Signature PRINTED NAME

PRELIMINARY EXAMINATION, PART II

Friday, August 26, 2022 1:30pm-4:00pm

This examination is based on Penn's code of academic integrity

Instructions:

Sign and print your name above.

This part of the examination consists of six problems, each worth ten points. You should work all of the problems. Show all of your work. Try to keep computations well-organized and proofs clear and complete — and justify your assertions. Each problem should be given its own page (or more than one page, if necessary).

If a problem has multiple parts, earlier parts may be useful for later parts. Moreover, if you skip some part, you may still use the result in a later part.

Be sure to write your name both on the exam and on any extra sheets you may submit.

Score (for faculty use only)	
7	
8	
9	
10	
11	
12	
GRADER	

7. Prove or give a counterexample of each of the statements a) and b) below.

   (a) (5 points) Let M be a metric space, and write $\overline{A}$ for the closure of a subset $A \subseteq M$. Then $\overline{A} \cap \overline{B} = \overline{A \cup B}$, for all subsets A, B of M.

   (b) (5 points) Let $f : X \to Y$ be a continuous map between metric spaces. If $B \subseteq Y$ is compact, then so is $f^{-1}(B) \subseteq X$.

Solution: a) This statement is false. For example, take $A = \mathbb{Q}$, and $B = \mathbb{R} \backslash \mathbb{Q}$. Then $A \cap B = \emptyset$, but $\overline{A} = \overline{B} = \mathbb{R}$ implying

$$\overline{A} \cap \overline{B} = \mathbb{R} \neq \emptyset = \overline{A \cap B}.$$

b) This statement is also false. For example, if $X = \mathbb{R}$, then for any constant function $f : \mathbb{R} \to Y$, $x \to c$, the preimage $f^{-1}(c) = \mathbb{R}$ is not compact.

8. Let $A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$.

   (a) Find the Jordan canonical form J of A.

   (b) Find a matrix P such that $A = PJP^{-1}$.

Solution: (a) The eigenvalues of A are $\lambda = 4$ and $\lambda = 1$ (with multiplicity 2). For $\lambda = 4$, we have

$$A - 4I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -3 & 0 \\ -1 & -1 & -3 \end{bmatrix},$$

so an eigenvector is $\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$.

For $\lambda = 1$, we have $A - I = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & -1 & 0 \end{bmatrix}$, which has rank 2, so there is only one linearly independent eigenvector, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. This means that the Jordan form of A is $J = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

(b) We have the eigenvectors for $\lambda = 4$ and $\lambda = 1$, and we need a generalized eigenvector for $\lambda = 1$. Since $(A-I)^2 = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 0 & 0 \\ -3 & 0 & 0 \end{bmatrix}$, we can take our generalized eigenvector to be $\mathbf{v_1} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ (since this is in $\ker(A-I)^2$ but not in $\ker(A-I)$). Then $\mathbf{v_2} = (A-I)\mathbf{v_1} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$ and so $P = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$

9. Let G be a group of order $48 = 2^4 \cdot 3$.

   (a) If P is a Sylow 2-subgroup of G, what is the index $n = [G : P]$? Justify.

   (b) Describe the non-trivial homomorphism $G \to S_n$ induced by G acting on G/P.

   (c) Prove that G is not a simple group.

   Solution

   (a) The order of a Sylow 2-subgroup of G is 2^4, so by Lagrange's theorem, we have $n = [G : P] = 2^4 \cdot 3/2^4 = 3$.

   (b) Let $e \in G$ denote the identity element in G and $\varepsilon \in S_3$ the identity permutation, and let $a, b \in G$ be two elements generating the two nontrivial cosets of P, so $G/P = \{P, aP, bP\}$. Let $f\colon G \to S_n$ denote the homomorphism. We will say that S_3 acts on $\{P, aP, bP\}$ via identifying P with 1, aP with 2 and bP with 3. Then for $x \in G$, $f(x)$ is the permutation that sends P to xP, aP to xaP, and bP to xbP. Since $f(a)$ doesn't stabilize P, we have $f(a) \neq \varepsilon$, so f is not the trivial homomorphism.

   (c) Since $|G| = 48 > 6 = |S_3|$, the homomorphism from Part 2 is not injective. By the result of Part 2, the kernel of f is not equal to G, and it is a normal subgroup. So G is not simple.

10. Prove that the two periodic functions agree:

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{-\infty}^{+\infty} \frac{1}{(z-n)^2}.$$

Hint: Start by showing the function

$$g(z) = \frac{\pi^2}{\sin^2 \pi z} - \sum_{-\infty}^{+\infty} \frac{1}{(z-n)^2}$$

is periodic and entire.

Solution: Let

$$g(z) = \frac{\pi^2}{\sin^2 \pi z} - \sum_{-\infty}^{+\infty} \frac{1}{(z-n)^2}.$$

Since $\lim_{z \in \to n} g(z)$ is finite for each n, the function g is entire. Note that $g(z) = g(z+1)$. For $z = x + iy$, we can check that $\frac{1}{\sin^2 \pi z}$ tend to 0 uniformly in $x \in [0,1]$ as $|y| \to \infty$. The same holds for the series $\sum_{-\infty}^{+\infty} \frac{1}{(z-n)^2}$. Therefore g is a bounded entire function, hence by Liouville Theorem must be a constant. Since $\lim_{y \to \infty} g(iy) = 0$ we have $g = 0$ everywhere.

11. Let $x = x(y,z), y = y(x,z), z = z(x,y)$ be the implicit functions determined by the equation $F(x,y,z) = 0$. Prove that

$$\frac{\partial x}{\partial y} \cdot \frac{\partial y}{\partial z} \cdot \frac{\partial z}{\partial x} = -1.$$

Solution: From the Implicit Function Theorem, one has that

$$\partial_y F(x,y,z) + \partial_x F(x,y,z) \cdot \frac{\partial x(y,z)}{\partial y} = 0,$$

in other words,

$$\frac{\partial x}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial x}}.$$

Similarly, one has

$$\frac{\partial y}{\partial z} = -\frac{\frac{\partial F}{\partial z}}{\frac{\partial F}{\partial y}}, \quad \frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}.$$

Therefore, the product of the three partial derivatives is equal to -1.

12. Let

$$I(R) = \iint_{R^2 \le x^2+y^2 \le 4R^2} \frac{1+x^2}{2022+x^4+y^4} dxdy$$

Find $\lim_{R \to \infty} I(R)$ or show this limit does not exist. [Use the fact that $\int_0^{2\pi} (\cos^4 t + \sin^4 t)^{-1} dt = 2\pi\sqrt{2}$.]

Solution: Using the symmetry we have

$$2I(R) = \iint_{R^2 \le x^2+y^2 \le 4R^2} \frac{2+x^2+y^2}{2022+x^4+y^4} dxdy.$$

Changing of variables with polar coordinate yields

$$2I(R) = \int_0^{2\pi} \int_R^{2R} \frac{2 + r^2}{2022 + r^4(\cos^4 t + \sin^4 t)} r dr dt.$$

As $R \to \infty$, the range of integration goes to infinity and the leading order behavior of the integrand is

$$\frac{2r + r^3}{2022 + r^4(\cos^4 t + \sin^4 t)} = \frac{1}{r(\cos^4 t + \sin^4 t)} + \mathcal{O}\left(\frac{1}{r^3}\right).$$

Therefore, we have

$$\lim_{R\to\infty} 2I(R) = \lim_{R\to\infty\infty} \int_R^{2R} \frac{dr}{r} \int_0^{2\pi} \frac{dt}{\cos^4 t + \sin^4 t}.$$

The first integral satisfies

$$\lim_{R\to\infty\infty} \int_R^{2R} \log(2R) - \log(R) = \log(2).$$

Therefore, the limit goes to

$$\lim_{R\to\infty} I(R) = \pi\sqrt{2}\log 2.$$