Signature PRINTED NAME

PRELIMINARY EXAMINATION, PART I

August 26, 2022 9:30am-12:30pm

This examination is based on Penn's code of academic integrity

Instructions:

Sign and print your name above.

This part of the examination consists of six problems, each worth ten points. You should work all of the problems. Show all of your work. Try to keep computations well-organized and proofs clear and complete — and justify your assertions. Each problem should be given its own page (or more than one page, if necessary).

If a problem has multiple parts, earlier parts may be useful for later parts. Moreover, if you skip some part, you may still use the result in a later part.

Be sure to write your name both on the exam and on any extra sheets you may submit.

Score (for faculty use only)	
1	
2	
3	
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6	
GRADER	

1. Let R be the region in the first quadrant of the x, y-plane (where $x \geq 0, y \geq 0$ bounded between the axes and the curve $y = 1 - x^2$. Let C be the boundary of R, oriented counterclockwise. Evaluate the line integral

$$\oint_C \left(e^{2x} + xy\right)\, dx + \left(\sin(y^2) + x^2\right)\, dy,$$

Solution: Let R_I denote the interior of R. By Green's Theorem, the integral equals

$$\begin{aligned}
&\int\int_{R_I} \partial_x \left(\sin(y^2) + x^2\right) - \partial_y \left(e^{2x} + xy\right)\, dA \\
&\qquad = \int_0^1 \int_0^{1-x^2} x\, dy\, dx = \int_0^1 x(1 - x^2)\, dx \\
&\qquad = \left(x^2/2 - x^4/4\right)\Big|_0^1 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.
\end{aligned}$$

2. Suppose that $f_n(x)$ is continuous on $[a, b]$, $n = 1, 2, \cdots$, and the series $\sum_{n=1}^{\infty} f_n(x)$ is uniformly convergent on (a, b). Is it true that $\sum_{n=1}^{\infty} f_n(x)$ also converges uniformly on $[a, b]$? If so, please give a proof; otherwise, find a counterexample.

Solution: Yes, the series must also converge uniformly on the closed interval $[a, b]$. To see this, for all $\epsilon > 0$, since the series converges uniformly on (a, b), there exists $N \in \mathbb{N}$ such that for all $n > N$ and all $k \in \mathbb{N}$,

$$|f_{n+1}(x) + f_{n+2}(x) + \cdots + f_{n+k}(x)| < \epsilon, \quad \forall x \in (a, b).$$

Letting $x \to a^+$ and by the continuity assumption, one has

$$|f_{n+1}(a) + f_{n+2}(a) + \cdots + f_{n+k}(a)| \leq \epsilon.$$

Similarly for the point b. In conclusion, we have showed that for all $\epsilon > 0$, there exists $N \in \mathbb{N}$, such that $\forall n > N$ and $k \in \mathbb{N}$,

$$|f_{n+1}(x) + f_{n+2}(x) + \cdots + f_{n+k}(x)| \leq \epsilon, \quad \forall x \in [a, b].$$

Therefore, the series converges uniformly on $[a, b]$.

3. Which of the following rings are fields? Justify your answers.

   (a) The quotient ring $\mathbb{Q}[x]/I$ when I is the ideal $\mathbb{Q}[x] \cdot (x^2 + 1)$.

   (b) The quotient ring $\mathbb{Q}[x]/J$ when $J$is the ideal $\mathbb{Q}[x] \cdot (x^2 - 1)$.

   (c) The ring of meromorphic functions f on $\mathbb{C}$.

(d) The ring of elements of $\mathrm{Mat}_2(\mathbb{Q})$ that commute with the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Solution: The ring in (1) is a field, since x^2+1 is irreducible in $\mathbb{Q}[x]$ and I is a maximal ideal. The ring in (2) is not a field since $x^2-1=(x-1)(x+1)$ so $\mathbb{Q}[x]\cdot(x^2-1)$ has zero divisors. The ring in (3) is a field since the inverse of a non-zero meromorphic function is meromorphic. To analyze the ring in (4), suppose

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b & -a \\ d & -c \end{pmatrix}$$

equals

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix}$$

This is true if and only if

$$b=-c \quad \text{and} \quad a=d.$$

So the ring in part (4) is the ring of all linear combinations

$$a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\cdot\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

with $a,b\in\mathbb{Q}$. Since

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2 = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

the ring in part (4) is isomorphic to $\mathbb{Q}(i)$, which is a field.

4. Compute

$$\int_0^\infty \frac{x\sin x}{x^2+a^2}\,\mathrm{d}x$$

for $a>0$. Justify your answer.

Solution: Let

$$f(z)=\frac{ze^{iz}}{z^2+a^2}.$$

For X_1, X_2, $Y>0$, consider the contour integral of f along the counterclockwise boundary of the rectangle with corners X_2, X_2+iY, $-X_1$, $-X_1+iY$. Since $|e^{iz}|=e^{-y}$ for $z=x+iy$, The integral on the two vertical lines are bounded by $\frac{C}{X_1}$ and $\frac{C}{X_2}$ respectively, for some constant $C>0$. The integral along the top line is bounded by $Ce^{-Y}(X_1+X_2)/Y$, which tends to zero as $Y\to\infty$ for fixed

X_1, X_2. By first sending $Y \to \infty$ and then $X_1, X_2 \to \infty$, the contour integral converges to $\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+a^2}\,\mathrm{d}x$. By Cauchy Residue Theorem,

$$\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2+a^2}\,\mathrm{d}x = 2\pi i \mathrm{Res}(f; ia) = 2\pi i \lim_{x \to ia} \frac{xe^{ix}}{x+ia} = i\pi e^{-a}.$$

So

$$\int_0^{\infty} \frac{x \sin x}{x^2+a^2}\,\mathrm{d}x = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+a^2}\,\mathrm{d}x = \frac{\pi e^{-a}}{2}.$$

5. (a) Is $\mathbb{F}_4$ a subfield of $\mathbb{F}_8$? Is $\mathbb{F}_9$ a subfield of $\mathbb{F}_{81}$? If yes in either case, what is the degree of the field extensions?

   (b) Take a power of a prime number $q = p^n$. Let k be a positive integer that divides n, and let $f \in \mathbb{F}_p$ be a monic irreducible polynomial of degree k. Prove that f divides $e_q(x) = x^q - x$.

   (c) Factor $e_{16}(x) = x^{16} - x$ over $\mathbb{F}_4$.

Solution

(a) $\mathbb{F}_4$ is not a subfield of $\mathbb{F}_8$ because 8 is not a power of 4. $\mathbb{F}_9$ is a subfield of $\mathbb{F}_{81}$ because $81 = 9^2$, and the degree of this field extension is 2.

(b) $\mathbb{F}_p[x]/(f(x)) \cong \mathbb{F}_{p^k}$. Since $k|n$ and $q = p^n$, we have $\mathbb{F}_{p^k} \subset \mathbb{F}_q$. The polynomial f, being monic and irreducible over $\mathbb{F}_p$, is the minimal polynomial of some element $\alpha \in \mathbb{F}_{p^k}$, and every element of $\mathbb{F}_q$ is a root of e_{16}, so $e_{16}(\alpha) = 0$, and thus $f|e_{16}$.

(c) Let ω be the primitive third root of unity, so $\mathbb{F}_4 = \mathbb{F}_2[\omega] = \{0, 1, \omega, \omega^2\}$, and $\omega^2 = \omega + 1$. By the previous problem, every irreducible polynomial $f \in \mathbb{F}_2$ of degree k with $k|4$ divides e_{16}, so we need to find all irreducible polynomials of degrees 1, 2 and 4:

- Degree one: x, $x+1$;
- Degree two: $x^2 + x + 1$;
- Degree four: $x^4 + x + 1$, $x^4 + x^3 + 1$, $x^4 + x^3 + x^2 + x + 1$.

So we get six polynomials of degrees summing up to 16, and each divides e_{16}, hence we get a decomposition

$$e_{16}(x) = x(x+1)(x^2+x+1)(x^4+x+1)(x^4+x^3+1)(x^4+x^3+x^2+x+1),$$

but not all of the factors are irreducible over $\mathbb{F}_4$, for example $x^2 + x + 1 = (x+\omega)(x+\omega^2)$. None of the linear polynomials in $\mathbb{F}_4[x]$ divides our degree four factors, so we need to check whether any of the degree two irreducible

polynomials in $\mathbb{F}_4[x]$ divide those. Here is the list of degree two irreducible polynomials in $\mathbb{F}_4[x]$:
$x^2+\omega x+1$, $x^2+\omega^2 x+1$, $x^2+x+\omega$, $x^2+\omega x+\omega$, $x^2+x+\omega^2$, $x^2+\omega^2 x+\omega^2$.
In fact, we can multiply some of the pairs and get:

$$\begin{aligned}(x^2+\omega x+1)(x^2+\omega^2 x+1) &= x^4+x^3+x^2+x+1,\\ (x^2+x+\omega)(x^2+x+\omega^2) &= x^4+x+1,\\ (x^2+\omega x+\omega)(x^2+\omega^2 x+\omega^2) &= x^4+x^3+1.\end{aligned}$$

Thus we can split e_{16} into a product of linear and quadratic terms that are irreducible in $\mathbb{F}_4[x]$:

$$\begin{aligned}e_{16}(x) = x(x+1)(x+\omega)(x+\omega^2)(x^2+x+\omega)(x^2+x+\omega^2)\\ (x^2+\omega x+\omega)(x^2+\omega^2 x+\omega^2)(x^2+\omega x+1)(x^2+\omega^2 x+1).\end{aligned}$$

6. Solve the following ordinary differential equation

$$\frac{d^2y}{dx^2} = 2y^3, \quad y(0)=1, \quad \left.\frac{dy}{dx}\right|_{x=0} = -1.$$

[Hint: Multiply dy/dx on both sides and perform integration.]

Solution: Multiplying dy/dx and performing integration gives

$$\frac{1}{2}\left(\frac{dy}{dx}\right)^2 = \frac{1}{2}y^4 + C_1,$$

and the constant is determined by the initial conditions, $C_1 = 0$. Since $dy/dx < 0$ and $y > 0$ at $x = 0$, we choose

$$\frac{dy}{dx} = -y^2.$$

Dividing both sides by $-y^2$ and integrating both sides gives

$$\frac{1}{y} = x + C_2,$$

and again the constant is determined by the initial condition, $C_2 = 1$. Therefore, we have the solution

$$y(x) = \frac{1}{1+x}.$$