say rk r, deg d

Hitchin: A = (∞-dim) space of holom. structures on VB V i.e. $\Omega^{0,1}(M, \mathrm{End} V)$

$T^*A^s = A^s \times \Omega^0(X, \mathrm{ad} P \otimes K)$
$= \{(\dot{A}, \dot{\Phi}) \mid \dot{\Phi} \in \Omega^{1,0}(\mathrm{Ad} P) = \Omega^0(\mathrm{Ad} P \otimes K)$
$\dot{A} \in \Omega^{0,1}(\mathrm{Ad} P)\}$

A^s = stable structures. an open set in A

$\tilde{G} = \mathrm{Aut}(V)$

$M_X(r,d) = A^s/\tilde{G}$ = moduli of stable rk r VB

Canonical 1-form $\theta(\dot{A}, \dot{\Phi})$ is $\int_X B(\dot{A} \wedge \dot{\Phi})$ i.e. apply killing form. End E is $\mathbb{C}$ ①

e.g. $G = \mathbb{C}^\times$, $M_X(1,d) = \mathrm{Jac}_d(X)$. $T^* M_X(1,0) = \mathrm{Jac}(X) \times H^0(X; K_X)$

i.e. T^* of $\mathrm{Jac}(X)$ ($= \mathcal{O}_{\mathrm{Jac} X}$)

$p_1, \dots, p_k$ basis of ring of inv. polynomial on $\mathfrak{g}$, get (degree $d_1 \cdots d_k$)

$$h: H^0(X; \mathrm{ad} P \otimes K) \to \bigoplus_{i=1}^k H^0(X, K^{d_i})$$

$$\dim H^0(X; K^{d_i}) = \deg K^{d_i} + (1-g) + \dim H^1(X; K^{1-d_i}) = (2g-2)d_i + (1-g) + \begin{cases} 0 & d_i > 1 \\ 1 & d_i = 1 \end{cases}$$

Gln case also work!

$d_i = 0$ thrown away?

G semisimple $\Rightarrow$ no d_i is 1 so $= (2d_i - 1)(g-1)$

$$\dim \oplus H^0(X, K^{d_i}) = \sum_{i=1}^k (2d_i - 1) \cdot (g-1)$$

Kostant: $\dim G = \sum_{i=1}^k (2d_i - 1)$ call it n.

They descend from maps $A^s \times \Omega^0(M; \mathrm{Ad} P \otimes K) \to \oplus \Omega^0(M; K^{d_i})$ which Poisson commute.

OTOH, $\dim H^0(X; \mathrm{ad} P \otimes K) = \dim G\,(g-1)$ if G semisimple (so $\dim H^1(\mathrm{Ad} P \otimes K) = 0$)

so (p_i) gives map $T^* M_X(r,d) \to \bigoplus_{i=1}^k H^0(M; K^{d_i})$ which are n Poisson-commute. functions on $2n$-dim symplectic mfd

WLOG 0

($T^* M_X$ is cplx symplectic, and functions actually holomorphic)

Case 1: $G = Gl(n, \mathbb{C})$, for $a_i \in H^0(X, K^i)$,

preimage is $\{(V, \Phi) \mid \det(x - \Phi) = x^n + \cdots + a_n\}$

① Note: Lie $\tilde{G} = \Omega^0(X, \mathrm{ad} P)$

Action fields: $\psi \in \mathrm{Lie}\, \tilde{G} \mapsto \bar{\partial}_E \psi \in \Omega^{0,1}(\mathrm{Ad} P)$

Moment map: $\mu(E, \Phi) = 0$ iff $\bar{\partial}_E \psi = 0$ i.e. $\dot{A}$

$Z := \mathrm{Tot}(K_X)$ $p^* K_X$ has tautological section λ

$\downarrow p$

X

$$\lambda^n + \cdots + a_n \in \Gamma(p^* K_X^{\otimes n})$$

Zero $(\lambda^n + \cdots + a_n)$ gives divisor on Z that descent to divisor on $\mathbb{P}(K_X \oplus \mathcal{O})$ i.e. the spectral curve S (see BNR, λ is their x, 1 is their y)

$= \mathbb{P}(\mathcal{O} \oplus T_X)$?

Adjunction in $\mathbb{P}(K_X \oplus \mathcal{O})$: $2g(S) - 2 = S \cdot S + K_{\mathbb{P}(K_X \oplus \mathcal{O})} \cdot S$

$S = nX$, $\mathrm{zero}(x) = \mathbb{P}(\mathcal{O}) \subseteq \mathbb{P}(K \oplus \mathcal{O})$ i.e. 0-section of K?

$H^*(\mathbb{P}(K_X \oplus \mathcal{O})) = H^*(X)[\xi]/(\xi^2 - c_1(K)\xi)$

$\xi|_{\text{Fiber}} = c_1(\mathcal{O}(1))|_{\text{Fiber}}$

$K_{\mathbb{P}(K_X \oplus \mathcal{O})} = \mathcal{O}(-2) \otimes p^* K_X$, $\mathcal{O}(1)$ is the divisor class of y

$X \cdot X = K_X \cdot K_X = -\chi(X) = 2g-2$.

$x \cdot y = 0$, $x \cdot p^* K_X \overset{?}{=} 0$

so $S \cdot S = 2n^2(g-1)$

so $K \cdot S = 0$ and $g(S) = n^2(g-1) + 1$

$\lambda|_S \in \Gamma(S, p^* K_X)$ is eigenvalue of $\Phi \in \Gamma(S, \mathrm{End}\, V \otimes p^* K_X)$

Note: S not singular!

$\ker(\lambda Id - \Phi) \subseteq p^* V$ is generically a LB except when eigenval. repeats, and $\exists!$ LB $L \subseteq \ker(\lambda Id - \Phi)$ giving pt in $Jac(S)$

$\mathcal{O}_S / I_{p^{-1}(x)} \otimes_{\mathcal{O}_S} L = \mathcal{O}_{X,\{x\}}/m_x \otimes_{\mathcal{O}_X} L$? as $\mathcal{O}_X$-mod. ?

For $L \in Jac(S)$, $(p_* L)_x = \mathcal{O}_S(L)/I_{p^{-1}(x)}$ ← ideal sheaf of $\pi^{-1}(x)$

x not branching pt : $= \bigoplus_{y \in p^{-1}(x)} L_y$

is branching pt : $= \bigoplus_{y \in p^{-1}(x)} J^{k(y)}(L)_y$ $\quad k(y) = $ ram. index at y

"V is essentially $p_* L$"

$J^k(L)_y \to L_y \to 0$ gives $0 \to \bigoplus_{y \in \pi^{-1}(x)} L_y^* \to (p_* L)_x^* \to \bigoplus J^{k(y)}(L)_y^* / L_y^* \to 0$

which gives $0 \to$ some locally free sheaf $\to \mathcal{O}(p_* L)^* \to S \to 0$

i.e. the $\mathcal{O}(V^*)$ ← what's this? holom. section of V^* $\quad$ ↑ supported at branch pts (Jet skyscrapers)

The locally free sheaf gives the VB, however may not be abelian.

Case 2. $G = Sp(2m, \mathbb{C})$, $\quad V$ rk $2m$ bundle w. symplectic form. ω

$\Phi \in H^0(X, End\, V \otimes K)$ s.t. $\omega(\Phi v, w) = \omega(v, \Phi w)$ $\quad$ i.e. in $Sp(2m,\mathbb{C})$'s Lie alg. $-JX^TJ = X$

eigenvals appear in pair, $\det(x\mathbb{1} - \Phi) = x^{2m} + a_2 x^{2m-2} + \cdots + a_{2m}$

Basis of inv. polynomial: $a_2 \cdots a_{2m}$

S curve of genus

G simple cplx Lie grp. ${}^LG$ Langlands dual,
$\overset{\cup\!\!|}{T}$ $\quad {}^L\overset{\cup\!\!|}{T}$ max. tori (complex)

W Weyl group.
then $\quad \mathrm{root}_{\mathfrak{g}} \subseteq \mathrm{char}_G \subseteq \mathrm{weight}_{\mathfrak{g}} \subseteq \mathfrak{t}^\vee$
$\quad \mathrm{coroot}_{\mathfrak{g}} \subseteq \mathrm{cochar}_G \subseteq \mathrm{coweight}_{\mathfrak{g}} \subseteq \mathfrak{t}$ Note: By def. $\mathrm{coroot}_{\mathfrak{g}}$ is $\mathrm{weight}_{\mathfrak{g}}^\vee$!

Langlands: $\quad \begin{matrix} \mathrm{root}_{{}^L\mathfrak{g}} & \subseteq \mathrm{char}_{{}^LG} & \subseteq \mathrm{weight}_{{}^L\mathfrak{g}} & \subseteq {}^L\mathfrak{t}^\vee \\ \| & \| & \| & \| \\ \mathrm{coroot}_{\mathfrak{g}} & \subseteq \mathrm{cochar}_G & \subseteq \mathrm{coweight}_{\mathfrak{g}} & \subseteq \mathfrak{t} \end{matrix}$ swapping short/long roots.

Higgs := space of semisstab. K_C-val. G-Higgs bundles on C. ← $g>0$ smooth cplx cpct curve.
i.e. (P, φ), P G-bundle on C
$\mathrm{ad}(P) = P \times_{\mathrm{Ad}} \mathrm{Lie}\, G$ VB on C.
$\varphi \in \Gamma(\mathrm{ad}(P) \otimes K_C)$ ← section: $\Gamma(K_C) \otimes \mathbb{C}[\mathfrak{g}]^G$ or atlease locally?

P is s.s. if $\forall$ parabolic $H \subseteq G$, Q a H-subbundle of P, $\deg Q \le 0$?
(P, φ) is s.s. if - - - - $\deg(Q, \varphi|_{\mathrm{ad}Q}) \le 0$

(Hitchin base) ← space of K_C-val cameral covers
$B := H^0(C, K_C \otimes \mathfrak{t}/W)$. i.e. W-inv. sections of $K_C \otimes \mathfrak{t}$
i.e. K_C-val. invariant polynomials e.g. tr. det. higher trace.

Def: A cameral cover of C is $\tilde{C}$ w. $p: \tilde{C} \to C$ morphism finite & flat. $\mathbb{C}[\mathfrak{g}]^G = \mathbb{C}[\mathfrak{h}]^W$
w. W acts on fibers of p
s.t. $p_*(\mathcal{O}_{\tilde{C}}) \cong \mathcal{O}_C \otimes \mathbb{C}[W]$ locally as $\mathcal{O}_C \otimes \mathbb{C}[W]$-mod
↑ group alg
it is a colimit of spectral covers.
use Chevalley Thm
is dimension correct?
& locally $\tilde{C}$ is pull back of $\mathfrak{t} \to \mathfrak{t}/W$ (at ramified pts?) just saying it embeds into some $\mathfrak{t}$-bundle's total space W-equivly.
A L-val. cameral cover is element in $H^0(C, L \otimes \mathfrak{t}/W)$ (i.e. embeds into total space of L?)

Def: A cameral cover has simple Galois ramification if all ramif. pts $x \in D$ has ramif. index 1. (a true cover?) what is ram. pt then?
↑ the ramif. divisor
In this case, ram. divisor is $\amalg$ of subdivisors labeled by roots of $\mathfrak{g}$. there're $|W|$ many preimages? yes.

Δ Discriminant := subset of B where the cameral cover is not simp. Gal. ramified
Hitchin fibers $h^{-1}(b)$ are identified w. generalized Prym-Variety Pr_b assoc. to $\tilde{C}_b \to C$

Hitchin fibration: $\text{Higgs} \xrightarrow{h} B$ (invariant polynomials applied to Higgs fields)

$\pi_0(\text{Higgs}) = H^2(C, \pi_1(G)) = \pi_1(G)$ = obstruction for trivializing the bundle P?

Higgs_0 = Higgs bundles induced from $\check{G}$-Higgs bundles
fin dim?

Over Higgs_0, $h : \text{Higgs}_0 \to B$ has a canonical section called Hitchin section. all order?

structure sheaf — sheaf of alg. differential operators

Geom. Langlands Conj: $D_{coh}(\text{Loc}, \mathcal{O}) \cong D_{coh}({}^L\text{Bun}, D)$ ← i.e. D-Mods

$D\,coh(X, A)$ $\Rightarrow D$(A-alg module w cohomology coherent)

moduli stack of alg. G-local system i.e. (P, ∇) on C, P prin. G-bundle, ∇ flat alg connex

moduli stack of ${}^L G$ bundle on C.

The eq of Cat. commutes w. endofunctors $W^{\mu,x} : D\,Coh(\text{Loc}, \mathcal{O}) \longrightarrow D\,Coh(\text{Loc}, \mathcal{O})$

aka Wilson loop operator — tensorization $\quad F \mapsto F \otimes \rho^{\mu}(V|_{\text{Loc} \times \{x\}})$

aka t'Hooft loop operator — ${}^L H^{\mu,x} : D\,Coh({}^L\text{Bun}, D) \longrightarrow D\,Coh({}^L\text{Bun}, D)$

Hecke functor $\quad \mathcal{M} \longmapsto q_!^{\mu,x}\left((p^{\mu,x})^* \mathcal{M} \otimes {}^L I^{\mu,x}\right)$

A Fourier-Mukai transform!

$x \in C$ closed pt

$\mu \in \text{cochar}^+({}^L G) = \text{char}^+(G)$ dominant char of G.

$\rho(-)$ means $- \times_P V$

ρ^{μ} irrep of G w. highest weight μ. $V \to \text{Loc} \times C$ universal local system (forget connex. just take bundle)

i.e. $V|_{\text{Loc} \times \{x\}}$ is assign all local system the fiber of P at $x \in C$, forming G-Bundle/Loc.

$\rho^{\mu}(V|_{\text{Loc}\{x\}})$ is the VB assoc. to G-Bundle via ρ^{μ}.

${}^L\text{Hecke}^{\mu,x} = \{(P, P', \beta) \mid P, P'$ ${}^L G$-Bundle, β isom $P|_{C-\{x\}} \cong P'|_{C-\{x\}}$

s.t. for ρ irrep of ${}^L G$ w. highest weight λ

$\rho(\beta) : \rho(P) \hookrightarrow \rho(P')(\langle \mu, \lambda \rangle \cdot x)$ ← twist by divisor?

(μ: cochar, λ: char)

$$\begin{tikzcd}
 & {}^L\text{Hecke}^{\mu,x} \arrow[dl, "p^{\mu,x}"'] \arrow[dr, "q^{\mu,x}"] & \\
{}^L\text{Bun} & & {}^L\text{Bun}
\end{tikzcd}
\qquad P \leftarrow (P,P',\beta) \mapsto P'$$

Both projections are proper representable locally trivial fiberbundles

?

${}^L I^{\mu,x} \in D\,Coh({}^L\text{Bun}, D)$ is $j_{!*}\,\mathbb{C}[\dim({}^L\text{Hecke}^{\mu,x})]$ of trivial local syst. on smooth part of ${}^L\text{Hecke}^{\mu,x}$, pushed forward via inclusion

shift?

Classical limit: ① Loc as quantization of Higgs:

$\exists$ family $\begin{matrix} H \\ \downarrow \\ \mathbb{C} \end{matrix}$. Loc = fiber over some pt in $\mathbb{C}^\times$

Higgs = fiber of 0

$H = \{(P, \nabla, z) \mid P \xrightarrow{\pi} C$ princ. G bundle on C, $z \in \mathbb{C}$, ∇ splitting

$0 \to \mathrm{ad}(P) \to \mathcal{E}(P) \to T_C \to 0$ s.t. $\sigma \circ \nabla = z \cdot \mathrm{id}_{TC}$ $\}$.

$\mathcal{E}(P)$: Atiyah alg. $\pi_*(TP)^G$; ∇: i.e. send tangent vector to horizontal lift.

$H \to \mathbb{C}$, $(P, \nabla, z) \mapsto z$

$H|_{\mathbb{C}^\times} \cong Loc \times \mathbb{C}^\times$ ($z=1$. it's G-bundle w. connex, why flat?) connex holomorphic, so locally it's $\partial + A$ w. $A \in A^{1,0}(C)$. $(\partial+A)^2 = \partial A + A \wedge A \in A^{2,0}(C) = 0$

$H|_{\{0\}} = \{(P, \nabla) \mid P$ G-bundle on C, ∇ splitting $TC \to \mathrm{ad}(P)$ i.e. section of $K_C \otimes \mathrm{ad}(P)\}$

② D on ${}^L\mathrm{Bun}$ as deformation of $\mathrm{Sym}^* T$ on ${}^L\mathrm{Bun}$.

R sheaf on ${}^L\mathrm{Bun} \times \mathbb{C}$ (Rees sheaf)

given by subsheaf of $p_1^* D$ pullback of D on ${}^L\mathrm{Bun}$

section of $p_1^* D$: $\sum z^i P_i$. P_i Diff operator on ${}^L\mathrm{Bun}$ → what're they like?

i.e. a powerseries w. coef P_i, z is coord. on $\mathbb{C}$

section of $R = \sum z^i P_i$, P_i Diff operator of degree $\le i$

$R|_{{}^L\mathrm{Bun} \times \{z\}} = D$ as all diff operator are $\sum P_i$?

$R|_{{}^L\mathrm{Bun} \times \{0\}} = \mathrm{Sym}^* T$ as sections are Diff operators of degree 0

How to interpret?

Classical limit of Geom Langlands: $D\mathrm{Coh}(\mathrm{Higgs}, 0) \xrightarrow{\simeq} D\mathrm{Coh}({}^L\mathrm{Higgs}, 0)$ equiv. of Cat

Tensorization functor: $D\mathrm{Coh}(\mathrm{Higgs}, 0) \to D\mathrm{Coh}(\mathrm{Higgs}, 0)$

$W^{\mu, x}: F \longmapsto F \otimes \rho^\mu(V|_{\mathrm{Higgs} \times \{x\}})$

$V \to \mathrm{Higgs} \times C$ the G-bundle behind the universal Higgs bundle.

Hecke functor: $D\mathrm{Coh}({}^L\mathrm{Higgs}, 0) \hookrightarrow D\mathrm{Higgs}({}^L\mathrm{Bun})$ $\xleftarrow{p^{\mu,x}}$ ${}^L\mathrm{Hecke}^{\mu,x} \times C$

$= \mathrm{Tot}(T^\vee\, {}^L\mathrm{Bun})$ $\qquad$?

$$\begin{array}{ccc} D\mathrm{Coh}({}^L\mathrm{Higgs}, 0) & \hookrightarrow & D\mathrm{Higgs}({}^L\mathrm{Bun}) \\ \downarrow & & \downarrow \\ D\mathrm{Coh}({}^L\mathrm{Higgs}, 0) & \hookrightarrow & D\mathrm{Higgs}({}^L\mathrm{Bun}) \end{array} \quad {}^L\mathrm{Hecke}^{\mu,x} \times C \xrightarrow{q^{\mu,x}} D\mathrm{Higgs}({}^L\mathrm{Bun})$$

given by FM-transform using ${}^L J^{\mu,x}$ the assoc. graded of ${}^L I^{\mu,x}$

(Saito: ${}^L I^{\mu,x}$ has structure of mixed Hodge module ?, Hodge filtration induces grading) sheaves on ${}^L\mathrm{Hecke}^{\mu,x} \times \mathbb{C}$: ${}^L\mathbb{I}^{\mu,x}$: ${}^L\mathbb{I}^{\mu,x}\big|_{{}^L\mathrm{Hecke}^{\mu,x}\times\{\hbar\}} \cong {}^L I^{\mu,x}$ as D-module.

${}^L\mathbb{I}^{\mu,x}$ mod over Rees sheaf ? of sheaf of diff. op. (i.e. D) on ${}^L\mathrm{Hecke}^{\mu,x}$ for each z?

${}^L\mathbb{I}^{\mu,x}/z\cdot{}^L\mathbb{I}^{\mu,x}\big|_{{}^L\mathrm{Hecke}^{\mu,x}\times\{0\}} =: {}^L J^{\mu,x}$ mod over $\mathrm{gr}\,D = \mathrm{Sym}^*T$, is Higgs sheaf on ${}^L\mathrm{Hecke}^{\mu,x}$

i.e. $\{(\mathcal{E},\varphi)$, $\mathcal{E}$ QCoh, $\varphi: \mathcal{E}\to\mathcal{E}\otimes\Omega^1$ O-linear map. $\varphi\wedge\varphi=0\}$ "flat"?

Thm (Duality for Hitchin Prym)

G simple, C as before,

then $\exists$ Isom $I^{base}: B \to {}^L B$, unique up to scalar

① I^{base} send Δ to ${}^L\Delta$

② induce $I^{cam}: \tilde{\mathcal{C}} \xrightarrow{\cong} {}^L\tilde{\mathcal{C}}$ universal cameral covers

③ $h^{-1}(b)$ is dual to ${}^Lh^{-1}(I^{base}(b))$ as polarized Ab. varieties. just the conn. component?

(①, ②: Choice of inner product gives isom $t \to t^\vee = {}^Lt$, W-invariant, taking reflection hyperplanes (walls of Weil chamber) to ref. hyperplanes.

corr. to diag-matrices w. "multiple eigenvalues"?

thus I^{base} preserve Δ.

$\tilde{\mathcal{C}}$ is

$$\begin{tikzcd} \tilde{\mathcal{C}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Tot}(K_C\otimes t) \arrow[d] \\ H^0(C,(K_C\otimes t)/W)\times C \arrow[r] & \mathrm{Tot}((K_C\otimes t)/W) \end{tikzcd}$$

and κ induces $id_{K_C}\otimes\kappa: K_C\otimes t \to K_C\otimes{}^Lt$

so induces $\tilde{\mathcal{C}} \xrightarrow{\cong} {}^L\tilde{\mathcal{C}}$

③: $\Lambda := \mathrm{cochar}(G) = \mathrm{Hom}(\mathbb{C}^\times, T)$, $p: \tilde{C}\to C$, a cameral cover corr. to b.

3 sheaves on C: $\overline{T} := p_*(\Lambda\otimes\mathcal{O}^\times_{\tilde{C}})^W$ W act on both?

$T = \{t\in\overline{T} \mid \alpha$ root of $\mathfrak{g}$, $\alpha(t)\in\mathcal{O}^\times_{\tilde{C}}$. $\alpha(t)|_{D^\alpha}=1$ is required

$D^\alpha \subseteq \tilde{C}$ fixed divisor for $\rho_\alpha\in W$ the reflection, acting on $\tilde{C}\downarrow C$ $\}$

$T^0 =$ conn. component of T (what is conn. comp of sheaf?)

$\overline{T}_{\mathbb{R}}, T_{\mathbb{R}}, T^0_{\mathbb{R}}$: replace $\mathcal{O}^\times_{\tilde{C}}$ by $\underline{S}^1$, i.e. $\mathbb{C}^\times$ by S^1

Fiber at unramified pt : $\Lambda\otimes\mathbb{R}/\mathbb{Z}$ (for $\overline{T}_{\mathbb{R}}$ At each pt in fiber of $c\in C$, it's $\Lambda\otimes S^1$, taking W invariant $\Rightarrow$ only one copy of $\Lambda\otimes S^1$)

(for $T_{\mathbb{R}}$, condition vacuous