaffian, $a_{2m} = p_m^2$

For $p_m \in H^0(X, K^m)$, it has $2m(g-1)$ zeros, corr. to $2m(g-1)$ ordinary double pts on S

where $\lambda = p_m = 0$ (generically)

virtual genus of S is $4m^2(g-1)+1$, normalization has genus = vir. genus $-2m(g-1)$

σ involution has fixed pts = singular pts

so act fix-pt freely on normalization.

actually $\mathbb{C}[s,t]/(t^n + \cdots + s_n(s))$

$\mathcal{O}_S =$ "$\mathbb{C}[s,t]/\det(t\,\mathrm{Id} - \Phi(s))$" is like

③ : fin gen mod / PID

i.e. of form $\mathbb{C}[s][\Phi]$

If $\mathbb{C}[s]$ is field then it decomposes.

in part. "$\mathbb{C}[s][\Phi]$" $= \bigoplus_i \mathbb{C}(s)[t]/(t - f_i(s))^{r_i}$

$$\dim \mathrm{Prym}(\hat{S}) = g(\hat{S}) - g(\hat{S}/\sigma) = m(2m-1)(g-1)$$

genuinely double covered by $\hat{S}$

④ Grothendieck RR:

$ch(\pi_* \mathcal{O})$

$= (1 + (-1)[X])\,\pi_*(1 + (\deg(\mathcal{O}) + 1)[X])$

$= 2\cdot(-1) + 0 + 1$

$= 2 - 1[X]$ so $\pi_* \mathcal{O}$ is deg -1

the $\mathbb{C}[s]$-mod E is also an $\mathcal{O}_S$-Mod.

Φ gives it the structure of $\mathcal{O}_S$-mod. The $\mathcal{O}_S$-mod structure dep. on Φ i.e. t acts as Φ on $\mathbb{C}[s] \oplus \cdots \oplus \mathbb{C}[s]$ making it $\mathbb{C}[s][\Phi(s)]$ $\mathbb{C}[s]$-mod generators are $\Phi(s), \Phi^2(s), \ldots$ ← as $\mathbb{C}[s]$-mod

e.g. say $E = \mathcal{O}_S$, then L is the $\mathcal{O}_S$-mod $\tilde{\mathcal{O}}_S$, say $\bar{t}$ is eigen of $\bar{s}$.

$L_{(\bar{t},\bar{s})} = (\mathbb{C}[t]/\det(t\,\mathrm{Id} - \Phi(\bar{s})))/(t - \bar{t})$

$= \bigoplus (\mathbb{C}[t]/(t - t_i(\bar{s}))^{r_i})/(t - \bar{t}) = \bigoplus_{\text{Jordan}} \mathbb{C}[t]/(t - \bar{t})^{r_i}$

Note on localization: m prime $(-)_m = A_m \otimes_A (-)$

$0 \to I \to A \to A/I \to 0$

$\Rightarrow 0 \to I_m \to A_m \to (A/I)_m \to 0$ i.e.

$A_m / I_m = (A/I)_m$.

m maximal, If $I = m$, $(A/m)_m = A/m = A_m/m_m$

i.e. $A/m \otimes_A (-)$ same as taking residue field.

$= (-)/m(-)$

Prop.:

assumed integral,

S spectral cover, it's smooth iff. zero (a_{n-1}) $\cap$ multiple zero $(a_n) = \emptyset$

BNR's proposition: X curve, L LB, $s = (s_i) \in \Gamma(L^i)$, $1 \le i \le n$, S the spectral cover. then Tor-free rk 1 sheaf on S $\updownarrow$ push forward (E, φ), E rk n VB, $\varphi: E \to L \otimes E$ homo w. char. coef.

assumed to be integral. may not be smooth

S smooth, then tor free rk 1 $\Rightarrow$ LB.

so you just push it down?

(as $\mathrm{grad}(t^n + a_1(s)t^{n-1} + \cdots + a_n(s))$

$= (nt^{n-1} + (n-1)a_1(s) \cdots,\ a_1'(s)t^{n-1} + \cdots + a_n'(s))$

If $\{t^n + \cdots + a_n(s),\ (nt^{n-1} + \cdots + a_{n-1}(s)),\ (a_1'(s)t^{n-1} + \cdots + a_n'(s))\}$

all vanish, then (t,s) is branching pt, WLOG

$(t,s) = (0,0)$, i.e. by swapping to $\sum a_i(s - \bar{s})(t - \bar{t})^{n-i}$

singular means LOT is 2nd order, i.e.

need to shift $\bar{t}$ to 0 i.e. $\sum t^i a_{n-i} = \sum (t - \bar{t})^i \tilde{a}_{n-i}$ and need $\tilde{a}_n, \tilde{a}_{n-1}, \tilde{a}_n'$ vanish.

the Taylor exp. $= \tilde{a}_n(\bar{s}) + \tilde{a}_{n-1}(\bar{s})(t - \bar{t}) + \tilde{a}_n'(\bar{s})(s - \bar{s})$

LOT vanish so $a_n(s) = a_n'(s) = a_{n-1}(\bar{s}) = 0$

$+ \tilde{a}_{n-2}(\bar{s})(t - \bar{t})^2 + \tilde{a}_{n-1}'(\bar{s})(s - \bar{s})(t - \bar{t})$

$+ \frac{\tilde{a}_n''(\bar{s})}{2}(s - \bar{s})^2 + \mathrm{HOT}.$)

e.g. Although here S is not integral, let $n=2$, $a_1 = 2\lambda$, $a_2 = \lambda^2$, $\lambda \in \mathbb{C}$,

$\mathcal{O}_S = \mathbb{C}[s,t]/(t-\lambda)^2$. $\Phi = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ gives $\mathcal{O}_S$-mod. $\mathbb{C}[s,t]/(t-\lambda) \oplus \mathbb{C}[s,t]/(t-\lambda)$ ..→ rk = ?

S is line w. multiplicity.

$\Phi = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ gives $\mathcal{O}_S$-mod $\mathbb{C}[s,t]/(t-\lambda)^2$

both fiber $\mathbb{C} \oplus \mathbb{C}$ at (s, λ) for any s.

e.g. $L = \mathcal{O}_{\mathbb{P}^1}(1)$ in BNR notation, $s_1 \in \Gamma(\mathcal{O}(1))$, $s_2^2 \in \Gamma(\mathcal{O}(2))$,

$Z(t^2 + s_1 t + s_2^2) \subseteq \mathrm{Tot}(\mathcal{O}(1))$ smooth, on chart $s_1 = 1$, it's $Z(t^2 + t + s_2^2) \subseteq U \times \mathbb{C}$

at $s_2 = \frac{1}{2}$, it's ramified. $\mathcal{O}_S \longleftrightarrow (E, \Phi) \begin{pmatrix} 0 & -s_2^2 \\ 1 & -s_1 \end{pmatrix} \in \Gamma(\mathrm{End}(\mathcal{O} \oplus \mathcal{O}(-1)) \otimes \mathcal{O}(1))$. unfortunately $\mathcal{O} \oplus \mathcal{O}(-1) =: E$

not stable.

← at ram. pt. Jordan block not trivial. → ④

The spectral cover looks like $\mathbb{P}^1 \to \mathbb{P}^1$, $z \mapsto z^2$

e.g. a_2 a quadratic differential, if it only has simple zero then.

$S = Z(\lambda^2 + a_2) \subseteq \mathrm{Tot}(K)$ is smooth spectral curve.

In a chart near zero of a_2, $\mathbb{C}_w \times \mathbb{C}_z \subseteq \mathrm{Tot}(K)$, $Z(\lambda^2+a_2)$ is param. by z and locally of from $w = z^2$. z the "tautological coordinate"

LB on S locally is like $\mathbb{C}[w,z]/w-z^2$ - $f_0(w)+zf_1(w)$, Higgs field. $\Phi = \begin{pmatrix} 0 & w \\ 1 & 0 \end{pmatrix}$, at $w=0$ non-trivial Jordan block.

... mult. by z takes $f_0(w)$ to $z f_0(w)$ so ...
$z f_1(w)$ to $w f_1(w)$

Q: If at ram. pt. Φ has form $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$, then . it doesn't corr. to LB on S, but Hitchin fiber no longer Jac? example w/ S is still smooth?

grp homo.

$Nm : \mathrm{Pic}(S) \to \mathrm{Pic}(X)$ $\ker Nm =: \mathrm{Prym}(S, X)$.

i.e. $\mathcal{O}(n_i p_i) \mapsto \mathcal{O}(n_i \pi(p_i))$ $[\sum n_i p_i] \mapsto [\sum n_i \pi(p_i)]$

BNR: $\det \pi_* L \cong Nm(L) \otimes K^{-\frac{n(n-1)}{2}}$. (follows from genus computation).

($Y \to X$ finite map of smooth curve. then Ram. div is defined by zero of section in $\pi^* K_X^\vee \otimes K_Y$.

relative duality: $(\pi_* \mathcal{O}_Y)^\vee = \pi_* (K_Y \otimes \pi^* K_X^\vee)$ so $\det(\pi_* \mathcal{O}_Y)^\vee = \det \pi_* \mathcal{O}(\mathrm{Ram Div})$

Is det needed?

$\det(\pi_* L) = Nm(L) \otimes \det(\pi_* \mathcal{O}_Y)$ in part. if $L = \mathcal{O}_Y(\sum n_i p_i)$, $\det(\pi_* L) = \det(\pi_* \mathcal{O}_Y) \otimes \mathcal{O}_X(\sum n_i \pi(p_i))$

$\det \pi_* \mathcal{O}(\mathrm{Ram\ div}) = Nm(\mathcal{O}(\mathrm{Ram\ div})) \otimes \det(\pi_* \mathcal{O}_Y) = \det(\pi_* \mathcal{O}_Y)^\vee$.

so $Nm\ \mathcal{O}_Y(\mathrm{Ram\ div}) = \mathcal{O}_X(\mathrm{Branch\ Div}) = \det(\pi_* \mathcal{O}_Y)^{-2}$)

If $\det \pi_* L = \mathcal{O}_X$, then $Nm(L) = K^{\frac{n(n-1)}{2}}$, i.e. $Nm(L \otimes \pi^* K^{-\frac{n-1}{2}}) = \mathcal{O}$ (i.e. ... is in Prym ...)

π^* has degree divided by n.

For $Sp(2n, \mathbb{C})$,

$$\begin{tikzcd} S \arrow[r, "p"] \arrow[dr] & S/\sigma \arrow[d] \\ & X \end{tikzcd}$$

. $p^* Nm_{S/S/\sigma}(x) = x + \sigma(x)$, $\mathrm{Prym}(S, \bar{S})$ = LB on S, $\sigma^* L = L$. ($\pi^* \mathrm{Jac}(X) \subseteq \mathrm{Prym}(S, \bar{S})$?) ?

Note: A symp. structure induce. $E \cong E^\vee$

(on S, $L \in \mathrm{Jac}(S)$, $0 \to L \otimes \pi^* K^{1-2n} \to \pi^* E \xrightarrow{\pi^* \Phi - \eta} \pi^*(E \otimes K^*) \to L \otimes \pi^* K \to 0$)

$\pi^*(\pi_* L) \otimes \pi^* K^* \xrightarrow{\text{evaluate}} L \otimes \pi^* K$

Prop: $L \otimes \sigma^* L \cong \underbrace{(K_S \otimes \pi^* K_X^\vee)^\vee}_{\text{"}\mathcal{O}(\mathrm{Ram\ Div})^\vee\text{"}}$ in $SO(2m)$ case. $L \otimes (K_S \otimes \pi^* K_X^\vee)^{\frac{1}{2}} \in \mathrm{Prym}(S, S/\sigma)$

$\cong (K_S \otimes \pi^* K_X^\vee)^\vee$ in $Sp(2m)$ case, $L \otimes (K_S \otimes \pi^* K_X^\vee)^{\frac{1}{2}} \in \mathrm{Prym}(S, S/\sigma)$

$\cong \cancel{K^*}\ \pi^* K_X$ in $SO(2m+1)$ case, $L \otimes \pi^* K_X^{-\frac{1}{2}} \in \mathrm{Prym}(S, S/\sigma)$

Summary :

					C_n	B_n	D_n
	G	$GL(n,\mathbb{C})$	$SL(n,\mathbb{C})$	$PSL(n,\mathbb{C})$	$Sp(2n,\mathbb{C})$ char $= x^{2n}+\cdots+a_{2n}$	$SO(2n+1,\mathbb{C})$ char $= x(x^{2n}+\cdots+a_{2n})$	$SO(2n,\mathbb{C})$
Inv. polynomial		$a_1,\cdots,a_n$	$a_2,\cdots,a_n$		$a_2,\cdots,a_{2n}$	$a_2,\cdots,a_{2n}$.	$a_2,\cdots,a_{2n-2},p_n$
Hitchin fiber		bij. to $\mathrm{Pic}(X)$, conn.comp $\mathrm{Jac}(X)$.	bij. to Subset of $\mathrm{Jac}(S)$ w. $\pi_*L=E$ $\Lambda^n\pi_*L=\mathcal{O}_X$ trivial. i.e. $\mathrm{Prym}(S,X)$'s coset.		bij. to $\mathrm{Prym}(S,S/\sigma)$	bij to $\mathrm{Prym}(S,S/\sigma)$	bij to $\mathrm{Prym}(\hat{S}/\hat{S}/\sigma)$
		$R=\{\pm(e_i-e_j)\mid 1\le i<j\le n\}$ $W=S_n$			$R=\{\pm e_i\pm e_j\}\cup\{\pm 2e_i\}$ $W=$ permutation & sign changes.	$R=\{\pm e_i\pm e_j\}\cup\{\pm e_i\}$ Same.	$R=\{\pm e_i\pm e_j\}$ $W=$ permutation & even # of sign change in coordinates.
						Mod of stab. bundle ~~Hitchin fiber~~ has 2 conn. components. corr. to $w_2\in H^2(\mathbb{Z}_2)$ S reducible.	S singular, need normalize.

G simple cplx Lie grp. ${}^LG$ Langlands dual.
$\cup\!|$ $\quad$ ${}^L\cup\!|$
T $\quad$ ${}^LT$ max. tori (complex)

W Weyl group.
then $\quad \text{root}_{\mathfrak{g}} \subseteq \text{char}_G \subseteq \text{weight}_{\mathfrak{g}} \subseteq \mathfrak{t}^\vee$
$\quad \text{coroot}_{\mathfrak{g}} \subseteq \text{cochar}_G \subseteq \text{coweight}_{\mathfrak{g}} \subseteq \mathfrak{t}$ Note: By def. $\text{coroot}_{\mathfrak{g}}$ is $\text{weight}_{\mathfrak{g}^\vee}$!

Langlands: $\quad \text{root}_{{}^L\mathfrak{g}} \subseteq \text{char}_{{}^LG} \subseteq \text{weight}_{{}^L\mathfrak{g}} \subseteq {}^L\mathfrak{t}^\vee$
$\quad \| \quad \| \quad \| \quad \|$ swapping short/long roots.
$\quad \text{coroot}_{\mathfrak{g}} \subseteq \text{cochar}_G \subseteq \text{coweight}_{\mathfrak{g}} \subseteq \mathfrak{t}$

Higgs := space of semisstab. K_C-val. G-Higgs bundles on C. ← $g>0$ smooth cplx cpct curve.
i.e. (P,φ), P G-bundle on C
$ad(P) = P \times_{Ad} \text{Lie}\, G$ VB on C.
$\varphi \in \Gamma(ad(P) \otimes K_C)$ ← section: $\Gamma(K_C) \otimes \mathbb{C}[\mathfrak{g}]^G$ or atleast locally?
P is s.s. if $\forall$ parabolic $H \subseteq G$, Q a H-subbundle of P. $\deg Q \leq 0$?
(P,φ) is s.s. if $\quad \deg(Q, \varphi|_{adQ}) \leq 0$

Hitchin base? ← space of K_C-val cameral covers (same data as spectral cover) $\quad$ ($\mathfrak{t}/_W = \mathbb{C}^{rk}$ literally is vector space!)
$B := H^0(C, K_C \otimes \mathfrak{t}/W)$. i.e. W-inv. sections of $K_C \otimes \mathfrak{t}$
i.e. K_C-val. invariant polynomials e.g. tr. det. higher trace.

Def: A cameral cover of C is $\tilde{C}$ w. $p: \tilde{C} \to C$ morphism finite & flat. $\mathbb{C}[\mathfrak{g}]^G = \mathbb{C}[\mathfrak{h}]^W$
w. W acts on fibers of p
s.t. $p_*(\mathcal{O}_{\tilde{C}}) \cong \mathcal{O}_C \otimes \mathbb{C}[W]$ locally as $\mathcal{O}_C \otimes \mathbb{C}[W]$-mod. $\quad$ it is a colimit of spectral covers.
group alg $\quad$ use Chevalley Thm
is dimension correct?
& locally $\tilde{C}$ is pull back of $\mathfrak{t} \to \mathfrak{t}/W$ (at ramified pts?) $\quad$ just saying it embeds into some $\mathfrak{t}$-bundle's total space W-equivly
A L-val. cameral cover is element in $H^0(C, L \otimes \mathfrak{t}/W)$ (i.e. embeds into total space of L?)

Def: A cameral cover has simple Galois ramification if $\quad$ (a true cover?)
all ramif. pts $x \in D$ has ramif. index 1. $\quad$ what is ram. pt then?
the ramif. divisor
In this case, ram. divisor is $\amalg$ of subdivisors labeled by roots of G. $\quad$ there're $|W|$ many preimages? yes.

Δ Discriminant := subset of B where the cameral cover is not simp. Gal. ramified.
Hitchin fibers $h^{-1}(b)$ are identified w. generalized Prym-Variety Pr_b assoc. to $\tilde{C}_b \to C$

Hitchin fibration: $\text{Higgs} \xrightarrow{h} B$ (invariant polynomials applied to Higgs fields)

$\pi_0(\text{Higgs}) = H^2(C, \pi_1(G)) = \pi_1(G)$ = obstruction for trivializing the bundle P?

Higgs_0 = Higgs bundles induced from $\tilde{G}$-Higgs bundles
↑ fin dim?

Over Higgs_0, $h : \text{Higgs}_0 \to B$ has a canonical section called Hitchin section. ↙ all order?

$(=GL_1(\mathbb{C}))$
$\mathrm{Hom}(\pi_1, \mathbb{C}^\times)$
$\mathbb{C}^\times)$
"is cotangent of"
Jac

↙ structure sheaf ↙ sheaf of alg. differential operators

Geom. Langlands Conj: $D_{coh}(Loc, \mathcal{O}) \xrightarrow{\cong} D_{coh}({}^L Bun, D)$ ← i.e. D-Mods

moduli stack of alg. G-local system i.e. (P, ∇) on C
P prin. G-bundle, ∇ flat alg connex

moduli stack of ${}^L G$ bundle on C.

$D coh(X, A) = D(A\text{-alg module w. cohomology coherent})$

un D)
of T^*Bun)
nds is
se of
symmetry

The eq of Cat. commutes w. endofunctors $W^{\mu,x} : DCoh(Loc, \mathcal{O}) \longrightarrow DCoh(Loc, \mathcal{O})$

aka Wilson loop operator; tensorization. $F \mapsto F \otimes \rho^\mu(V|_{Loc \times \{x\}})$

aka t'Hooft loop operator; ${}^L H^{\mu,x} : DCoh({}^L Bun, D) \longrightarrow DCoh({}^L Bun, D)$
Hecke functor. $M \longmapsto q_!^{\mu,x}((p^{\mu,x})^* M \otimes {}^L I^{\mu,x})$

A Fourier-Mukai transform!

$x \in C$ closed pt
$\mu \in \text{cochar}^+({}^L G) = \text{char}^+(G)$ dominant char of G.

$\rho(-)$ means $- \times_\rho V$

ρ^μ irrep of G w. highest weight μ. $V \to Loc \times C$ universal local system (forget connex. just take bundle)

i.e. $V|_{Loc \times \{x\}}$ is assign all local system the fiber of P at $x \in C$, forming G-Bundle$/Loc$.

$\rho^\mu(V|_{Loc\{x\}})$ is the VB assc. to G-Bundle via ρ^μ.

${}^L Hecke^{\mu,x} = \{(P, P', \beta) \mid P, P'\ {}^L G\text{-Bundle}, \beta \text{ isom } P|_{C-\{x\}} \cong P'|_{C-\{x\}}$

s.t. for ρ irrep of ${}^L G$ w. highest weight λ
$\rho(\beta) : \rho(P) \hookrightarrow \rho(P')(\langle \mu, \lambda \rangle \cdot x)$ ← twist by divisor?
(cochar: μ, char: λ)

$$\begin{tikzcd} & {}^L Hecke^{\mu,x} \arrow[dl] \arrow[d, "p^{\mu,x}"'] \arrow[dr, "q^{\mu,x}"] \arrow[drr] & & \\ P & {}^L Bun & {}^L Bun & P' \end{tikzcd}$$

Both projections are proper representable locally trivial fiberbundles

${}^L I^{\mu,x} \in DCoh({}^L Bun, D)$ is $j_{!*}$? $\mathbb{C}[\dim({}^L Hecke^{\mu,x})]$ of trivial local syst. on smooth part of ${}^L Hecke^{\mu,x}$, pushed forward via inclusion
↑ shift?

Classical limit: ① Loc as quantization of Higgs:

$\exists$ family $H \to \mathbb{C}$. Loc = fiber over some pt in $\mathbb{C}^\times$
Higgs = fiber of 0

$H = \{(P, \nabla, z) \mid P \xrightarrow{\pi} C$ princ. G bundle on C, $z \in \mathbb{C}$, ∇ splitting

$0 \to \mathrm{ad}(P) \to \mathcal{E}(P) \rightleftarrows_{\nabla} T_C \to 0$ s.t. $\sigma \circ \nabla = z \cdot \mathrm{id}\, T_C \}$

Atiyah alg. $\pi_*(TP)^G$ i.e. send tangent vector to horizontal lift.

$H \to \mathbb{C}$, $\quad z$

connex holomorphic, so locally it's $\partial + A$ w. $A \in A^{1,0}(C)$. $(\partial + A)^2 = \partial A + A \wedge A \in A^{2,0}(C) = 0$

$H|_{\mathbb{C}^\times} \cong \mathrm{Loc} \times \mathbb{C}^\times$ ($z=1$, it's G-bundle w. connex, why flat?)

$H|_{\{0\}} = \{(P, \nabla) \mid P$ G-bundle on C, ∇ splitting $TC \to \mathrm{ad}(P)$ i.e. section of $K_C \otimes \mathrm{ad} P)\}$

② D on ${}^L\mathrm{Bun}$ as deformation of $\mathrm{Sym}^* T$ on ${}^L\mathrm{Bun}$.

R sheaf on ${}^L\mathrm{Bun} \times \mathbb{C}$ (Rees sheaf)

given by subsheaf of $p_1^* D$ pullback of D on ${}^L\mathrm{Bun}$

section of $p_1^* D$: $\sum z^i P_i$. P_i Diff operator on ${}^L\mathrm{Bun}$ → what're they like?

i.e. a powerseries w. coef P_i, z is coord. on $\mathbb{C}$

section of $R = \sum z^i P_i$, P_i Diff operator of degree $\le i$

$R|_{{}^L\mathrm{Bun} \times \{z\}} = D$ as all diff operator are $\sum P_i$ ~?

$R|_{{}^L\mathrm{Bun} \times \{0\}} = \mathrm{Sym}^* T$ as sections are Diff operators of degree 0

How to interpret?

$R := D$ is graded,
$I := D^+$, $\mathrm{gr}_I R = \oplus I^n / I^{n+1}$
$R := \bigoplus_{n=0}^{\infty} I^n t^n \subseteq D[t]$
$R/tR = $ fiber@0 $= \oplus I^n / I^{n+1}$
$R/(t-1)R = D = $ fiber@1
$\sum i^n t^n \mapsto \sum i^n$

Classical limit of Geom Langlands: $D\mathrm{Coh}(\mathrm{Higgs}, \mathcal{O}) \xrightarrow[\text{equiv. of Cat}]{\simeq} D\mathrm{Coh}({}^L\mathrm{Higgs}, \mathcal{O})$

Tensorization functor: $D\mathrm{Coh}(\mathrm{Higgs}, \mathcal{O}) \to D\mathrm{Coh}(\mathrm{Higgs}, \mathcal{O})$

$W^{\mu,x}: \mathcal{F} \longmapsto \mathcal{F} \otimes \rho^\mu(V|_{\mathrm{Higgs} \times \{x\}})$

$V \to \mathrm{Higgs} \times C$ the G-bundle behind the universal Higgs bundle.

Hecke functor:

$$\begin{tikzcd} \underbrace{D\mathrm{Coh}({}^L\mathrm{Higgs}, \mathcal{O})}_{=\mathrm{Tot}(T^\vee\, {}^L\mathrm{Bun})} \arrow[r, hook] \arrow[d] & D\mathrm{Higgs}({}^L\mathrm{Bun}) \arrow[d] & {}^L\mathrm{Hecke}^{\mu,x} \times C \arrow[l, "p^{\mu,x}"'] \arrow[dl, "q^{\mu,x}"] \\ D\mathrm{Coh}({}^L\mathrm{Higgs}, \mathcal{O}) \arrow[r, hook] & D\mathrm{Higgs}({}^L\mathrm{Bun}) & \end{tikzcd}$$

↪?

given by FM-transform using ${}^L J^{\mu,x}$ the assoc. graded of ${}^L I^{\mu,x}$

(Saito: ${}^L I^{\mu,x}$ has structure of mixed Hodge module?, Hodge filtration induces grading) sheaves on ${}^L\mathrm{Hecke}^{\mu,X} \times \mathbb{C}$: ${}^L\mathbb{I}^{\mu,X}$: ${}^L\mathbb{I}^{\mu,X}\big|_{{}^L\mathrm{Hecke}^{\mu,X}\times\{1\}} \cong {}^L I^{\mu,X}$ as D-module.

i.e. D

${}^L\mathbb{I}^{\mu,X}$ mod over Rees sheaf ? of sheaf of diff. op. on ${}^L\mathrm{Hecke}^{\mu,X}$ for each z?

${}^L\mathbb{I}^{\mu,X}/z\cdot{}^L\mathbb{I}^{\mu,X}\big|_{{}^L\mathrm{Hecke}^{\mu,X}\times\{0\}} =: {}^L J^{\mu,X}$ mod over

$gr D = Sym^* T$, is Higgs sheaf on ${}^L\mathrm{Hecke}^{\mu,X}$

i.e. $\{(\mathcal{E},\varphi)$, $\mathcal{E}$ QCoh,

$\varphi : \mathcal{E} \to \mathcal{E}\otimes\Omega^1$ O-linear map.

$\varphi\wedge\varphi = 0$ $\}$ "flat"?

Thm (Duality for Hitchin Prym)

G simple, C as before,

unique up to scalar.

then $\exists$ Isom $I^{base} : B \to {}^L B$,

① I^{base} send Δ to ${}^L\Delta$

② induce $I^{cam} : \tilde{\mathcal{C}} \xrightarrow{\cong} {}^L\tilde{\mathcal{C}}$ universal cameral covers.

③ $h^{-1}(b)$ is dual to ${}^Lh^{-1}(I^{base}(b))$ as polarized Ab. varieties. just the conn. component?

(①, ②: Choice of inner product gives isom $t \to t^\vee = {}^L t$, W-invariant, taking reflection hyperplanes (Walls of Weil chamber) to ref. hyperplanes.

corr. to diag-matrices w. "multiple" eigenvalues?

thus I^{base} preserve Δ.

$\tilde{\mathcal{C}}$ is

$$\begin{tikzcd}
\tilde{\mathcal{C}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathrm{Tot}(K_C\otimes t) \arrow[d] \\
H^0(C,(K_C\otimes t)/W)\times C \arrow[r] & \mathrm{Tot}((K_C\otimes t)/W)
\end{tikzcd}$$

and κ induces $id_{K_C}\otimes\kappa : K_C\otimes t \to K_C\otimes{}^L t$

so induces $\tilde{\mathcal{C}} \xrightarrow{\cong} {}^L\tilde{\mathcal{C}}$

③ : $\Lambda := \mathrm{cochar}(G) = \mathrm{Hom}(\mathbb{C}^\times, T)$, $p : \tilde{C} \to C$ a cameral cover corr. to b.

3 sheaves on C : $\overline{T} := p_*(\Lambda\otimes\mathcal{O}^\times_{\tilde{C}})^W$. W act on both?

$T = \{ t\in\overline{T} \mid \alpha$ root of $\mathfrak{g}$, $\alpha(t)\in\mathcal{O}^\times_{\tilde{C}}$. $\alpha(t)|_{D^\alpha} = 1$ is required

$D^\alpha \subseteq \tilde{C}$ fixed divisor for $\rho_\alpha\in W$ the reflection, acting on $\tilde{C}\downarrow C$ $\}$

what is conn. comp. of sheaf?

$T^0 =$ conn. component of T

$\overline{T}_{\mathbb{R}}, T_{\mathbb{R}}, T^0_{\mathbb{R}}$: replace $\mathcal{O}^\times_{\tilde{C}}$ by $\underline{S}^1$, i.e. $\mathbb{C}^\times$ by S^1

Fiber at unramified pt : $\Lambda\otimes\mathbb{R}/\mathbb{Z}$ (for $\overline{T}_{\mathbb{R}}$ At each pt in fiber of $c\in C$, it's $\Lambda\otimes S^1$, taking W invariant $\Rightarrow$ only one copy of $\Lambda\otimes S^1$)

(for $T_{\mathbb{R}}$, condition vacuous