Signature PRINTED NAME

PRELIMINARY EXAMINATION, PART I

Monday, August 30, 2021 9:30am-12:30pm

This examination is based on Penn's code of academic integrity

Instructions:

Sign and print your name above.

This part of the examination consists of six problems, each worth ten points. You should work all of the problems. Show all of your work. Try to keep computations well-organized and proofs clear and complete — and justify your assertions. Each problem should be given its own page (or more than one page, if necessary).

If a problem has multiple parts, earlier parts may be useful for later parts. Moreover, if you skip some part, you may still use the result in a later part.

Be sure to write your name both on the exam and on any extra sheets you may submit.

Score (for faculty use only)	
1	
2	
3	
4	
5	
6	
GRADER	

1. Given real numbers a, b, c, d, consider the differential equation $E(a, b, c, d)$ given by $y'' + (a \sin x - 3)y' + (be^x + 2)y + c \cos x = d$.

   (a) Find the set S of all $(a, b, c, d) \in \mathbb{R}^4$ such that the solutions to the differential equation $E(a, b, c, d)$ form a real vector space $V(a, b, c, d)$ under addition and scalar multiplication of functions.

   (b) Pick some $(a, b, c, d) \in S$ (your choice), and find a basis for the vector space $V(a, b, c, d)$.

   Explain your assertions.

Extra page for work for problem 1.

2. (a) Give an example of a non-abelian group G, generated by two elements g, h, such that the center of G is non-trivial.

   (b) Show that no such example can exist if one additionally requires that g is in the center of G.

Extra page for work for problem 2.

3. Let $f(x) = 1/x$ for $x \neq 0$. On which of the following intervals is the function f uniformly continuous? Explain your assertions.

   (i) $1 \leq x \leq 2$.

   (ii) $1 < x < 2$.

   (iii) $0 < x < 1$.

Extra page for work for problem 3.

4. For each of the following either give an example of a real square matrix M with the given properties or explain why none exists:

   (a) M is not similar over $\mathbb{R}$ to an upper triangular matrix.

   (b) M is similar over $\mathbb{R}$ to an upper triangular matrix but is not similar over $\mathbb{R}$ to a diagonal matrix.

   (c) M is not similar over $\mathbb{C}$ to an upper triangular matrix.

   (d) M is similar over $\mathbb{C}$ to an upper triangular matrix but is not similar over $\mathbb{C}$ to a diagonal matrix.

Extra page for work for problem 4.

5. Let $f(x) = x^2 + 2y^2 - 2xy + 2x$, let D be the closed disc $x^2 + y^2 \leq 10$, and let D' be the interior of D.

   (a) Does the restriction of f to D achieve a maximum? (I.e., is there a point $(a, b) \in D$ such that $f(a, b) \geq f(x, y)$ for all $(x, y) \in D$?) Similarly, does it achieve a minimum on D? Justify your assertions. (You are not asked to *find* the maximum and minimum if you assert that they exist.)

   (b) Does the restriction of f to D' achieve a maximum, and does it achieve a minimum? If it does, find where the maximum (resp. minimum) is achieved. Justify your assertions.

Extra page for work for problem 5.

6. Let $f(x,y) = e^{xy^5} + x^{10} + \cos(y^2)$, let $g = \partial f/\partial x$, and let $h = \partial f/\partial y$. Let C be the path in the plane from the origin to the point $(1,0)$ given by the portion of the graph of $y = \sin^3(\pi x)$ over the interval $0 \leq x \leq 1$. Evaluate $\int_C g\,dx + h\,dy$. Explain your computations. [Hint: This does not require a brute force calculation of the integral.]

Extra page for work for problem 6.

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PRELIMINARY EXAMINATION, PART II

Monday, August 30, 2021 2-5pm

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Score (for faculty use only)	
7	
8	
9	
10	
11	
12	
GRADER	

7. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, with graph Γ.

   (a) Show that $\{(x, y) \mid y > f(x)\}$ is an open subset of $\mathbb{R}^2$.

   (b) Show that the complement of Γ in $\mathbb{R}^2$ is disconnected.

Extra page for work for problem 7.

8. Let V be the span of the four vectors $(1,-1,0,1)$, $(2,-1,1,6)$, $(-1,2,1,3)$, $(1,0,1,5)$ in $\mathbb{R}^4$. With respect to the usual inner product on $\mathbb{R}^4$, find an orthogonal basis of V, and find the point on V closest to $(1,1,1,1)$.

Extra page for work for problem 8.

9. Let $\{a_n\}$ be a sequence of real numbers. For each of the following, give either a proof or a counter-example:

   (a) If $\sum_{n=1}^{\infty} a_n$ is convergent but not absolutely convergent, then $\sum_{n=1}^{\infty} n a_n$ is divergent.

   (b) If $\sum_{n=1}^{\infty} a_n$ is convergent but not absolutely convergent, then $\sum_{n=1}^{\infty} n^2 a_n$ is divergent.

Extra page for work for problem 9.

10. (a) Show that if $\mathfrak{m}$ is a maximal ideal in $\mathbb{Q}[x]$, then $\mathbb{Q}[x]/\mathfrak{m}$ is a field extension of $\mathbb{Q}$ of finite degree.

    (b) Conversely, show that if K is a field extension of $\mathbb{Q}$ of finite degree, then K is isomorphic to $\mathbb{Q}[x]/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ of $\mathbb{Q}[x]$.

Extra page for work for problem 10.

11. Let $f(x)$ be a differentiable function on the real line such that $f(0) = 0$ and $f'(0) = 1$. Prove directly, from the definition of the derivative, that there exists a positive real number c such that $f(x) > 0$ for all x with $0 < x < c$.

Extra page for work for problem 11.

12. Let v, w be elements of a finite dimensional real vector space V. Prove that there is a linear transformation $T : V \to \mathbb{R}^2$ such that $T(v) = (1, 0)$ and $T(w) = (0, 1)$ if and only if v, w are linearly independent vectors.

Extra page for work for problem 12.